Higher-order ODEs: overview

- General aspects
- Reduction of order
- Homogeneous linear equation with constant coefficients
- Method of undetermined coefficients
- Method of variation of parameters
- Linear models
- Examples

HIGHER ORDER DIFFERENTIAL EQUATIONS

Theory of linear equations

Initial-value and boundary-value problem

nth-order initial value problem is

Solve: $$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \ \ldots \ + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x)$$

Subject to: $$y(x_0) = y_0,\ y'(x_0) = y_1,\ \ldots, y^{(n-1)} = y_{n-1} \qquad (1)$$

we seek a function defined on an interval I, containing x_0, that satisfies the DE and the n initial conditions above.

Existence and uniqueness

Theorem: Existence of a unique solution

Let $a_n(x)$, $a_{n-1}(x)$, ... , $a_1(x)$, $a_0(x)$ and $g(x)$ be continuous on an interval I and let $a_n(x) \neq 0$ for every x in this interval. If $x = x_0$ in any point in this interval, then a solution $y(x)$ of the initial value problem (1) exists on the interval and is unique.

Example: Unique solution of an IVP

$$3y''' + 5y'' - y' + 7y = 0, \qquad y(1) = 0, y'(1) = 0, y''(1) = 0$$

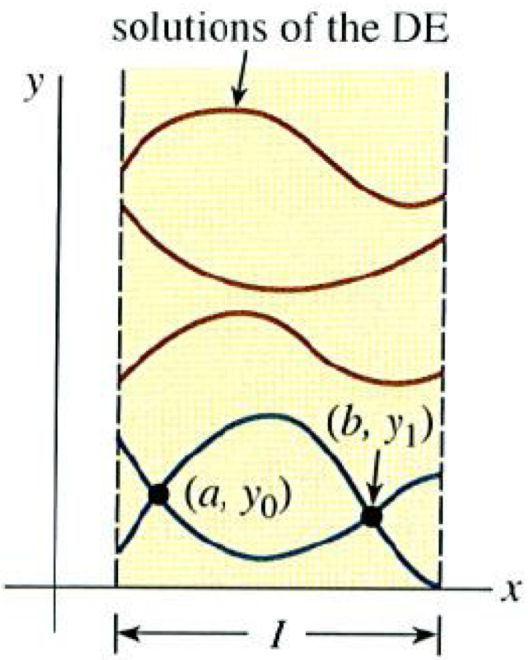
has the trivial solution $y = 0$. Since the DE is linear with constant coefficients, all the conditions of the theorem are fulfilled, and thus $y = 0$ is the *only* solution on any interval containing $x = 1$.

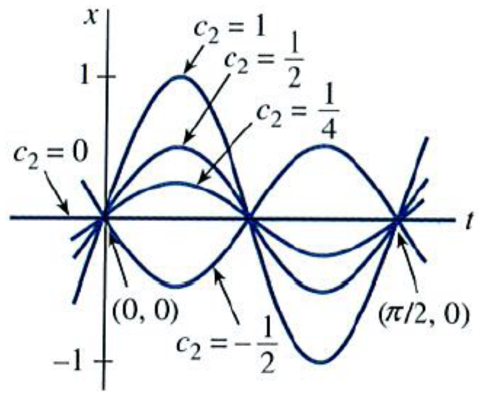
Boundary-value problem

consists of solving a linear DE of order two or greater in which the dependent variable y or its derivatives are specified at *different points*. Example: a two-point BVP

Solve: $$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Subject to boundary conditions: $$y(x_0) = y_0,\ y(b) = y_1 \qquad (2)$$





A BVP can have many, one or no solutions:

The DE $x'' + 16x = 0$ has the two-parameter family of solutions $x = c_1 \cos 4t + c_2 \sin 4t$. Consider the BVPs:

(1) $x(0) = 0$, and $x(\pi/2) = 0 \Rightarrow c_1 = 0$ and the solution satisfies the DE for any value of c_2, thus the solution of this BVP is the one-parameter family $x = c_2 \sin 4t$.

