Fourier integral

Fourier series were used to represent a function f defined of a *finite* interval $(-p, p)$ or $(0, L)$. It converged to f and to its periodic extension. In this sense Fourier series is associated with *periodic* functions.

Fourier integral represents a certain type of *nonperiodic* functions that are defined on either $(-\infty, \infty)$ or $(0, \infty)$.

From Fourier series to Fourier integral

Let a function f be defined on $(-p, p)$. The Fourier series of the function is then

$$f(x) = \frac{1}{2p}\int_{-p}^{p} f(t)\,dt + \tag{1}$$
$$+\frac{1}{p}\sum_{n=1}^{\infty}\left[\left(\int_{-p}^{p} f(t)\ \cos\frac{n\pi}{p}t\,dt\right)\cos\frac{n\pi}{p}x + \left(\int_{-p}^{p} f(t)\ \sin\frac{n\pi}{p}t\,dt\right)\sin\frac{n\pi}{p}x\right]$$

If we let $\alpha_n = n\pi/p$, $\Delta\alpha = \alpha_{n+1} - \alpha_n = \pi/p$, we get

$$f(x) = \frac{1}{2\pi}\left(\int_{-p}^{p} f(t)\,dt\right)\Delta\alpha + \tag{2}$$
$$+\frac{1}{\pi}\sum_{n=1}^{\infty}\left[\left(\int_{-p}^{p} f(t)\ \cos\alpha_n t\,dt\right)\cos\alpha_n x + \left(\int_{-p}^{p} f(t)\ \sin\alpha_n t\,dt\right)\sin\alpha_n x\right]\Delta\alpha$$

We now expand the interval $(-p, p)$ by taking $p \to \infty$ which implies that $\Delta\alpha \to 0$. Consequently,

$$\lim_{\Delta\alpha \to 0} \sum_{n=1}^{\infty} F(\alpha_n)\, \Delta\alpha \to \int_0^{\infty} F(\alpha)\, d\alpha$$

Thus, the limit of the first term in the Fourier series $\int_{-p}^{p} f(t)\, dt$ vanishes, and the limit of the sum becomes

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \left[\left(\int_{-\infty}^{\infty} f(t) \ \cos\alpha t\, dt \right) \cos\alpha x + \left(\int_{-\infty}^{\infty} f(t) \ \sin\alpha t\, dt \right) \sin\alpha x \right] d\alpha$$

This is the **Fourier** integral of f on the interval $(-\infty, \infty)$.

Definition: Fourier integral

The Fourier integral of a function f defined on the interval $(-\infty, \infty)$ is given by

$$f(x) = \frac{1}{\pi}\int_0^{\infty} [A(\alpha)\ \cos\alpha x + B(\alpha)\ \sin\alpha x]\, d\alpha \tag{3}$$

where

$$A(\alpha) = \int_{-\infty}^{\infty} f(x)\ \cos\alpha x\, dx \tag{4}$$

$$B(\alpha) = \int_{-\infty}^{\infty} f(x)\ \sin\alpha x\, dx \tag{5}$$

Convergence of a Fourier integral

Theorem: Conditions for convergence

Let f and f' be piecewise continuous on every finite interval, and let f be absolutely integrable on $(-\infty, \infty)$ (i.e. the integral $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges). Then the Fourier integral of f on the interval converges for $f(x)$ at a point of continuity. At a point of dicontinuity, the Fourier integral will converge to the average

$$\frac{f(x+) + f(x-)}{2}$$

where $f(x+)$ and $f(x-)$ denote the limit of f at x from the right and from the left, respectively.

