Translation theorems

Translation on the s-axis

Theorem: First translation theorem

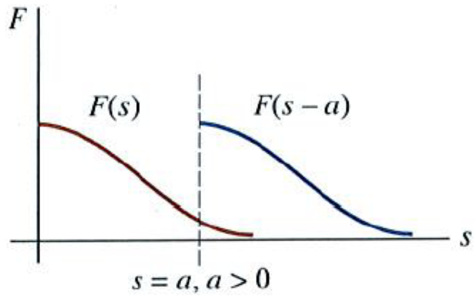
If $\mathcal{L}\{f(t)\} = F(s)$ and a is any real number, then

$$\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a) \tag{1}$$

Proof:

$$\mathcal{L}\left\{e^{at}f(t)\right\} = \int_0^\infty e^{-st}e^{at}f(t)dt = \int_0^\infty e^{-(s-a)t}f(t)dt = F(s-a)$$

It is sometimes useful to use the notation $\mathcal{L}\left\{e^{at}f(t)\right\} = \mathcal{L}\{f(t)\}_{s\to s-a}$.



Example 1:

(a)

$$\mathcal{L}\left\{e^{5t}t^3\right\} = \mathcal{L}\left\{t^3\right\}_{s\to s-5} = \left.\frac{3!}{s^4}\right|_{s\to s-5} = \frac{6}{(s-5)^4}$$

(b)

$$\mathcal{L}\left\{e^{-2t}\cos 4t\right\} = \mathcal{L}\left\{\cos 4t\right\}_{s\to s-(-2)} = \left.\frac{s}{s^2+16}\right|_{s\to s+2} = \frac{s+2}{(s+2)^2+16}$$

Inverse form of the theorem

To compute the inverse of $F(s-a)$, we must recognize $F(s)$, take its inverse to find $f(t)$, and then multiply by e^{at}:

$$\mathcal{L}^{-1}\{F(s-a)\} = \mathcal{L}^{-1}\{F(s)|_{s\to s-a}\} = e^{at}f(t)$$

where $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

Example 2:

(a) $\mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\}$

$$\frac{2s+5}{(s-3)^2} = \frac{A}{s-3} + \frac{B}{(s-3)^2}$$

Putting both terms on the common denominator and solving for A and B yields $A = 2$ and $B = 11$. Therefore

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} &= 2\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + 11\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\} \\ &= 2\mathcal{L}^{-1}\left\{\left.\frac{1}{s}\right|_{s\to s-3}\right\} + 11\mathcal{L}^{-1}\left\{\left.\frac{1}{s^2}\right|_{s\to s-3}\right\} \\ &= 2e^{3t} + 11e^{3t}t \end{aligned}$$

(b) $\mathcal{L}^{-1}\left\{\frac{s/2+5/3}{s^2+4s+6}\right\}$

The quadratic polynomial s^2+4s+6 has no real zeros, so by completing the square we get

$$\begin{aligned}\frac{s/2+5/3}{s^2+4s+6} &= \frac{s/2+5/3}{(s+2)^2+2} \\ &= \frac{1/2(s+2)+2/3}{(s+2)^2+2} = \frac{1}{2}\frac{s+2}{(s+2)^2+2}+\frac{2}{3}\frac{1}{(s+2)^2+2}\end{aligned}$$

The inverse Laplace transform is then

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s/2+5/3}{s^2+4s+6}\right\} &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+2}\right\}+\frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+2}\right\} \\ &= \frac{1}{2}e^{-2t}\cos\sqrt{2}t+\frac{\sqrt{2}}{3}e^{-2t}\sin\sqrt{2}t\end{aligned}$$

