# Special functions

- Bessel's equations

## Special Functions

The differential equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

is called Bessel's equation of order $\nu$. It occurs frequently in advanced studies in applied mathematics, physics and engineering.

Its solutions are called Bessel functions.

In following we will assume that $\nu \geq 0$ and we will seek series solutions of Bessel's equation about $x = 0$ which is its regular singular point.

**The solution:**

Since $x = 0$ is a regular singular point of Bessel's equation, there is at least one solution of the form $y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$.

Substituting this into the equation gives

$$
\begin{aligned}
x^2y'' \quad + \quad & xy' + (x^2 - \nu^2)y = \\
= \quad & \sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} c_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} - \nu^2 \sum_{n=0}^{\infty} c_n x^{n+r} \\
= \quad & c_0(r^2 - r + r - \nu^2)x^r + x^r \sum_{n=1}^{\infty} c_n[(n+r)(n+r-1) + (n+r) - \nu^2]x^n + x^r \sum_{n=0}^{\infty} c_n x^{n+2} \\
= \quad & c_0(r^2 - \nu^2)x^r + x^r \sum_{n=1}^{\infty} c_n[(n+r)^2 - \nu^2]x^n + x^r \sum_{n=0}^{\infty} c_n x^{n+2}.
\end{aligned}
$$

We see that the indicial equation is $r^2 - \nu^2 = 0$, so the roots are $r_1 = \nu$ and $r_2 = -\nu$.

When $r_1 = \nu$, the equation above becomes

$$\begin{aligned} & x^\nu \sum_{n=1}^{\infty} c_n n(n+2)x^n + x^\nu \sum_{n=0}^{\infty} c_n x^{n+2} \\ = \; & x^\nu \left[ (1+2\nu)c_1 x + \sum_{n=2}^{\infty} c_n n(n+2\nu)x^n + \sum_{n=0}^{\infty} c_n x^{n+2} \right] \\ = \; & x^\nu \left[ (1+2\nu)c_1 x + \sum_{k=0}^{\infty} [(k+2)(k+2+2\nu)c_{k+2} + c_k] x^{k+2} \right] = 0. \end{aligned}$$

Therefore $(1+2\nu)c_1 = 0$ and $(k+2)(k+2+2\nu)c_{k+2} + c_k = 0$.

The relation $(k+2)(k+2+2\nu)c_{k+2} + c_k = 0$ imply the recurrence

$$c_{k+2} = \frac{-c_k}{(k+2)(k+2+2\nu)}, \quad k = 0, 1, 2 \ldots .$$

From $(1+2\nu)c_1 = 0$, the choice $c_1 = 0$ implies $c_3 = c_5 \cdots = 0$.

For $k = 0, 2, 4 \ldots$ we find letting $k+2 = 2n$, $n = 1, 2, 3, \ldots,$ that

$$c_{2n} = -\frac{c_{2n-2}}{2^2 n(n+\nu)}.$$

Thus explicitly

$$
\begin{aligned}
c_2 &= -\frac{c_0}{2^2.1.(1+\nu)} \\
c_4 &= -\frac{c_2}{2^2.2(2+\nu)} = -\frac{c_0}{2^4.1.2(1+\nu)(2+\nu)} \\
c_6 &= -\frac{c_2}{2^2.3(3+\nu)} = -\frac{c_0}{2^6.1.2.3(1+\nu)(2+\nu)(3+\nu)} \\
&\vdots \\
c_{2n} &= -\frac{(-1)^n c_0}{2^{2n} n!(1+\nu)(2+\nu)\ldots(n+\nu)}, \quad n = 1, 2, 3 \ldots .
\end{aligned}
$$

It is standard to choose $c_0$ to be a specific value, namely

$$
c_0 = \frac{1}{2^\nu \Gamma(1+\nu)},
$$

where $\Gamma(1+\nu)$ is the gamma function.

