# Green's function methods
# - boundary value problems

## $2^{nd}$-order ODEs

## Boundary value problems

A boundary value problem for a second order differential equation involves conditions on $y(x)$ and $y'(x)$ that are specified at two different points $x = a$ and $x = b$.

Conditions such as

$$y(a) = 0, y(b) = 0; \quad y(a) = 0, y'(b) = 0; \quad y'(a) = 0, y'(b) = 0;$$

are special cases of the more general homogeneous boundary conditions

$$\begin{aligned} A_1 y(a) + B_1 y'(a) &= 0 \\ A_2 y(b) + B_2 y'(b) &= 0 \end{aligned}$$

where $A_1$, $A_2$, $B_1$, and $B_2$ are constants.

The goal is to find an integral solution $y_p(x)$ for nonhomogeneous boundary value problems defined as

$$
\begin{aligned}
y'' + P(x)y' + Q(x)y &= f(x), \\
A_1 y(a) + B_1 y'(a) &= 0 \\
A_2 y(b) + B_2 y'(b) &= 0
\end{aligned}
$$

Here we assume in addition to that $P(x)$, $Q(x)$ and $f(x)$ are continuous on the interval $[a, b]$, also that the homogeneous problem

$$
\begin{aligned}
y'' + P(x)y' + Q(x)y &= 0, \\
A_1 y(a) + B_1 y'(a) &= 0 \\
A_2 y(b) + B_2 y'(b) &= 0
\end{aligned}
$$

has only the trivial solution $y = 0$.

This condition is sufficient to guarantee that a unique solution of the nonhomogeneous boundary value problem, defined above, exists and is given by an integral

$$y_p(x) = \int_a^b G(x,t) f(t)\, dt,$$

where $G(x,t)$ is a Green's function.

The starting point for the construction of $G(x,t)$ are again the formulas for the variation of parameters

$$y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x)$$

where $y_1(x)$ and $y_2(x)$ are linearly independent on the interval $[a,b]$, and the functions $u_1(x)$ and $u_2(x)$ are determined from the equations

$$u_1'(x) = -\frac{y_2(x) f(x)}{W}, \qquad u_2'(x) = \frac{y_1(x) f(x)}{W}.$$

where $W$ is the Wronskian.

## Another Green's function

Suppose $y_1(x)$ and $y_2(x)$ are linearly independent on $[a,b]$ of the associated homogeneous differential equation and that $x \in [a,b]$. we now integrate the equation for $u_1'$ on $[b,x]$ and the equation for $u_2'$ on $[a,x]$ (we will see later why)

$$u_1(x) = -\int_b^x \frac{y_2(t)f(t)}{W}\,dt, \qquad u_2(x) = \int_a^x \frac{y_1(t)f(t)}{W}\,dt.$$

A particular solution is then

$$\begin{aligned} y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) &= y_1(x)\int_x^b \frac{y_2(t)f(t)}{W}\,dt + y_2(x)\int_a^x \frac{y_1(t)f(t)}{W}\,dt \\ &= \int_a^x \frac{y_2(x)y_1(t)}{W}f(t)\,dt + \int_x^b \frac{y_1(x)y_2(t)}{W}f(t)\,dt. \end{aligned}$$

The r.h.s.

$$y_p(x) = \int_a^x \frac{y_2(x)y_1(t)}{W} f(t)\, dt + \int_x^b \frac{y_1(x)y_2(t)}{W} f(t)\, dt$$

can be written compactly as a single integral

$$y_p(x) = \int_a^b G(x,t) f(t)\, dt,$$

where the function $G(x,t)$ is

$$G(x,t) = \begin{cases} \frac{y_1(t)y_2(x)}{W}, & a \le t \le x \\ \\ \frac{y_1(x)y_2(t)}{W}, & x \le t \le b. \end{cases}$$

This piecewise defined function is called a **Green's function** for the boundary value problem that is defined above. It can be proved that $G(x,t)$ is a continuous function on the interval $[a,b]$.

If the solutions $y_1(x)$ and $y_2(x)$ are chosen such that

at $x = a$, $y_1(x)$ satisfies $A_1 y(a) + B_1 y'(a) = 0$, and
at $x = b$, $y_2(x)$ satisfies $A_2 y(b) + B_2 y'(b) = 0$,

then $y_p(x)$ defined above satisfies both homogeneous boundary conditions.

To see this, we use

$$\begin{aligned} y_p(x) &= u_1(x)y_1(x) + u_2(x)y_2(x) \\ y'_p(x) &= u_1(x)y'_1(x) + y_1(x)u'_1(x) + u_2(x)y'_2(x) + y_2(x)u'_2(x) \\ &= u_1(x)y'_1(x) + u_2(x)y'_2(x) \end{aligned}$$

where we applied the assumption $y_1 u'_1 + y_2 u'_2 = 0$ of the variation of parameters.