(2) $x(0) = 0$, and $x(\pi/8) = 0 \Rightarrow c_1 = 0$ and $c_2 = 0$, so the only solution to this BVP is $x = 0$.

(3) $x(0) = 0 \Rightarrow c_1 = 0$ again but the second condition $x(\pi/2) = 1$ leads to the contradiction: $1 = c_2 \sin 2\pi = c_2.0 = 0$.

Homogeneous equations

nth-order **homogeneous** differential equation

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \ \dots \ + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = 0 \tag{3}$$

nth-order **nonhomogeneous** differential equation ($g(x) \neq 0$)

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \ \dots \ + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x) \tag{4}$$

Examples:

(1) Homogeneous DE: $2y'' + 3y' - 5y = 0$

(2) Nonhomogeneous DE: $x^2 y''' + 6y' + 10y = e^x$.

To solve a nonhomogeneous DE, we must first be able to solve the **associated homogeneous equation**.

We will soon proceed to the general theory of nth-order linear equations which we will present through a number of definitions and theorems. To avoid needless repetition, we make (and remeber) the following assumptions:

on some common interval I

- the coefficients $a_i(x)$, $i = 0, 1, 2, ..., n$ are continuous;

- the function $g(x)$ on r. h. s. is continuous; and

- $a_n(x) \neq 0$ for every x in the interval.

Differential operators

Examples:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}y = Dy \quad \text{or} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = D\,(Dy) = D^2y \quad \text{and in general} \quad \frac{\mathrm{d}^ny}{\mathrm{d}x^n} = D^ny$$

nth-order differential operator:

polynomial expressions involving D are also differential operators

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \ \ldots\ + a_1(x)D + a_0(x)$$

An nth-order differential operator is a **linear operator**, that is, it satisfies

$$L\,[\alpha f(x) + \beta g(x)] = \alpha L\,(f(x)) + \beta L\,(g(x)) \tag{5}$$

Differential equations

Any linear differential equation can be expressed in terms of the D notation.

Example

$$\begin{aligned} y'' + 5y' + 6y &= 5x - 3 \\ D^2 y + 5Dy + 6y &= 5x - 3 \\ (D^2 + 5D + 6)y &= 5x - 3 \end{aligned}$$

The nth-order linear differential equations can be written compactly as

Homogeneous:	$L(y)$	$=$	0
Non-homogeneous:	$L(y)$	$=$	$g(x)$

Superposition principle

Theorem: Superposition principle - homogeneous equations

Let y_1, y_2, ... , y_k be solutions of the homogeneous nth-order DE (3) on an interval I, then the linear combination

$$y = c_1 y_1(x) + c_2 y_2(x) + ... + c_k y_k(x) = \sum_{i=1}^{k} c_i y_i(x),$$

where the c_i, $i = 1, 2, ..., k$ are arbitrary constants, is also a solution.

Proof: The case $k = 2$. Let $y_1(x)$ and $y_2(x)$ be solutions of $L(y) = 0$, then also

$$L(y) = L\left[c_1 y_1(x) + c_2 y_2(x)\right] = c_1 L(y_1) + c_2 L(y_2) = 0$$

Corollaries

(a) A constant multiple $y = c_1 y_1(x)$ of a solution $y_1(x)$ of a homogeneous linear DE is also a solution.

(b) A homogeneous linear DE always possesses the trivial solution $y = 0$.

Example: Superposition - homogeneous DE
Let $y_1 = x^2$ and $y_2 = x^2 \ln x$ be both solutions of the homogeneous linear DE $x^3 y''' - 2xy' + 4y = 0$ on the interval $I = (0, \infty)$.

Show that by superposition principle, the linear combination

$$y = c_1 x^2 + c_2 x^2 \ln x$$

is also a solution of the equation on the interval.