Example 3: an IVP

$$y'' - 6y' + 9y = t^2e^{3t}, \quad y(0) = 2, \quad y'(0) = 17$$

$$\begin{aligned}
\mathcal{L}\left\{y''\right\} - 6\mathcal{L}\left\{y'\right\} + 9\mathcal{L}\left\{y\right\} &= \mathcal{L}\left\{t^2e^{3t}\right\} \\
s^2Y(s) - sy(0) - y'(0) - 6\left[sY(s) - y(0)\right] + 9Y(s) &= \frac{2}{(s-3)^3} \\
\left(s^2 - 6s + 9\right)Y(s) &= 2s + 5 + \frac{2}{(s-3)^3} \\
(s-3)^2Y(s) &= 2s + 5 + \frac{2}{(s-3)^3} \\
Y(s) &= \frac{2s+5}{(s-3)^2} + \frac{2}{(s-3)^5}
\end{aligned}$$

The partial fraction decomposition of the first term on r.h.s. was done already in Example 2 (a):

$$Y(s) = \frac{2}{s-3} + \frac{11}{(s-3)^2} + \frac{2}{(s-3)^5}$$

Thus

$$\begin{aligned} y(t) &= 2\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + 11\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\} + \frac{2}{4!}\mathcal{L}^{-1}\left\{\frac{4!}{(s-3)^5}\right\} \\ &= 2e^{3t} + 11te^{3t} + \frac{1}{12}t^4e^{3t} \end{aligned}$$

Example 4:

$$y'' + 4y' + 6y = 1 + e^{-t}, \qquad y(0) = 0, \quad y'(0) = 0 \tag{2}$$

Solution:

$$s^2 Y(s) - sy(0) - y'(0) + 4\left[sY(s) - y(0)\right] + 6Y(s) = \frac{1}{s} + \frac{1}{s+1}$$

$$Y(s) = \frac{2s+1}{s(s+1)(s^2+4s+6)} = \frac{1/6}{s} + \frac{1/3}{s+1} - \frac{s/2 + 5/3}{s^2+4s+6}$$

The inverse Laplace transform of the last term on r.h.s. was carried out in Example 2 (b); thus the final result is

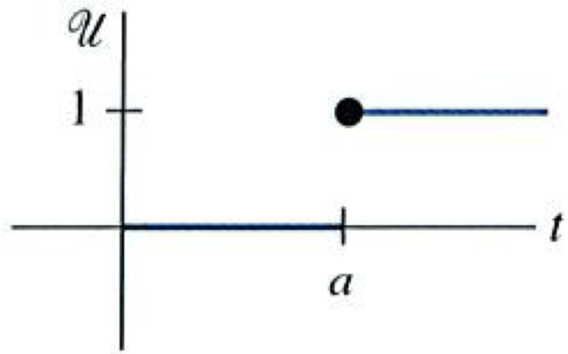
$$y(t) = \frac{1}{6} + \frac{1}{3}e^{-t} - \frac{1}{2}e^{-2t}\cos\sqrt{2}t - \frac{\sqrt{2}}{3}e^{-2t}\sin\sqrt{2}t$$

Translation on the t-axis

Definition: Unit step function / Heaviside function

The **unit step function** $\mathcal{U}(t-a)$ is defined to be

$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a. \end{cases}$$



Remarks:

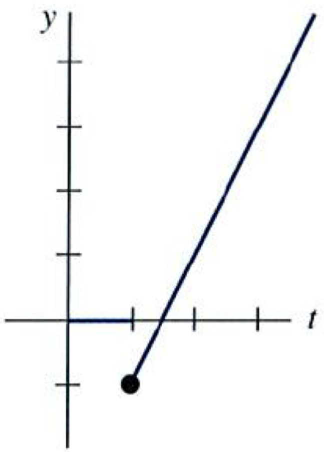
(i) We define the function $\mathcal{U}(t-a)$ only on the non-negative t-axis since we are concerned with the Laplace transform.

(ii) When a function $f(t)$ is multiplied by $\mathcal{U}(t-a)$, the unit step function *turns off* a portion of the graph of that function.