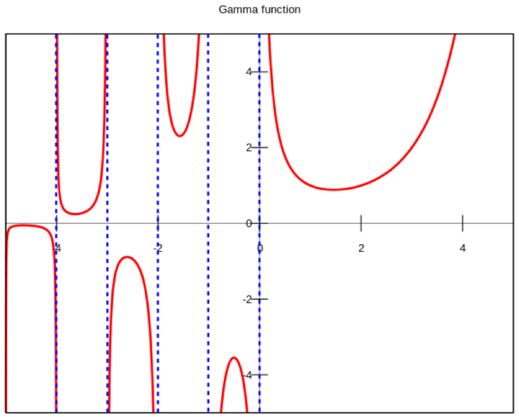
## Gamma function

Euler's integral definition:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt.$$

Though this integral does not converge for $x < 0$ , it can be shown by other definitions that the gamma function is defined for all real and complex values except

$$x = -n, \quad n = 0, 1, 2, \dots \,.$$




The gamma function has a convenient property

$$\Gamma(1+\alpha) = \alpha\Gamma(\alpha).$$

When $n$ is a positive integer

$$\Gamma(n+1) = n!$$

so the gamma function is often called **generalized factorial function**.

The recurrence property gamma function $\Gamma(1+\alpha) = \alpha\Gamma(\alpha)$ allows us to reduce the product in the denominator of $c_{2n}$ to one term. For example

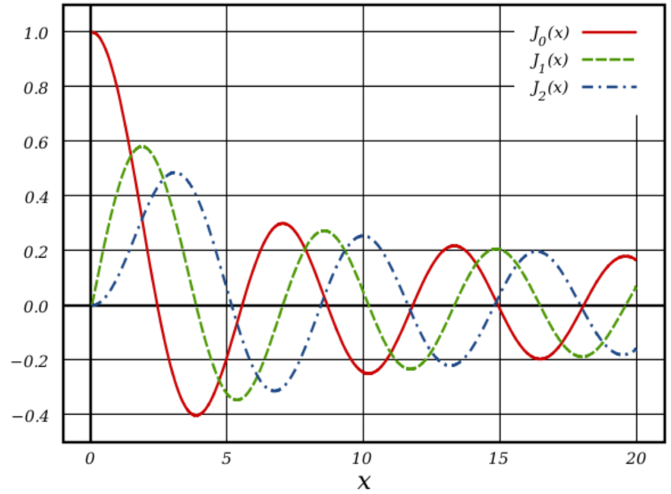
$$\begin{aligned}
\Gamma(1+\nu+1) &= (1+\nu)\Gamma(1+\nu) \\
\Gamma(1+\nu+2) &= (2+\nu)\Gamma(2+\nu) = (2+\nu)(1+\nu)\Gamma(1+\nu).
\end{aligned}$$

so we can write

$$c_{2n} = \frac{(-1)^n}{2^{2n+\nu} n!(1+\nu)(2+\nu)\ldots(n+\nu)\Gamma(1+\nu)} = \frac{(-1)^n}{2^{2n+\nu} n!\Gamma(1+\nu+n)}$$

for $n = 0, 1, 2 \ldots$ .

## Bessel functions of the first kind




The series solution $y = \sum_{n=0}^{\infty} c_{2n} x^{2n+\nu}$ is usually denoted

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}.$$

If $\nu \geq 0$, the series converges at least on the interval $[0, \infty)$.

Also for the second exponent $r_2 = -\nu$ we obtain in the same manner

$$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1-\nu+n)} \left(\frac{x}{2}\right)^{2n-\nu}.$$

The functions $J_\nu(x)$ and $J_{-\nu}(x)$ are called **Bessel functions of the first kind** of order $\nu$ and $-\nu$ respectively. Depending on the value of $\nu$, the expression for $J_{-\nu}(x)$ may contain negative powers of $x$ and hence converge on the interval $(0, \infty)$.

**General solution**

When $\nu = 0$ than the expressions for $J_\nu(x)$ and $J_{-\nu}(x)$ are the same.

If $\nu > 0$ and $r_1 - r_2 = \nu - (-\nu) = 2\nu$ is not a positive integer, than it follows from the Case I, that $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent solutions on $(0, \infty)$ and so the general solution is

$$y = c_1 J_\nu(x) + c_2 J_{-\nu}(x).$$

From the Case II, when $r_1 - r_2 = 2\nu$ is a positive integer, a second solution of Bessel's equation may exist. We distinguish two possibilities:

(i) When $\nu = m$ is a positive integer, $J_{-\nu}(x)$ and $J_\nu(x)$ are not linearly independent solutions, and specifically it can be show that $J_{-m}(x)$ is a constant multiple of $J_m(x)$.