Linear dependence and linear independence

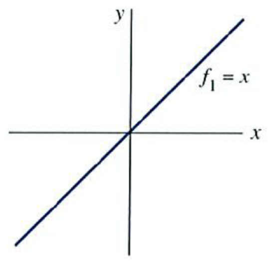
Definition:
A set of functions $f_1(x), f_2(x), ..., f_n(x)$ is said to be **linearly dependent** on an interval I if there exist constants $c_1, c_2, ..., c_n$, not all zero, s.t.

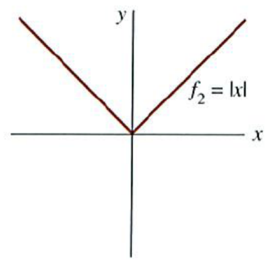
$$c_1 f_1(x) + c_2 f_2(x) + ... + c_n f_n(x) = 0 \qquad (6)$$

for every x in the interval. If the set of functions is not linearly dependent on the interval, it is said to be **linearly independent**.

Example: If two functions are linearly dependent, then one is simply a constant multiple of the other: assuming $c_1 \neq 0$, $c_1 f_1(x) + c_2 f_2(x) = 0 \Rightarrow f_1(x) = -(c_2/c_1) f_2(x)$. For example $f_1(x) = \sin(x)\cos(x)$ and $f_2(x) = \sin(2x) = 2f_1(x)$.

Two functions are linearly independent when neither is a constant multiple of the other on an interval. For example $f_1(x) = x$ and $f_2(x) = |x|$ on $I = (-\infty, \infty)$.





Solutions of differential equations

We are primarily interested in linearly independent solutions of linear DEs.

How to decide whether n solutions $y_1, y_2, ..., y_n$ of a homogeneous linear nth-order DE (3) are linearly independent?

Definition: Wronskian

Suppose each of the functions $f_1(x), f_2(x), ..., f_n(x)$ possesses at least $n-1$ derivatives. The determinant

$$W(f_1, f_2, ..., f_n) = \begin{vmatrix} f_1 & f_2 & ... & f_n \\ f_1' & f_2' & ... & f_n' \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ f_1^{(n-1)} & f_2^{(n-1)} & ... & f_n^{(n-1)} \end{vmatrix}$$

is called the **Wronskian** of the functions.

Theorem: Criterion for linearly independent solutions

Let $y_1, y_2, ..., y_n$ be n solutions of the homogeneous linear nth-order DE (3) on an interval I. Then the set of solutions is **linearly independent** on I if and only if $W(y_1, y_2, ..., y_n) \neq 0$ for every x in the interval.

Definition: Fundamental set of solutions

Any set $y_1, y_2, ..., y_n$ of n linearly independent solutions of the homogeneous linear nth-order DE (3) on an interval I is said to be a **fundamental set of solutions** on the interval.

Theorem: Existence of a fundamental set

There exists a fundamental set of solutions for the homogeneous linear nth-order DE (3) on an interval I.

Theorem: General solution - homogeneous equations

Let $y_1, y_2, ..., y_n$ be a fundamental set of solutions of the homogeneous linear nth-order DE (3) on an interval I. Then the **general** solution of the equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x)$$

where c_i, $i = 1, 2, ..., n$ are arbitrary constants.
For proof for the case $n = 2$ see D.G. Zill et al., Advanced Engineering Mathematics, 4th Edition, p. 104.

Example 1:

The functions $y_1 = e^{3x}$ and $y_2 = e^{-3x}$ are both solutions of the homogeneous linear DE $y'' - 9y = 0$ on $(-\infty, \infty)$.

Calculate the Wronskian and determine whether the functions form a fundamental set of solutions. If yes, determine a general solution.

Example 2:

The function $y = 4\sinh 3x - 5e^{3x}$ is a solution of the DE in Example 1 above. Verify this.

