Method of Variation of Parameters

1^{st}-order ODEs

$$\frac{dy}{dx} + P(x)y = f(x)$$

The procedure: Variation of parameters

Idea: to find a function u so that $y_p = u(x)y_1(x) = u(x)e^{-\int P(x)dx}$ is a solution of (5).

Substituting $y_p = uy_1$ into the equation gives

$$\begin{aligned} u\frac{dy_1}{dx} + y_1\frac{du}{dx} + P(x)uy_1 &= f(x) \\ u\left[\frac{dy_1}{dx} + P(x)y_1\right] + y_1\frac{du}{dx} &= f(x) \end{aligned}$$

and since y_1 is the solution of the homogeneous equation, the expression in the square bracket is zero and

$$y_1\frac{du}{dx} = f(x)$$

$$y_1 \frac{\mathrm{d}u}{\mathrm{d}x} = f(x)$$

Separating variables and integrating then gives

$$du = \frac{f(x)}{y_1(x)} dx \qquad \Rightarrow \qquad u = \int \frac{f(x)}{y_1(x)} dx$$

Since $y_1(x) = e^{-\int P(x)dx}$, $1/y_1(x) = e^{\int P(x)dx}$, and therefore

$$y_p = uy_1 = \left(\int \frac{f(x)}{y_1(x)} dx \right) e^{-\int P(x)dx} = e^{-\int P(x)dx} \int e^{\int P(x)dx} f(x)dx$$

and the solution of (5) is then of the form

$$y = y_c + y_p = ce^{-\int P(x)dx} + e^{-\int P(x)dx} \int e^{\int P(x)dx} f(x)dx$$

Method of Variation of Parameters

2^{nd}-order ODEs

The method of variation of parameters

Advantage: the method always yields a particular solution y_p, provided the associated homogeneous equation can be solved. Also it is not limited to certain types of $g(x)$.

First we put a linear second-order DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x) \tag{18}$$

into the standard form by dividing by $a_2(x)$

$$y'' + P(x)y' + Q(x)y = f(x) \tag{19}$$

We seek a solution of the form

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where y_1 and y_2 form a fundamental set of solutions on I of the associated homogeneous form of (18). Using the product rule to differentiate y_p twice gives

$$\begin{aligned} y_p' &= u_1 y_1' + y_1 u_1' + u_2 y_2' + y_2 u_2' \\ y_p'' &= u_1 y_1'' + y_1' u_1' + y_1 u_1'' + u_1' y_1' + u_2 y_2'' + y_2' u_2' + y_2 u_2'' + u_2' y_2' \end{aligned}$$

Substituting these into the standard form (19) yields

$$\begin{aligned} y_p'' + P(x) y_p' + Q(x) y_p &= u_1 \left[y_1'' + P y_1' + Q y_1 \right] + u_2 \left[y_2'' + P y_2' + Q y_2 \right] \\ &\quad + y_1 u_1'' + u_1' y_1' + y_2 u_2'' + u_2' y_2' + P \left[y_1 u_1' + y_2 u_2' \right] + y_1' u_1' + y_2' u_2' \\ &= \frac{\mathrm{d}}{\mathrm{d}x} \left[y_1 u_1' \right] + \frac{\mathrm{d}}{\mathrm{d}x} \left[y_2 u_2' \right] + P \left[y_1 u_1' + y_2 u_2' \right] + y_1' u_1' + y_2' u_2' \\ &= \frac{\mathrm{d}}{\mathrm{d}x} \left[y_1 u_1' + y_2 u_2' \right] + P \left[y_1 u_1' + y_2 u_2' \right] + y_1' u_1' + y_2' u_2' = f(x) \end{aligned} \tag{20}$$

We need two equations for two unknown functions u_1 and u_2. Assuming that these functions satisfy $y_1u_1' + y_2u_2' = 0$, the equation above reduces to $y_1'u_1' + y_2'u_2' = f(x)$. By Cramer's rule, the solution of the system

$$\begin{aligned} y_1u_1' + y_2u_2' &= 0 \\ y_1'u_1' + y_2'u_2' &= f(x) \end{aligned}$$

can be expressed in terms of determinants:

$$u_1' = \frac{W_1}{W} = -\frac{y_2 f(x)}{W} \quad \text{and} \quad u_2' = \frac{W_2}{W} = \frac{y_1 f(x)}{W} \tag{21}$$

where

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}.$$

The functions u_1 and u_2 are found by integrating the result in (21). The determinant W is the Wronskian of y_1 and y_2 whose linear independence ensures that $W \neq 0$.