(ii) In addition, $r_1 - r_2 = 2\nu$ can be a positive integer when $\nu$ is half an odd positive integer. In this case, it can be shown that $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent, and the general solution on $(0, \infty)$ is

$$y = c_1 J_\nu(x) + c_2 J_{-\nu}(x), \quad \nu \neq \text{ integer.}$$

**Example 1:** General solution: $\nu$ is not an integer

Consider Bessel's differential equation

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0;$$

by identifying $\nu^2 = \frac{1}{4}$ and $\nu = \frac{1}{2}$, we get the general solution on $(0, \infty)$ as

$$y = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x).$$

## Bessel functions of the second kind

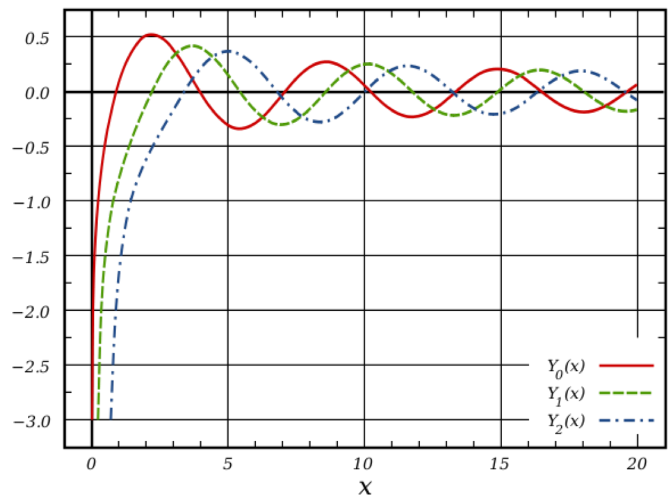
If $\nu \neq$ integer, the function defined by the linear combination

$$Y_\nu(x) = \frac{\cos(\nu\pi)J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}$$

where the functions $J_\nu(x)$ are linearly independent solutions of Bessel's equation, another form of the general solution of this equation is

$$y(x) = c_1 J_\nu(x) + c_2 Y_\nu(x)$$

provided $\nu$ is not an integer.



$$Y_\nu(x) = \frac{\cos(\nu\pi)J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}$$

As $\nu \to m$, where $m$ is an integer, the expression above has the indeterminate form $0/0$. Using L'Hospital's rule, it can be shown that

$$Y_m(x) = \lim_{\nu\to m} Y_\nu(x)$$

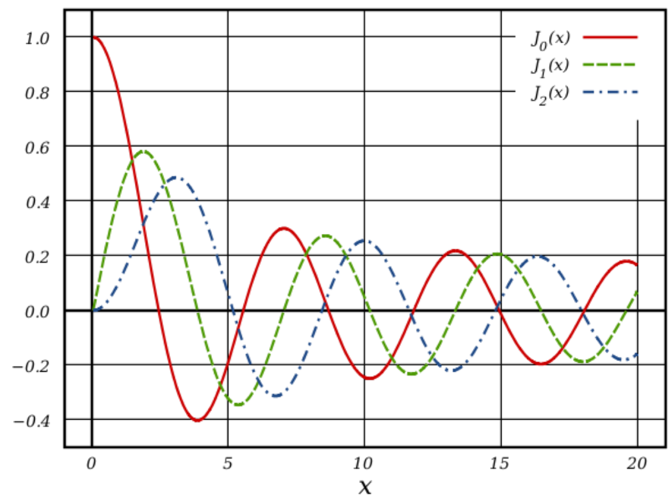
exists; moreover both $Y_m(x)$ and $J_m(x)$ are linearly independent solutions of