We must be able to obtain this solution from the general solution $y = c_1e^{3x} + c_2e^{-3x}$. What values the constants c_1 and c_2 have to have to get the solution above.

Example 3:

The functions $y_1 = e^x$, $y_2 = e^{2x}$, and $y_3 = e^{3x}$ satisfy the third order DE $y''' - 6y'' + 11y' - 6y = 0$. Determine whether these functions form the fundamental set of solutions on $(-\infty, \infty)$, and write down the general solution.

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ \dots \ + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (4)$$

Nonhomogeneous equations

Any function y_p free of any arbitrary parameters that satisfies (4) is said to be a **particular solution** of the equation.

For example, $y_p = 3$ is a particular solution of $y'' + 9y = 27$.

Theorem: General solution - nonhomogeneous equations

Let y_p be any particular solution of the nonhomogeneous linear nth-order DE (4) on an interval I, and let $y_1, y_2, ..., y_n$ be a fundamental set of solutions of the associated homogeneous DE (3) on I. Then the **general solution** of the equation on I is

$$y = c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x) + y_p \qquad (7)$$

where the $c_i, i = 1, 2, ..., n$ are arbitrary constants.

Complementary function

The general solution of a homogeneous linear equation consists of the sum of two functions

$$y = c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x) + y_p(x) = y_c(x) + y_p(x)$$

The linear combination $y = c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x)$ which is the general solution of the homogeneous DE (3), is called the **complementary solution** for equation (4).

Thus to solve the nonhomogeneous linear DE, we first solve the associated homogeneous equation and then find any particular solution of the nonhomogeneous equation. The general solution is then

y = complementary function + any particular solution.

Another superposition principle

Theorem: Superposition principle - nonhomogeneous equations

Let $y_{p_1}, y_{p_2}, ..., y_{p_k}$ be k particular solutions of the nonhomogeneous linear nth-order DE (4) on an interval I corresponding, in turn, to k distinct functions $g_1, g_2, ..., g_k$. That is, suppose y_{p_i} denotes a particular solution of the corresponding DE

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ . \ . \ . \ + a_1(x)y' + a_0(x)y = g_i(x) \qquad (8)$$

where $i = 1, 2, ..., k$. Then

$$y_p = y_{p_1}(x) + y_{p_2}(x) + ... + y_{p_k}(x) \qquad (9)$$

is a particular solution of

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ . \ . \ . \ + a_1(x)y' + a_0(x)y = g_1(x) + g_2(x) + ... + g_k(x) \qquad (10)$$

For proof for the case $k = 2$ see D.G. Zill et al., Advanced Engineering Mathematics, 4th Edition, p. 104.

Example:

Verify that

$$
\begin{aligned}
y_{p_1} &= -4x^2 && \text{is a particular solution of} \quad y'' - 3y' + 4y = -16x^2 + 24x - 8 \\
y_{p_2} &= e^{2x} && \text{is a particular solution of} \quad y'' - 3y' + 4y = 2e^{2x} \\
y_{p_3} &= xe^{x} && \text{is a particular solution of} \quad y'' - 3y' + 4y = 2xe^{x} - e^{x}
\end{aligned}
$$

and that $y = y_{p_1} + y_{p_2} + y_{p_3} = -4x^2 + e^{2x} + xe^{x}$ is a solution of

$$y'' - 3y' + 4y = -16x^2 + 24x - 8 + 2e^{2x} + 2xe^{x} - e^{x}$$

Remarks:

A dynamical system whose mathematical model is a linear nth-order DE

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \ \ldots \ + a_1(t)y' + a_0(t)y = g(t)$$

is said to be a **linear system**. The set of n time dependent functions $y(t), y'(t)$, ... , $y^{(n-1)}(t)$ are the **state variables** of the system. Their values at some time t give the **state of the system**. The function g is called the **input function**, **forcing function**, or **excitation function**. A solution $y(t)$ of the DE is said to be the **output** or **response of the system**. The output or response $y(t)$ is uniquely determined by the input and the state of the system prescribed at a time t_0; that is, by the initial conditions $y(t_0), y'(t_0), ..., y^{(n-1)}(t_0)$.