Complex form

The Fourier integral (3) also possesses an equivalent **complex form**, or **exponential form**:

$$
\begin{aligned}
f(x) &= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)[\cos\alpha t\cos\alpha x + \sin\alpha t\sin\alpha x]\; dt\; d\alpha \\
&= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos\alpha(t-x)\; dt\; d\alpha \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)\cos\alpha(t-x)\; dt\; d\alpha \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)[\cos\alpha(t-x) - i\sin\alpha(t-x)]\; dt\; d\alpha \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)\; e^{-i\alpha(t-x)}\; dt\; d\alpha \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty \left(\int_{-\infty}^\infty f(t)\; e^{-i\alpha t}\; dt\right) e^{i\alpha x}\; d\alpha
\end{aligned}
$$

Fourier transform

We will now

- introduce a new integral transforms called **Fourier transforms**;
- expand on the concept of transform pair: an integral transform and its inverse;
- see that the inverse of an integral transform is itself another integral transform.
- present operational properties of the Fourier transform.

Transform pairs

Integral transforms appear in **transform pairs**: if $f(x)$ is transformed into $F(\alpha)$ by an integral transform

$$F(\alpha) = \int_a^b f(x)\, K(\alpha, x)\, dx$$

then the function f can be recovered by another integral transform

$$f(x) = \int_a^b F(\alpha)\, H(\alpha, x)\, dx$$

called the **inverse transform**. The functions K and H in the integrands above are called the **kernels** of their respective transforms. For example $K(s,t) = e^{-st}$ is the kernel of the Laplace transform.

Definition: Fourier transform pairs

(i)

Fourier transform:

$$\mathcal{F}\left\{f(x)\right\} = \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx = F(\alpha)$$

Inverse Fourier transform:

$$\mathcal{F}^{-1}\left\{F(\alpha)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\, e^{i\alpha x}\, d\alpha = f(x)$$

The Parseval-Plancherel formula:

The function and its Fourier transform have the same norm

$$\int_{-\infty}^{+\infty} |f(x)|^2\, dx = \int_{-\infty}^{+\infty} |F(\alpha)|^2\, d\alpha$$

(ii)

Fourier sine transform:

$$\mathcal{F}_s\{f(x)\} = \int_0^\infty f(x)\ \sin \alpha x\, dx = F(\alpha)$$

Inverse Fourier sine transform:

$$\mathcal{F}_s^{-1}\{F(\alpha)\} = \frac{2}{\pi}\int_0^\infty F(\alpha)\ \sin \alpha x\, d\alpha = f(x)$$

(iii)

Fourier cosine transform:

$$\mathcal{F}_c\{f(x)\} = \int_0^{\infty} f(x)\ \cos \alpha x \, dx = F(\alpha)$$

Inverse Fourier cosine transform:

$$\mathcal{F}_c^{-1}\{F(\alpha)\} = \frac{2}{\pi} \int_0^{\infty} F(\alpha)\ \cos \alpha x \, d\alpha = f(x)$$

Existence

The existence conditions for the Fourier transform are more stringent than those for the Laplace transform. For example, $\mathcal{F}\{1\}$, $\mathcal{F}_s\{1\}$ and $\mathcal{F}_c\{1\}$ do not exist.

Sufficient conditions for existence are that f be absolutely integrable on the appropriate interval and that f and f' are piecewise continuous on every finite interval.

Operational properties

Transforms of derivatives.

(i) Fourier transform

Supose that f is continuous and absolutely integrable on the interval $(-\infty, \infty)$ and f' is piecewise continuous on every finite interval. If $f(x) \to 0$ as $x \to \pm\infty$, then integration by parts gives

$$\begin{aligned} \mathcal{F}\left\{f'(x)\right\} &= \int_{-\infty}^{\infty} f'(x)\, e^{-i\alpha x}\, dx = \left[f(x)\, e^{-i\alpha x}\right]_{-\infty}^{\infty} + i\alpha \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx \\ &= i\alpha \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx \end{aligned}$$