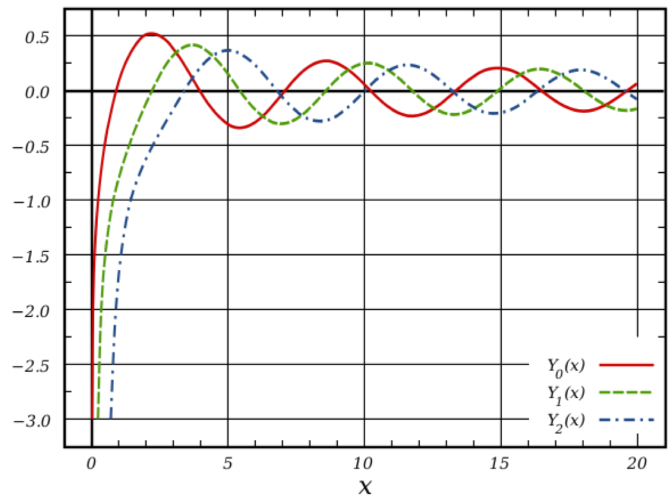
$$x^2y'' + xy' + (x^2 - m^2)y = 0'$$

Hence for any value of $\nu$, the general solution of Bessel's equation on the interval $(0, \infty)$ can be written using the **Bessel functions of the second kind** of order $\nu$ $Y_\nu(x)$ as

$$y = c_1J_\nu(x) + c_2Y_\nu(x).$$

| $x$ | $J_0(x)$ | $J_1(x)$ | $Y_0(x)$ | $Y_1(x)$ |
|---|---|---|---|---|
| 0 | 1.0000 | 0.0000 | — | — |
| 1 | 0.7652 | 0.4401 | 0.0883 | −0.7812 |
| 2 | 0.2239 | 0.5767 | 0.5104 | −0.1070 |
| 3 | −0.2601 | 0.3391 | 0.3769 | 0.3247 |
| 4 | −0.3971 | −0.0660 | −0.0169 | 0.3979 |
| 5 | −0.1776 | −0.3276 | −0.3085 | 0.1479 |
| 6 | 0.1506 | −0.2767 | −0.2882 | −0.1750 |
| 7 | 0.3001 | −0.0047 | −0.0259 | −0.3027 |
| 8 | 0.1717 | 0.2346 | 0.2235 | −0.1581 |
| 9 | −0.0903 | 0.2453 | 0.2499 | 0.1043 |
| 10 | −0.2459 | 0.0435 | 0.0557 | 0.2490 |
| 11 | −0.1712 | −0.1768 | −0.1688 | 0.1637 |
| 12 | 0.0477 | −0.2234 | −0.2252 | −0.0571 |
| 13 | 0.2069 | −0.0703 | −0.0782 | −0.2101 |
| 14 | 0.1711 | 0.1334 | 0.1272 | −0.1666 |
| 15 | −0.0142 | 0.2051 | 0.2055 | 0.0211 |



**Example 2:** General solution: $\nu$ is an integer

Consider Bessel's differential equation

$$x^2y'' + xy' + \left(x^2 - 9\right)y = 0;$$

by identifying $\nu^2 = 9$ and $\nu = 3$, we get the general solution on $(0, \infty)$ as

$$y = c_1J_3(x) + c_2Y_3(x).$$

**Differential equations solvable in terms of the Bessel functions**

We can in some cases transform a differential equation into Bessel's equation by changing a variable. For example, let $t = \alpha x$, $\alpha > 0$ in

$$x^2 y'' + xy' + \left(\alpha^2 x^2 - \nu^2\right) y = 0;$$

and then by the chain rule

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \alpha\frac{\mathrm{d}y}{\mathrm{d}t}, \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}t}{\mathrm{d}x} = \alpha^2\frac{\mathrm{d}^2y}{\mathrm{d}t^2}.$$

The equation above then becomes

$$\left(\frac{t}{\alpha}\right)^2 \alpha^2\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \left(\frac{t}{\alpha}\right)\alpha\frac{\mathrm{d}y}{\mathrm{d}t} + \left(t^2 - \nu^2\right) y = 0 \quad \text{or} \quad t^2\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + t\frac{\mathrm{d}y}{\mathrm{d}t} + \left(t^2 - \nu^2\right) y = 0.$$

The last equation is Bessel's equation of order $\nu$ with solution $y = c_1 J_\nu(t) + c_2 Y_\nu(t)$.