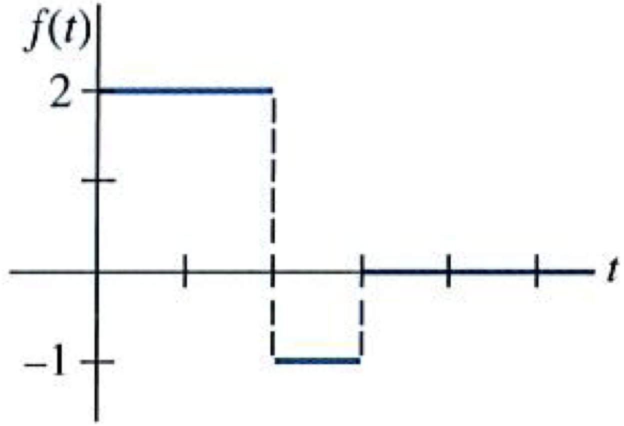
Example: $f(t) = 2t - 3$ multiplied by $\mathcal{U}(t-1)$ has the portion of $f(t)$ on the interval $0 \le t < 1$ turned off (zero); the function is on for $t \ge 1$.

(iii) The unit step function can be used to write piecewise-defined functions in a compact form.

Example: Considering $0 \le t < 2, 2 \le t < 3, t \ge 3$ and the corresponding values of $\mathcal{U}(t-2)$ and $\mathcal{U}(t-3)$, the piecewise-defined function in the figure can be written



$$f(t) = 2 - 3\mathcal{U}(t-2) + \mathcal{U}(t-3)$$



A general piecewise-defined function

$$f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a. \end{cases}$$

is the same as

$$f(t) = g(t) - g(t)\mathcal{U}(t-a) + h(t)\mathcal{U}(t-a)$$

Similarly

$$f(t) = \begin{cases} 0, & 0 \le t < a \\ g(t), & a \le t < b \\ 0, & t \ge b. \end{cases}$$

is the same as

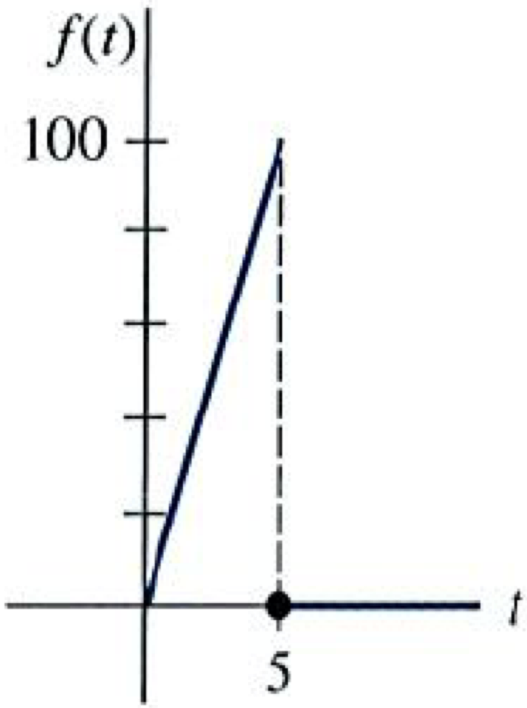
$$f(t) = g(t)\left[\mathcal{U}(t-a) - \mathcal{U}(t-b)\right]$$

Example 5:

$$f(t) = \begin{cases} 20t, & 0 \le t < 5 \\ 0, & t \ge 5. \end{cases}$$

$a = 5$, $g(t) = 20t$, $h(t) = 0$:

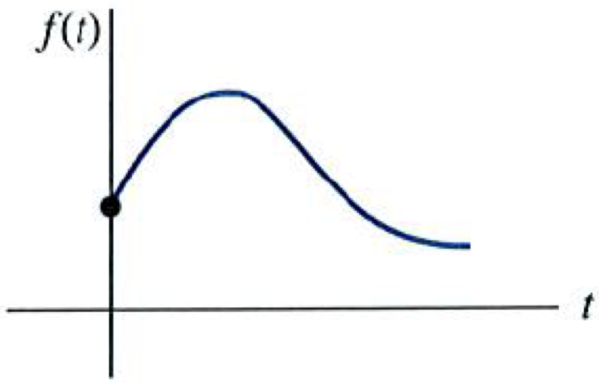
$$f(t) = 20t - 20t\,\mathcal{U}(t-5)$$



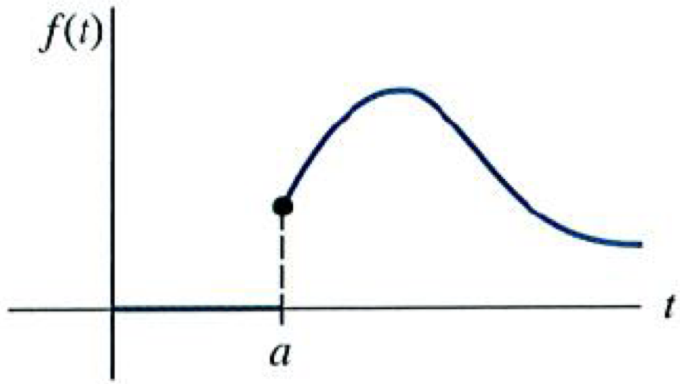
Consider a general function $y = f(t)$. For $a > 0$, the graph of the piecewise-defined function

$$f(t-a)\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ f(t-a), & t \ge a. \end{cases}$$

coincides with the graph $y = f(t-a)$ for $t \ge a$ (which is the entire graph of $f(t)$, $t \ge 0$ shifted a units to the right) but is identically zero for $0 \le t < a$.



(a) $f(t)$, $t \ge 0$



(b) $f(t-a)\,\mathcal{U}(t-a)$

Theorem: Second translation theorem

If $F(s) = \mathcal{L}\{f(t)\}$ and $a > 0$, then

$$\mathcal{L}\{f(t-a)\,\mathcal{U}(t-a)\} = e^{-as}F(s) \tag{3}$$

<u>Proof:</u>

$$\begin{aligned}
\mathcal{L}\{f(t-a)\,\mathcal{U}(t-a)\} &= \int_0^a e^{-st} f(t-a)\mathcal{U}(t-a)\;dt + \int_a^\infty e^{-st} f(t-a)\mathcal{U}(t-a)\;dt \\
&= \int_a^\infty e^{-st} f(t-a)\,dt
\end{aligned}$$

Using the substitution $v = t - a$ and $dv = dt$ in the last integral, we get

$$\mathcal{L}\{f(t-a)\,\mathcal{U}(t-a)\} = \int_0^\infty e^{-s(v+a)} f(v)\,dv = e^{-as}\int_0^\infty e^{-sv} f(v)\,dv = e^{-as}\mathcal{L}\{f(t)\}$$

The Laplace transform of a unit step function, i.e. $f(t-a) = 1$

$$\mathcal{L}\{\mathcal{U}(t-a)\} = \frac{e^{-as}}{s} \tag{4}$$

Example: $f(t) = 2 - 3\mathcal{U}(t-2) + \mathcal{U}(t-3)$

$$2\mathcal{L}\{1\} - 3\mathcal{L}\{\mathcal{U}(t-2)\} + \mathcal{L}\{\mathcal{U}(t-3)\} = 2\,\frac{1}{s} - 3\,\frac{e^{-2s}}{s} + \frac{e^{-3s}}{s}$$

Inverse form of the second translation theorem:

If $f(t) = \mathcal{L}^{-1}\{F(s)\}$, the inverse of the theorem, with $a > 0$, is

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\} = f(t-a)\mathcal{U}(t-a) \tag{5}$$

Example 6:

(a) $\mathcal{L}^{-1}\left\{\frac{1}{s-4}e^{-2s}\right\}$: with the identification $a = 2$, $F(s) = 1/(s-4)$, $\mathcal{L}^{-1}\{F(s)\} = e^{4t}$, we have

$$\mathcal{L}^{-1}\left\{\frac{1}{s-4}e^{-2s}\right\} = e^{4(t-2)}\,\mathcal{U}(t-2)$$

(b) $\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}e^{-\pi s/2}\right\}$: with $a = \pi/2$, $F(s) = s/(s^2+9)$, $\mathcal{L}^{-1}\{F(s)\} = \cos 3t$, we get

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}e^{-\pi s/2}\right\} = \cos 3(t-\pi/2)\,\mathcal{U}(t-\pi/2)$$

Verify that using the addition formula for the cosine the result is the same as $-\sin 3t\,\mathcal{U}(t-\pi/2)$.

Alternative form of the second translation theorem

How do we find the Laplace transform of $g(t)\mathcal{U}(t-a)$?

Using the substitution $u = t - a$ and the definition of $\mathcal{U}(t-a)$

$$\mathcal{L}\{g(t)\,\mathcal{U}(t-a)\} = \int_a^\infty e^{-st} g(t)\,dt = \int_0^\infty e^{-s(u+a)} g(u+a)\,du$$

$$\mathcal{L}\{g(t)\,\mathcal{U}(t-a)\} = e^{-as}\,\mathcal{L}\{g(t+a)\} \tag{6}$$

Example:

$$\mathcal{L}\left\{t^2\mathcal{U}(t-2)\right\} = e^{-2s}\mathcal{L}\left\{(t+2)^2\right\} = e^{-2s}\mathcal{L}\left\{t^2+4t+4\right\} = e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}\right)$$

Example 7: $\mathcal{L}\{\cos t\, \mathcal{U}(t-\pi)\}$

Here $g(t) = \cos t$, $a = \pi$, and then $g(t+\pi) = \cos(t+\pi) = -\cos t$ by the addition formula for the cosine. Thus

$$\mathcal{L}\{\cos t\, \mathcal{U}(t-\pi)\} = -e^{-\pi s}\mathcal{L}\{\cos t\} = -\frac{s}{s^2+1}e^{-\pi s}$$

Example 8: an IVP

$$y' + y = f(t), \qquad y(0) = 5, \quad \text{where} \quad f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 3\cos t, & t \ge \pi. \end{cases}$$

The function $f(t)$ can be written as $f(t) = 3\cos t\, \mathcal{U}(t-\pi)$ and so by linearity, the results of Example 7 and the usual partial fractions, we have

$$\begin{aligned} \mathcal{L}\left\{y'\right\} + \mathcal{L}\{y\} &= 3\mathcal{L}\{\cos t\, \mathcal{U}(t-\pi)\} \\ sY(s) - y(0) + Y(s) &= -3\frac{s}{s^2+1}e^{-\pi s} \\ Y(s) &= \frac{5}{s+1} - \frac{3}{2}\left[-\frac{1}{s+1}e^{-\pi s} + \frac{1}{s^2+1}e^{-\pi s} + \frac{s}{s^2+1}e^{-\pi s}\right] \end{aligned}$$

It follows with $a = \pi$ that the inverses of the terms in the bracket are

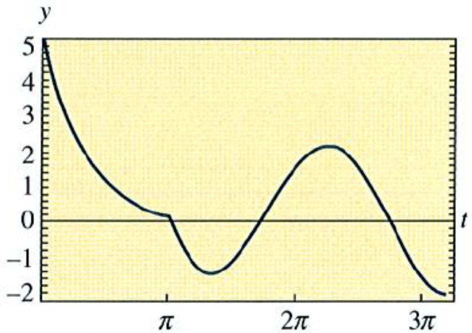
$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s+1}e^{-\pi s}\right\} &= e^{-(t-\pi)}\mathcal{U}(t-\pi) \\
\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}e^{-\pi s}\right\} &= \sin(t-\pi)\mathcal{U}(t-\pi) \\
\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}e^{-\pi s}\right\} &= \cos(t-\pi)\mathcal{U}(t-\pi)
\end{aligned}$$