That is: $\mathcal{F}\left\{f'(x)\right\} = i\alpha F(\alpha)$

$$\mathcal{F}\left\{f'(x)\right\} = i\alpha F(\alpha)$$

Similarly, under the added assumptions that f' is continuous on $(-\infty, \infty)$, $f''(x)$ is piecewise continuous on every finite interval, and $f'(x) \to 0$ as $x \to \pm\infty$, we have

$$\mathcal{F}\left\{f''(x)\right\} = (i\alpha)^2 F(\alpha)$$

In general, under analogous conditions, we have

$$\mathcal{F}\left\{f^{(n)}(x)\right\} = (i\alpha)^n F(\alpha)$$

where $n = 0, 1, 2, ...$

It is important to realize that the sine and cosine transforms are not suitable for transforming the first derivatives and in fact any odd-order derivatives:

$$\mathcal{F}_s\left\{f'(x)\right\} = -\alpha\mathcal{F}_c\left\{f(x)\right\} \qquad \text{and} \qquad \mathcal{F}_c\left\{f'(x)\right\} = \alpha\mathcal{F}_s\left\{f(x)\right\} - f(0)$$

as these are not expressed in terms of the original integral transform.

(ii) Fourier sine transform (optional)

Suppose f and f' are continuous, f is absolutely integrable on $[0, \infty)$ and f'' is piecewise continuous on every finite interval. If $f \to 0$ and $f' \to 0$ as $x \to \infty$, then

$$\begin{aligned}
\mathcal{F}_s\left\{f''(x)\right\} &= \int_0^\infty f''(x)\ \sin\alpha x\, dx = \left[f'(x)\ \sin\alpha x\right]_0^\infty - \alpha\int_0^\infty f'(x)\ \cos\alpha x\, dx \\
&= -\alpha\left[f(x)\ \cos\alpha x\right]_0^\infty - \alpha^2\int_0^\infty f(x)\ \sin\alpha x\, dx = \alpha f(0) - \alpha^2\mathcal{F}_s\{f(x)\} \\
\mathcal{F}_s\left\{f''(x)\right\} &= -\alpha^2 F(\alpha) + \alpha f(0)
\end{aligned}$$

(iii) Fourier cosine transform (optional)

Under the same assumptions, we find the Fourier the Fourier cosine transform of $f''(x)$ to be

$$\mathcal{F}_c\left\{f''(x)\right\} = -\alpha^2 F(\alpha) - f'(0)$$

Properties of the Fourier transform

Let us identify time t with the variable x and the angular frequency ω with α. Then the Fourier transform of a function of time $f(t)$, a signal, produces the spectrum of the signal in the representation given by the angular frequency ω.

1. **Linearity**
   The Fourier transform is a linear operator:
   $$\mathcal{F}\left\{k_1 f_1(t) + k_2 f_2(t)\right\} = k_1 F_1(\omega) + k_2 F_2(\omega)$$
   where $\mathcal{F}\left\{f_1(t)\right\} = F_1(\omega)$ and $\mathcal{F}\left\{f_2(t)\right\} = F_2(\omega)$.

2. **Time translation/shifting**
   Time translation or shifting by an amount t_0 leads to a phase shift in the Fourier

transform:

$$\mathcal{F}\left\{f(t-t_0)\right\} = e^{-i\omega t_0}F(\omega)$$

3. **Frequency translation/shifting**

$$\mathcal{F}\left\{e^{i\omega_0 t}f(t)\right\} = F(\omega-\omega_0)$$

The multiplication of $f(t)$ by $e^{i\omega_0 t}$ is called the **complex modulation**. Thus, the complex modulation in the time domain corresponds to a shift in the frequency domain.

4. **Time scaling**

$$\mathcal{F}\left\{f(kt)\right\} = \frac{1}{|k|}\, F\left(\frac{\omega}{k}\right)$$

Therefore if t is directly scaled by a factor k, then the frequency variable is inversely scaled by the factor k. Consequently, for $k > 1$ we have a time-compression resulting in a frequency spectrum expansion. For $k < 1$ there is a time-expansion and a resulting frequency spectrum compression.