$$y = c_1 J_\nu(t) + c_2 Y_\nu(t).$$

By re-substituting $t = \alpha x$ into the general solution, we find that the general solution on the interval $(0, \infty)$ is

$$y = c_1 J_\nu(\alpha x) + c_2 Y_\nu(\alpha x).$$

The equation

$$x^2 y'' + xy' + \left(\alpha^2 x^2 - \nu^2\right) y = 0;$$

is called the **parametric Bessel equation of order** $\nu$ and its general solutions are very important in the study of certain boundary-value problems involving partial differential equations that are expressed in cylindrical coordinates.

Another equation similar to Bessel's equation of order $\nu$

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

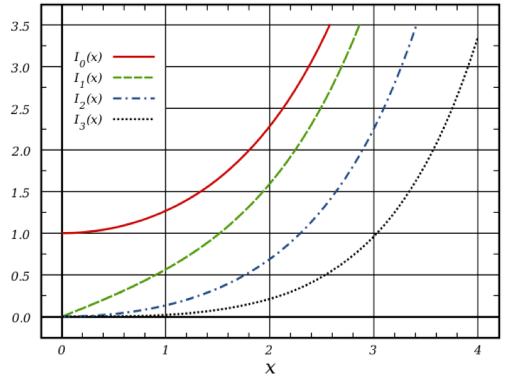
is the **modified Bessel equation of order** $\nu$

$$x^2y'' + xy' - \left(x^2 + \nu^2\right)y = 0;$$

which can also be solved by substitution. Specifically with $t = ix$ where $i = \sqrt{-1}$ we get

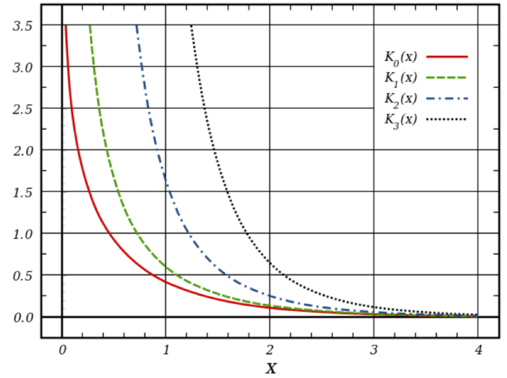
$$t^2\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + t\frac{\mathrm{d}y}{\mathrm{d}t} + \left(t^2 - \nu^2\right)y = 0.$$

Since the solution of the last differential equation are $J_\nu(t)$ and $Y_\nu(t)$, *complex-valued* solutions of the modified Bessel equation are $J_\nu(ix)$ and $Y_\nu(ix)$.




Real valued solution, called the **modified Bessel function of the first kind** of order $\nu$, is defined in terms of $J_\nu(ix)$:

$$I_\nu(x) = i^{-\nu} J_\nu(ix).$$




The **modified Bessel function of the second kind** of order $\nu \neq$ integer is defined to be

$$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi}$$

and for integral $\nu = n$

$$K_n(x) = \lim_{\nu \to n} K_\nu(x).$$

Since $I_\nu(x)$ and $K_\nu(x)$ are linearly independent on the interval $(0, \infty)$ for any value of $\nu$, the general solution of the modified Bessel equation can be written as

$$y = c_1 I_\nu(x) + c_2 K_\nu(x).$$

The general solution of the **parametric form** of the modified Bessel equation of order $\nu$

$$x^2y'' + xy' - (\alpha^2x^2 + \nu^2)y = 0$$

on the interval $(0, \infty)$ can be written as

$$y = c_1I_\nu(\alpha x) + c_2K_\nu(\alpha x).$$

**Another equation**, which is important because many differential equations fit into its form by appropriate choice of parameters, is

$$y'' + \frac{1-2a}{x}y' + \left(b^2c^2x^{2c-2} + \frac{a^2-p^2c^2}{x^2}\right)y = 0$$

whose general solution can be found in the form

$$y = x^a\left[c_1J_p(bx^c) + c_2Y_p(bx^c)\right].$$

by means of change of both the independent and dependent variables:

$$z = bx^c, \qquad y(x) = \left(\frac{z}{b}\right)^{a/c} w(z).$$

and if $p$ is not an integer, then $Y_p(x)$ above can be replaced by $J_{-p}$.