Reduction of order

Suppose $y(x)$ denotes a known solution of a homogeneous linear second-order equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{11}$$

we seek the second solution $y_2(x)$ so that y_1 and y_2 are linearly independent on some interval I. That is we are looking for y_2 s. t. $y_2/y_1 = u(x)$, or $y_2(x) = u(x)y_1(x)$.

The idea is to find $u(x)$ by substituting $y_2(x) = u(x)y_1(x)$ into the DE. This method is called **reduction of order** since we must solve a first-order equation to find u.

Example:

Given $y_1 = e^x$ is a solution of $y'' - y = 0$ on $(-\infty, \infty)$, use the reductions of order to find a second solution y_2.

If $y = u(x)y_1(x) = u(x)e^x$ then

$$\begin{aligned} y' &= ue^x + e^x u' \\ y'' &= ue^x + 2e^x u' + e^x u'' \end{aligned}$$

and substituting into the original ODE and using the substitution $w = u'$ we get

$$y'' - y = e^x(u'' + 2u') = 0 \quad \Rightarrow \quad u'' + 2u' = 0 \quad \Rightarrow \quad w' + 2w = 0.$$

Using the integrating factor e^{2x}, and get $w = u' = c_1 e^{-2x}$ and integrating again yields $u = -\frac{1}{2}c_1 e^{-2x} + c_2$, so the second solution is e^{-x}:

$$y = u(x)e^x = -\frac{c_1}{2}e^{-x} + c_2 e^x.$$

General case

We put the equation (11) into the standard form by dividing by $a_2(x)$:

$$y'' + P(x)y' + Q(x)y = 0 \tag{12}$$

where $P(x)$ and $Q(x)$ are continuous on some interval I. Assume that $y_1(x)$ is a known solution of (12) on I and that $y_1(x) \neq 0$ for every $x \in I$. We define $y = u(x)y_1(x)$

$$y' = uy_1' + y_1u', \qquad y'' = uy_1'' + 2y_1'u' + y_1u''$$
$$y'' + Py' + Qy = u\left[y_1'' + Py_1' + Qy_1\right] + y_1u'' + \left(2y_1' + Py_1\right)u' = 0$$

where the term in the square bracket equals to zero.

This implies

$$y_1 u'' + \left(2y_1' + Py_1\right) u' = 0 \quad \text{or} \quad y_1 w' + \left(2y_1' + Py_1\right) w = 0$$

where we used $w = u'$. The last equation can be solved by separating variables and integrating

$$\frac{dw}{w} + 2\frac{y_1'}{y_1}dx + Pdx = 0$$
$$\ln\left|wy_1^2\right| = -\int Pdx + c$$

or

$$wy_1^2 = c_1 e^{-\int Pdx}$$

Solving the last equation for w, and using $w = u'$ and integrating again gives

$$u = c_1 \int \frac{e^{-\int P dx}}{y_1^2} dx + c_2$$

By choosing $c_1 = 1$ and $c_2 = 0$ and by using $y = u(x)y_1(x)$ we find the second solution of the equation (12):

$$y_2 = y_1(x) \int \frac{e^{-\int P dx}}{y_1(x)^2} dx \tag{13}$$

Example:

$y_1 = x^2$ is a solution of $x^2y'' - 3xy' + 4y = 0$. Find the general solution on $(0, \infty)$.
From the standard form of the equation

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$$

we find using the formula above

$$y_2 = x^2 \int \frac{e^{3\int dx/x}}{x^4} dx = x^2 \int \frac{dx}{x} = x^2 \ln x$$

The general solution on $(0, \infty)$ is given by

$$y = c_1 y_1 + c_2 y_2 = c_1 x^2 + c_2 x^2 \ln x$$