5. **Time reversal**

This property follows from the time scaling for $k = -1$

$$\mathcal{F}\left\{f(-t)\right\} = F(-\omega)$$

6. **Symmetry**
   This property is very useful in evaluation of certain Fourier transforms

$$\mathcal{F}\{F(t)\} = 2\pi f(-\omega)$$

7. **Fourier transform and inverse Fourier transform of a derivative**

$$\mathcal{F}\left\{\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right\} = i\omega F(\omega)$$

$$\mathcal{F}^{-1}\left\{\frac{\mathrm{d}F(\omega)}{\mathrm{d}\omega}\right\} = -itf(t)$$

8. **Fourier transform of an integral**

$$\mathcal{F}\left\{\int_{-\infty}^{t} f(u)\,du\right\} = \pi F(0)\delta(\omega) + \frac{1}{i\omega}F(\omega)$$

9. **Fourier transform of a convolution**

$$\mathcal{F}\left\{f_1(t) * f_2(t)\right\} = \mathcal{F}\left\{\int_0^t f_1(\tau)\, f_2(t-\tau)\, d\tau\right\} = F_1(\omega)\, F_2(\omega)$$

The counterpart of convolution in the time domain is multiplication in the frequency domain.

10. **Fourier transform of a product**

$$\mathcal{F}\left\{f_1(t)\, f_2(t)\right\} = \frac{1}{2\pi} F_1(\omega) * F_2(\omega)$$

Example 1: Fourier transform of a simple piecewise continuous function

$$f(t) = \begin{cases} -2, & -\pi \leq t < 0 \\ 2, & 0 \leq t < \pi \\ 0, & \text{Otherwise} \end{cases}$$

Solution:

$$\begin{aligned} F(\omega) &= \int_{-\pi}^{0} (-2) e^{-i\omega t}\, dt + \int_{0}^{\pi} (2) e^{-i\omega t}\, dt = \frac{2}{i\omega}\left[e^{-i\omega t}\right]_{-\pi}^{0} - \frac{2}{i\omega}\left[e^{-i\omega t}\right]_{0}^{\pi} \\ &= \frac{2}{i\omega}\left[\left(1 - e^{i\omega\pi}\right) - \left(e^{-i\omega\pi} - 1\right)\right] = \frac{2}{i\omega}\left[2 - 2\cos(\omega\pi)\right] \\ F(\omega) &= \frac{4}{i\omega}\left[1 - \cos(\omega\pi)\right] \end{aligned}$$

Example 2: Fourier transform and ODEs

Find a solution of the Airy equation

$$y'' - xy = 0$$

subject to the far field condition $\lim_{|x|\to\infty} y(x) = 0$.

Solution:

The Fourier transform of the equation gives the first order ODE

$$-k^2 Y(k) - iY'(k) = 0.$$

The equation

$$-k^2 Y(k) - iY'(k) = 0$$

can be solved using the method of separation of variables. This gives the following result

$$Y(k) = Ce^{ik^3/3}$$

whose inverse Fourier transfom is

$$y(x) = \frac{C}{2\pi} \int_{-\infty}^{+\infty} \exp\left[i(kx + k^3/3)\right]\, dk.$$

This integral cannot be reduced any further. For $C = 1$, the result is known as **Airy function** and is denoted as $\mathrm{Ai}(x)$.

Fourier transform in quantum mechanics

Consider a (one-dimensional) wave function $\psi(x)$. Its Fourier transform is defined (with a slightly different convention) as

$$\bar{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{-ipx/\hbar}\, \psi(x)\, dx$$

and its inverse as

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{ipx/\hbar}\, \bar{\psi}(p)\, dp$$

where

$$\bar{\psi}(p) = \frac{1}{\sqrt{\hbar}} \tilde{\psi}(k) = \frac{1}{\sqrt{\hbar}} \tilde{\psi}\left(\frac{p}{\hbar}\right)$$

as $p = \hbar k$, and where $\tilde{\psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx/\hbar}\, \psi(x)\, dx$.

δ-function

The Fourier transform of the δ-function

$$\bar{\delta}_{x_0}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{-ipx/\hbar}\, \delta(x - x_0)\, dx = \frac{1}{\sqrt{2\pi\hbar}}\, e^{-ipx_0/\hbar}$$

and in particular

$$\bar{\delta}_0(p) = \frac{1}{\sqrt{2\pi\hbar}}.$$

The inverse Fourier transform yields a very useful expression for the δ-function

$$\delta(x - x_0) = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} e^{ip(x-x_0)/\hbar}\, dp = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ik(x-x_0)}\, dk.$$