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Consider a linear n-th order ODE

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \dots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x)$$

If the coefficients satisfy $a_n(x) \neq 0$, and $a_i = 0$ for all $i < n$, the equation can be written in the following form and solved by direct integration

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = \frac{g(x)}{a_n(x)}.$$

Example 1: $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 1$

$$\Rightarrow \int \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\mathrm{d}x = \frac{\mathrm{d}y}{\mathrm{d}x} = \int \mathrm{d}x = x + A \quad \Rightarrow \quad y = \int \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int (x + A)\mathrm{d}x = \frac{1}{2}x^2 + Ax + B$$

where the integration constants A and B are determined from the initial conditions.

Example 2: $\frac{d^5y}{dx^5} = \sin x$

$$\begin{aligned}
\Rightarrow \quad & \frac{d^4y}{dx^4} = -\cos x + k_1 \\
\Rightarrow \quad & \frac{d^3y}{dx^3} = -\sin x + k_1 x + k_2 \\
\Rightarrow \quad & \frac{d^2y}{dx^2} = \cos x + \frac{1}{2}k_1 x^2 + k_2 x + k_3 \\
\Rightarrow \quad & \frac{dy}{dx} = \sin x + \frac{1}{6}k_1 x^3 + \frac{1}{2}k_2 x^2 + k_3 x + k_4 \\
\Rightarrow \quad & y = -\cos x + \frac{1}{24}k_1 x^4 + \frac{1}{6}k_2 x^3 + k_3 x^2 + k_4 x + k_5.
\end{aligned}$$

Example 3: $m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 0$

$$\Rightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}t} = A \quad \Rightarrow \quad x = At + B.$$

Using the initial conditions

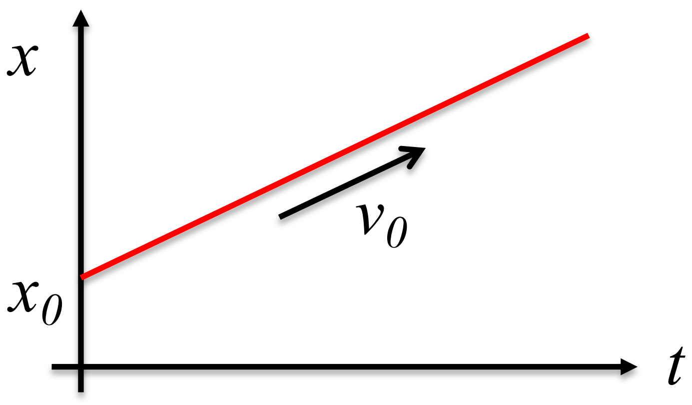
$$x(0) \;=\; B = x_0$$

and

$$\dot{x}(0) \;=\; A = v_0$$

we can write

$$x(t) = v_0 t + x_0$$



First-order differential equations

To find either explicit or implicit solution, we need to

(i) recognize the *kind* of differential equation, and then

(ii) apply to it an equation-specific method of solution.

Separable variables

Solution by integration

The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x) \tag{2}$$

is the simplest ODE. It can be solved by integration:

$$y(x) = \int g(x)dx = G(x) + c$$

where $G(x)$ is an indefinite integral of $g(x)$.

Example:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + e^{2x}$$

has the solution

$$y = \int \left(1 + e^{2x}\right) dx = \frac{1}{2}e^{2x} + x + c.$$

This ODE and its method of solution is a special case when f is a product of a function of x and a function of y.

Definition: Separable equation

A first-order differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)h(y) \qquad (3)$$

is said to be **separable** or to have **separable variables**.

Method of solution:

A one parameter family of solutions, usually given implicitly, is obtained by first rewriting the equation in the form

$$p(y)dy = g(x)dx$$

where $p(y) = 1/h(y)$, and integrating both sides of the equation. We get the solution in the form

$$H(y) = G(x) + c$$

where $H(y) = \int p(y)dy$ and $G(y) = \int g(x)dx$ and c is the combined constant of integration.