**Example 3**

Find the general solution of $xy'' + 3y' + 9y = 0$ on the interval $[0, \infty)$.

Solution:

We can rewrite the equation into its standard form

$$y'' + \frac{3}{x}y' + \frac{9}{x}y = 0$$

and make the identifications

$$1 - 2a = 3, \quad b^2c^2 = 9, \quad 2c - 2 = -1, \quad \text{and} \quad a^2 - p^2c^2 = 0$$

which imply $a = -1$ and $c = \frac{1}{2}$. From the remaining the equations, we get $b = 6$ and $p = 2$.

The general solution on the interval $[0, \infty)$ is then

$$y = x^{-1}\left[c_1 J_2(6x^{1/2}) + c_2 Y_2(6x^{1/2})\right].$$

**Example 4:** Free undamped motion of mass $m$ on an aging spring:

$$mx'' + ke^{-\alpha t}x = 0, \quad \alpha > 0.$$

Solution:
The change of variables $s = \frac{2}{\alpha}\sqrt{\frac{k}{m}}e^{-\alpha t/2}$ transforms the differential equation into

$$s^2\frac{\mathrm{d}^2 x}{\mathrm{d}s^2} + s\frac{\mathrm{d}x}{\mathrm{d}s} + s^2 x = 0,$$

which is Bessel's equation with $\nu = 0$. The general solution of this equation is $x = c_1 J_0(s) + c_2 Y_0(s)$ which after re-substitution of $s$ gives

$$x(t) = c_1 J_0\left(\frac{2}{\alpha}\sqrt{\frac{k}{m}}e^{-\alpha t/2}\right) + c_2 Y_0\left(\frac{2}{\alpha}\sqrt{\frac{k}{m}}e^{-\alpha t/2}\right).$$

Another example

$$mx'' + ktx = 0 \quad \text{or} \quad x'' + \frac{k}{m}tx = 0$$

is an Airy equation

$$y'' + \alpha^2 xy = 0.$$

Its general equation can also be written in terms of Bessel functions.

**Properties of Bessels functions**
of the first and second kind of order $m$, $m = 0, 1, 2, \ldots$:
(i)

$$J_{-m}(x) = (-1)^m J_m(x),$$

(ii) $J_m(x)$ is an even function if $m$ is an even integer, and an odd function if $m$ is an odd integer:

$$J_m(-x) = (-1)^m J_m(x),$$

(iii)

$$J_m(0) = \begin{cases} 0, & m > 0 \\ 1, & m = 0 \end{cases},$$

(iv) $Y_m(x)$ is unbounded at the origin:

$$\lim_{x \to 0^+} Y_m(x) = -\infty.$$

The solutions of the Bessel equation of order $0$ can be obtained using the Case III solutions

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} c_n x^{n+r_1}, c_0 \neq 0, \\
y_2(x) &= y_1(x)\ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r_2}.
\end{aligned}
$$

It can be shown that

$$
\begin{aligned}
y_1(x) &= J_0(x) \\
y_2(x) &= J_0(x)\ln x - \sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)\left(\frac{x}{2}\right)^{2k}.
\end{aligned}
$$

The Bessel function of the second kind of order $0$, $Y_0(x)$, is then defined to be the linear combination

$$\begin{aligned} Y_0(x) &= \frac{2}{\pi}(\gamma - \ln 2)y_1(x) + \frac{2}{\pi}y_2(x), \quad \text{for } x > 0 \\ &= \frac{2}{\pi}J_0(x)\left[\gamma + \ln\frac{x}{2}\right] - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{(k!)^2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)\left(\frac{x}{2}\right)^{2k}, \end{aligned}$$

where $\gamma = 0.57721566\ldots$ is **Euler's constant**. Due to the presence of the logarithmic term, $Y_0(x)$ is discontinuous at $x = 0$.

**Differential recurrence relations**

$$xJ'_\nu(x) = \nu J_\nu(x) - xJ_{\nu+1}(x)$$

relate Bessel functions of different orders.