Thus the inverse of Y(s) is

$$\begin{aligned}
y(t) &= 5e^{-t} + \frac{3}{2}e^{-(t-\pi)}\mathcal{U}(t-\pi) - \frac{3}{2}\sin(t-\pi)\mathcal{U}(t-\pi) - \frac{3}{2}\cos(t-\pi)\mathcal{U}(t-\pi) \\
&= 5e^{-t} + \frac{3}{2}\left[e^{-(t-\pi)} + \sin t + \cos t\right]\mathcal{U}(t-\pi)
\end{aligned}$$

or

$$y(t) = \begin{cases} 5e^{-t}, & 0 \le t < \pi \\ 5e^{-t} + \frac{3}{2}e^{-(t-\pi)} + \frac{3}{2}\sin t + \frac{3}{2}\cos t, & t \ge \pi. \end{cases}$$



Additional operational properties

How to find the Laplace transform of a function $f(t)$ that is multiplied by a monomial t^n, the transform of a special type of integral, and the transform of a periodic function?

Multiplying a function by t^n

$$\frac{\mathrm{d}}{\mathrm{d}s}F(s) = \frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty e^{-st}f(t)\,dt = \int_0^\infty \frac{\partial}{\partial s}\left[e^{-st}f(t)\right]\,dt = -\int_0^\infty e^{-st}\;t\;f(t)\,dt = -\mathcal{L}\{t\;f(t)\}$$

that is

$$\mathcal{L}\{t\;f(t)\} = -\frac{\mathrm{d}}{\mathrm{d}s}F(s)$$

Similarly

$$\mathcal{L}\left\{t^2\;f(t)\right\} = \mathcal{L}\{t.t\;f(t)\} = -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{t\;f(t)\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(-\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{f(t)\}\right) = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{L}\{f(t)\}$$

Theorem: Derivatives of transforms

If $F(s) = \mathcal{L}\{f(t)\}$ and $n = 1, 2, 3, ...$ then

$$\mathcal{L}\{t^n\ f(t)\} = (-1)^n\ \frac{\mathrm{d}^n}{\mathrm{d}s^n}F(s) \tag{7}$$

Example 1: $\mathcal{L}\{t\ \ \sin kt\}$

With $f(t) = \sin kt$, $F(s) = k/(s^2 + k^2)$, and $n = 1$, the theorem above gives

$$\mathcal{L}\{t\ \ \sin kt\} = -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{\sin kt\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{k}{s^2 + k^2}\right) = \frac{2ks}{(s^2 + k^2)^2}$$

Evaluate $\mathcal{L}\left\{t^2\ \ \sin kt\right\}$ and $\mathcal{L}\left\{t^3\ \ \sin kt\right\}$.

Example 2: $x'' + 16x = \cos 4t, \quad x(0) = 0, \quad x'(0) = 1$

The Laplace transform of the DE gives

$$(s^2 + 16)X(s) = 1 + \frac{s}{s^2 + 16}$$
$$X(s) = \frac{1}{s^2 + 16} + \frac{s}{(s^2 + 16)^2}$$

In the example 1 we have got

$$\mathcal{L}^{-1}\left\{\frac{2ks}{(s^2 + k^2)^2}\right\} = t \ \sin kt$$

and so with the identification $k = 4$, we obtain

$$x(t) = \frac{1}{4}\,\mathcal{L}^{-1}\left\{\frac{4}{s^2 + 16}\right\} + \frac{1}{8}\,\mathcal{L}^{-1}\left\{\frac{8s}{(s^2 + 16)^2}\right\} = \frac{1}{4}\sin 4t + \frac{1}{8}\,t \ \sin 4t$$