Example: A separable ODE

Solve

$$(1+x)dy - ydx = 0$$

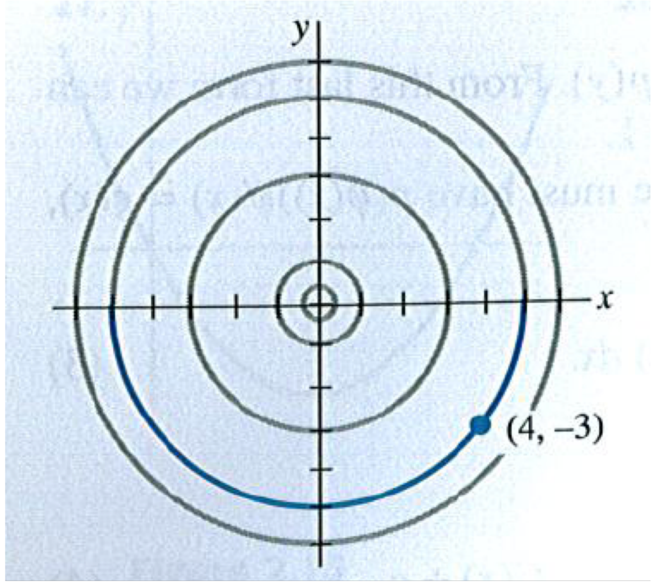
Dividing by $(1+x)y$ we get $dy/y = dx/(1+x)$ and can integrate

$$\begin{aligned}
\int \frac{dy}{y} &= \int \frac{dx}{1+x} \\
\ln|y| &= \ln|1+x| + c_1 \\
y &= e^{\ln|1+x|+c_1} = e^{\ln|1+x|}.e^{c_1} \\
&= |1+x|\, e^{c_1} \\
&= \pm e^{c_1}(1+x) = c(1+x)
\end{aligned}$$

Example: Solution curve

Solve the initial value problem

$$\frac{dy}{dx} = -\frac{x}{y}, \qquad y(4) = -3$$



By rewriting the equation as $ydy = -xdx$, we get

$$\int ydy = -\int xdx$$
$$\frac{y^2}{2} = -\frac{x^2}{2} + c_1$$

We can rewrite the result as $x^2 + y^2 = c^2$, where $c^2 = 2c_1$. This family of solutions represents a family of concerning circles centered at the origin. The IVP determines the circle $x^2 + y^2 = 25$ with radius 5.

Losing a solution

Some care should be exercised when separating variables, since the variable divisors could be zero at a point.

If r is a zero of $h(y)$, then substituting $y = r$ into $dy/dx = g(x)h(y)$ makes both sides zero, i.e. $y = r$ is a constant solution of the DE.

This solution, which is a singular solution, can be missed in the course of the solving the ODE.

Example:

Solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - 4$$

We put the equation into the following form by using partial fractions

$$\frac{dy}{y^2 - 4} = \left[\frac{1/4}{y-2} - \frac{1/4}{y+2}\right] dy = dx$$

and integrate

$$\frac{1}{4}\ln|y-2| - \frac{1}{4}\ln|y+2| = x + c_1$$

$$\ln\left|\frac{y-2}{y+2}\right| = 4x + c_2$$

$$\frac{y-2}{y+2} = e^{4x+c_2}$$

We substitute $c = e^{c_2}$ and get the one-parameter family of solutions

$$y = 2\frac{1 + ce^{4x}}{1 - ce^{4x}}$$

Actually, if we factor the r.h.s. of the ODE as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (y - 2)(y + 2)$$

we see that $y = 2$ and $y = -2$ are two constant (equilibrium solutions). The earlier is a member of the family of solutions defined above corresponding to $c = 0$. However $y = -2$ is a singular solution and in this example it was lost in the course of the solution process.

Example: an IVP

Solve

$$\cos x \left(e^{2y} - y\right) \frac{\mathrm{d}y}{\mathrm{d}x} = e^{y} \sin 2x, \qquad y(0) = 0$$

By dividing the equation we get

$$\frac{e^{2y} - y}{e^{y}} dy = \frac{\sin 2x}{\cos x} dx$$

We use the trigonometric identity $\sin 2x = 2 \sin x \cos x$ on r.h.s. and integrate

$$\int \left(e^{y} - y e^{-y}\right) dy = 2 \int \sin x dx$$

$$e^{y} + y e^{-y} + e^{-y} = -2 \cos x + c$$

The initial condition $y(0) = 0$ implies $c = 4$, so we get the solution of the IVP

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$$\int\left(e^{y}-ye^{-y}\right)dy = 2\int \sin x dx$$

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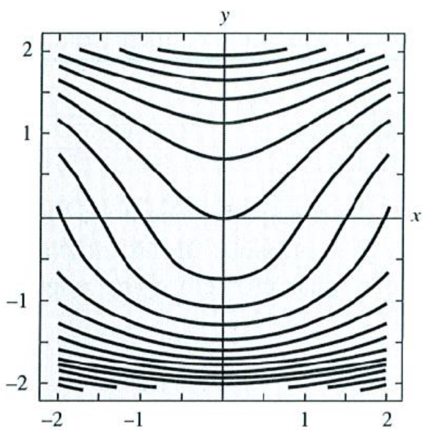

$$e^{y}+ye^{-y}+e^{-y} = 4-2\cos x$$