**Example 5:** Derivation using the series definition

Solution:

It follows from

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}$$

that

$$
\begin{aligned}
xJ_\nu'(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n(2n+\nu)}{n!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} \\
&= \nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} + 2 \sum_{n=0}^{\infty} \frac{(-1)^n n}{n!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} \\
&= \nu J_\nu(x) + x \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu-1} \\
&= \nu J_\nu(x) - x \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(2+\nu+k)} \left(\frac{x}{2}\right)^{2k+\nu+1} = \nu J_\nu(x) - xJ_{\nu+1}(x),
\end{aligned}
$$

where $k = n - 1$.

The recurrence relation can be written in an alternative form: dividing the recurrence relation by $x$ gives

$$J_\nu'(x) - \frac{\nu}{x} J_\nu(x) = -J_{\nu+1}(x).$$

which is a linear first-order differential equation in $J_\nu(x)$. Multiplying both sides by by the integrating factor $x^{-\nu}$ yields

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-\nu} J_\nu(x)\right] = -x^{-\nu} J_{\nu+1}(x).$$

It can similarly be shown that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{\nu} J_\nu(x)\right] = x^{\nu} J_{\nu-1}(x).$$

The differential recurrence relations above are also valid for the Bessel function of the second kind $Y_\nu(x)$. When $\nu = 0$, the first differential recurrence relation gives

$$J_0'(x) = -J_1(x) \quad \text{and} \quad Y_0'(x) = -Y_1(x).$$

**Sperical Bessel functions**

When the order $\nu$ is half an odd integer, $\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots,$ the Bessel functions of the first kind $J_\nu(x)$ can be expressed in terms of the elementary functions $\sin x$, $\cos x$ and powers of $x$ and are called **spherical Bessel functions**.

Consider the case $\nu = \frac{1}{2}$:

$$J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma\left(1+\frac{1}{2}+n\right)} \left(\frac{x}{2}\right)^{2n+1/2}.$$

Using that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and the property $\Gamma(1+\alpha) = \alpha\Gamma(\alpha)$, we can obtain the values of $\Gamma\left(1+\frac{1}{2}+n\right)$ for $n = 0$, $n = 1$, $n = 3$ etc.

The values of $\Gamma\left(1+\frac{1}{2}+n\right)$ for $n=0$, $n=1$, $n=3$:

$$\begin{aligned}
\Gamma\left(\frac{3}{2}\right) &= \Gamma\left(1+\frac{1}{2}\right)=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{1}{2}\sqrt{\pi},\\
\Gamma\left(\frac{5}{2}\right) &= \Gamma\left(1+\frac{3}{2}\right)=\frac{3}{2}\Gamma\left(\frac{3}{2}\right)=\frac{3}{2^2}\sqrt{\pi},\\
\Gamma\left(\frac{7}{2}\right) &= \Gamma\left(1+\frac{5}{2}\right)=\frac{5}{2}\Gamma\left(\frac{5}{2}\right)=\frac{5.3}{2^3}\sqrt{\pi}=\frac{5.4.3.2.1}{2^3.4.2}\sqrt{\pi}=\frac{5!}{2^5 2!}\sqrt{\pi},\\
\Gamma\left(\frac{9}{2}\right) &= \Gamma\left(1+\frac{7}{2}\right)=\frac{7}{2}\Gamma\left(\frac{7}{2}\right)=\frac{7.5!}{2^6 2!}\sqrt{\pi}=\frac{7.6.5!}{2^6.6.2!}\sqrt{\pi}=\frac{7!}{2^7 3!}\sqrt{\pi}.
\end{aligned}$$

In general

$$\Gamma\left(1+\frac{1}{2}+n\right)=\frac{(2n+1)!}{2^{2n+1}n!}\sqrt{\pi}.$$

Consequently

$$
\begin{aligned}
J_{1/2}(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{\frac{(2n+1)!}{2^{2n+1} n!}\sqrt{\pi}} \left(\frac{x}{2}\right)^{2n+1/2}. \\
&= \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.
\end{aligned}
$$

The infinite series above is the Maclaurin series for $\sin x$ and thus

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x.$$

Similarly it can be shown that

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$$