Observe from

$$u_1(x) = -\int_b^x \frac{y_2(t)f(t)}{W}\,dt, \qquad u_2(x) = \int_a^x \frac{y_1(t)f(t)}{W}\,dt.$$

that $u_1(b) = 0$ and $u_2(a) = 0$. From the latter we can show that

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

satisfies $A_1y(a) + B_1y'(a) = 0$ whenever $y_1(x)$ satisfies the same boundary condition:

$$\begin{aligned} A_1y_p(a) + B_1y_p'(a) &= A_1\left[u_1(a)y_1(a) + u_2(a)y_2(a)\right] + B_1\left[u_1(a)y_1'(a) + u_2(a)y_2'(a)\right] \\ &= u_1(a)\left[A_1y_1(a) + B_1y_1'(a)\right] = 0 \end{aligned}$$

Likewise, $u_1(b) = 0$ implies that $y_p(x)$ satisfies $A_2y(b) + B_2y'(b) = 0$ whenever $y_2(x)$ does

$$\begin{aligned} A_2y_p(b) + B_1y_p'(b) &= A_2\left[u_1(b)y_1(b) + u_2(b)y_2(b)\right] + B_2\left[u_1(b)y_1'(b) + u_2(b)y_2'(b)\right] \\ &= u_2(b)\left[A_2y_2(b) + B_2y_2'(b)\right] = 0. \end{aligned}$$

**Theorem:** Solution of a BVP

Let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of

$$y'' + P(x)y' + Q(x)y = 0$$

on the interval $[a, b]$, and suppose $y_1(x)$ and $y_2(x)$ satisfy $A_1 y(a) + B_1 y'(a) = 0$ and $A_2 y(b) + B_2 y'(b) = 0$ respectively. Then the function

$$y_p(x) = \int_a^b G(x,t) f(t)\, dt, \qquad \text{where} \quad G(x,t) = \begin{cases} \frac{y_1(t) y_2(x)}{W}, & a \le t \le x \\ \frac{y_1(x) y_2(t)}{W}, & x \le t \le b. \end{cases}$$

is the solution of the boundary value problem

$$\begin{aligned} y'' + P(x)y' + Q(x)y &= f(x), \\ A_1 y(a) + B_1 y'(a) &= 0 \\ A_2 y(b) + B_2 y'(b) &= 0. \end{aligned}$$

**Example 7:**

Solve the boundary value problem

$$y'' + 4y = 3, \quad y'(0) = 0, y(\pi/2) = 0.$$

The solution of the associate homogeneous equation $y'' + 4y = 0$ are $y_1(x) = \cos 2x$ and $y_2(x) = \sin 2x$ and $y_1'(0) = 0$ and $y_2(\pi/2) = 0$. The Wronskian is $W = 2$ so the Green's function for the BVP is

$$G(x,t) = \begin{cases} \frac{1}{2}\cos 2t \sin 2x, & 0 \leq t \leq x \\ \\ \frac{1}{2}\cos 2x \sin 2t, & x \leq t \leq \pi/2. \end{cases}$$

It follows from the theorem above that the solution of the BVP with $a = 0$ and $b = \pi/2$, and $f(t) = 3$ is

$$\begin{aligned} y_p(x) &= 3\int_0^{\pi/2} G(x,t)\,dt = 3.\frac{1}{2}\sin 2x \int_0^x \cos 2t\,dt + 3.\frac{1}{2}\cos 2x \int_x^{\pi/2} \sin 2t\,dt \\ &= \frac{3}{4}\sin 2x\,[\sin 2t]_0^x + \frac{3}{4}\cos 2x\,[-\cos 2t]_x^{\pi/2} \\ &= \frac{3}{4} + \frac{3}{4}\cos 2x. \end{aligned}$$

**Example 8:** BVP

Solve the boundary value problem

$$x^2y'' - 3xy' + 3y = 24x^5, \quad y(1) = 0, y(2) = 0.$$

This differential equation is a Cauchy-Euler differential equation.

We assume a solution of the associated homogeneous problem $x^2y'' - 3xy' + 3y = 0$ in the form $y = x^m$, and obtain the auxiliary equation

$$m(m-1) - 3m + 3 = (m-1)(m-3) = 0$$

of which the roots are $m_1 = 1$ and $m_2 = 3$. The general solution of the associated homogeneous differential equation is then

$$y = c_1x + c_2x^3.$$

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We apply $y(1) = 0$ to the general solution; this gives $c_1 = -c_2$ and by choosing $c_2 = -1$ we get

$$y_1 = x - x^3.$$

Applying $y(2) = 0$, we get $2c_1 + 8c_2 = 0$ or $c_1 = -4c_2$ and by choosing $c_2 = -1$ we get

$$y_2 = 4x - x^3.$$

The Wronskian of these two solutions is

$$W(y_1(x), y_2(x)) = \begin{vmatrix} x - x^3 & 4x - x^3 \\ 1 - 3x^2 & 4 - 3x^2 \end{vmatrix} = 6x^3.$$

The Green's function for the boundary value problem is then

$$G(x,t) = \begin{cases} \dfrac{\left(t-t^3\right)\left(4x-x^3\right)}{6t^3}, & 0 \le t \le x \\ \dfrac{\left(x-x^3\right)\left(4t-t^3\right)}{6t^3}, & x \le t \le 2. \end{cases}$$

In order to identify the correct forcing function we have to write the differential equation in the standard form:

$$y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 24x^3.$$

so the forcing function is $f(t) = 24t^3$.

The particular solution is then

$$
\begin{aligned}
y_p(x) &= 24 \int_1^2 G(x,t)\, t^3\, dt \\
&= 4\left(4x - x^3\right) \int_1^x \left(t - t^3\right)\, dt + 4\left(x - x^3\right) \int_x^2 \left(4t - t^3\right)\, dt \\
&= 12x - 15x^3 + 3x^5.
\end{aligned}
$$

Remark:
Notice that the boundary conditions

$$
\begin{aligned}
A_1 y(a) + B_1 y'(a) &= 0 \\
A_2 y(b) + B_2 y'(b) &= 0
\end{aligned}
$$

do not uniquely determine the functions $y_1(x)$ and $y_2(x)$. There is a certain arbitrariness in the selection of these functions.