Example: General solution using variation of parameters

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

From the auxiliary equation $m^2 - 4m + 4 = (m-2)^2 = 0$ we have $y_c = c_1 e^{2x} + c_2 x e^{2x}$. We identify $y_1 = e^{2x}$ and $y_2 = xe^{2x}$ and evaluate the Wronskian

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = e^{4x}$$

The DE above is already in the standard form, so $f(x) = (x+1)e^{2x}$, W_1 and W_2 are then

$$W_1 = \begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = -(x+1)xe^{4x}, \quad W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix} = (x+1)e^{4x}$$

and so

$$u_1' = -\frac{(x+1)xe^{4x}}{e^{4x}} = -x^2 - x, \qquad u_2' = \frac{(x+1)e^{4x}}{e^{4x}} = x + 1$$

It follows that $u_1 = -\frac{1}{3}x^3 - \frac{1}{2}x^2$ and $u_2 = \frac{1}{2}x^2 + x$, and hence

$$y_p = \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right)e^{2x} + \left(\frac{1}{2}x^2 + x\right)xe^{2x} = \frac{1}{6}x^3e^{2x} + \frac{1}{2}x^2e^{2x}$$

The general solution is then

$$y = y_c + y_p = c_1e^{2x} + c_2xe^{2x} + \frac{1}{6}x^3e^{2x} + \frac{1}{2}x^2e^{2x}$$

Generalization to linear n–th order equations with $n > 2$

Standard form:

$$y^{(n)} + P_{n-1}(x)y^{(n-1)} + \ldots + P_1(x)y' + P_0(x)y = f(x)$$

If $y_c = c_1y_1 + c_2y_2 + \ldots + c_ny_n$ is the complementary function for the equation above then a particular solution is

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) + \ldots + u_n(x)y_n(x)$$

where u_k', $k = 1, 2, \ldots, n$ are determined by the n equations

$$\begin{array}{ccccccccc}
y_1u_1' & + & y_2u_2' & + & \ldots & + & y_nu_n' & = & 0 \\
y_1'u_1' & + & y_2'u_2' & + & \ldots & + & y_n'u_n' & = & 0 \\
\vdots & & & & & & & & \vdots \\
y_1^{(n-1)}u_1' & + & y_2^{(n-1)}u_2' & + & \ldots & + & y_n^{(n-1)}u_n' & = & f(x)
\end{array}$$

The first $n - 1$ equations, like $y_1 u_1' + y_2 u_2' = 0$ in the second order case, are assumptions made to simplify the resulting equation after

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) + \ldots + u_n(x)y_n(x)$$

is substituted in the higher order ODE.

Application of Cramer's rule gives

$$u_k' = \frac{W_k}{W}, \qquad k = 1, 2, \ldots, n,$$

where W is the Wronskian of $y_1, y_2, \ldots, y_n$ and W_k is the determinant obtained by replacing the kth column of the Wronskian by the column $(0, 0, \ldots, f(x))$.

Example: $n = 3$

The particular solution is

$$y_p = u_1 y_1 + u_2 y_2 + u_3 y_3$$

where y_1, y_2, and y_3 are linearly independent solutions of the associated homogeneous ODE, and u_1, u_2, u_3 are determined from

$$u_1' = \frac{W_1}{W}, \qquad u_2' = \frac{W_2}{W}, \qquad u_3' = \frac{W_3}{W},$$

where

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}, W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ f(x) & y_2'' & y_3'' \end{vmatrix}, W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & f(x) & y_3'' \end{vmatrix}, W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & f(x) \end{vmatrix}.$$