# HIGHER ORDER DIFFERENTIAL EQUATIONS

## Theory of linear equations

### Initial-value and boundary-value problem

$n$th-order initial value problem is

*Solve:* $$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ \dots \ + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

*Subject to:* $$y(x_0) = y_0,\ y'(x_0) = y_1,\ \dots, y^{(n-1)} = y_{n-1} \qquad (1)$$

we seek a function defined on an interval $I$, containing $x_0$, that satisfies the DE and the $n$ initial conditions above.

**Existence and uniqueness**

**Theorem: Existence of a unique solution**

Let $a_n(x)$, $a_{n-1}(x)$, ... , $a_1(x)$ , $a_0(x)$ and $g(x)$ be continuous on an interval $I$ and let $a_n(x) \neq 0$ for every $x$ in this interval. If $x = x_0$ in any point in this interval, then a solution $y(x)$ of the initial value problem (1) exists on the interval and is unique.

Example: Unique solution of an IVP

$$3y''' + 5y'' - y' + 7y = 0, \qquad y(1) = 0, y'(1) = 0, y''(1) = 0$$

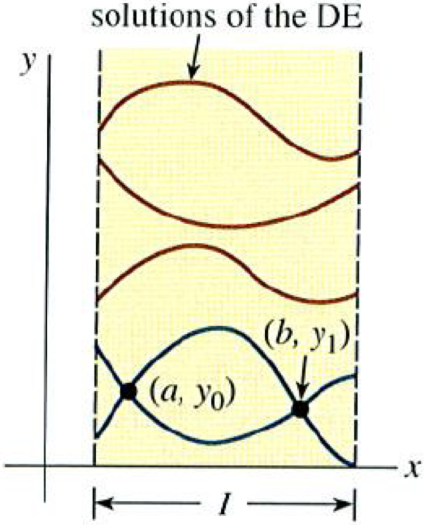
has the trivial solution $y = 0$. Since the DE is linear with constant coefficients, all the conditions of the theorem are fulfilled, and thus $y = 0$ is the *only* solution on any interval containing $x = 1$.

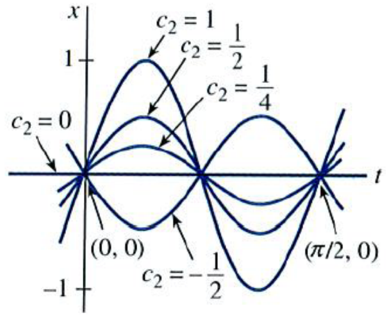
**Boundary-value problem**

consists of solving a linear DE of order two or greater in which the dependent variable $y$ or its derivatives are specified at *different points*. Example: a two-point BVP

*Solve:* $$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

*Subject to boundary conditions:* $$y(x_0) = y_0,\ y(b) = y_1 \qquad (2)$$






A BVP can have many, one or no solutions:

The DE $x'' + 16x = 0$ has the two-parameter family of solutions $x = c_1 \cos 4t + c_2 \sin 4t$. Consider the BVPs:

(1) $x(0) = 0$, and $x(\pi/2) = 0 \Rightarrow c_1 = 0$ and the solution satisfies the DE for any value of $c_2$, thus the solution of this BVP is the one-parameter family $x = c_2 \sin 4t$.

(2) $x(0) = 0$, and $x(\pi/8) = 0 \Rightarrow c_1 = 0$ and $c_2 = 0$, so the only solution to this BVP is $x = 0$.

(3) $x(0) = 0 \Rightarrow c_1 = 0$ again but the second condition $x(\pi/2) = 1$ leads to the contradiction: $1 = c_2 \sin 2\pi = c_2.0 = 0$.

# Method of Variation of Parameters

## $2^{nd}$-order ODEs

## The method of variation of parameters

Advantage: the method always yields a particular solution $y_p$, provided the associated homogeneous equation can be solved. Also it is not limited to certain types of $g(x)$.

First we put a linear second-order DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x) \tag{18}$$

into the standard form by dividing by $a_2(x)$

$$y'' + P(x)y' + Q(x)y = f(x) \tag{19}$$

We seek a solution of the form

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where $y_1$ and $y_2$ form a fundamental set of solutions on $I$ of the associated homogeneous form of (18). Using the product rule to differentiate $y_p$ twice gives

$$\begin{aligned} y_p' &= u_1y_1' + y_1u_1' + u_2y_2' + y_2u_2' \\ y_p'' &= u_1y_1'' + y_1'u_1' + y_1u_1'' + u_1'y_1' + u_2y_2'' + y_2'u_2' + y_2u_2'' + u_2'y_2' \end{aligned}$$

Substituting these into the standard form (19) yields

$$\begin{aligned} y_p'' + P(x)y_p' + Q(x)y_p &= u_1\left[y_1'' + Py_1' + Qy_1\right] + u_2\left[y_2'' + Py_2' + Qy_2\right] \\ &\quad + y_1u_1'' + u_1'y_1' + y_2u_2'' + u_2'y_2' + P\left[y_1u_1' + y_2u_2'\right] + y_1'u_1' + y_2'u_2' \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left[y_1u_1'\right] + \frac{\mathrm{d}}{\mathrm{d}x}\left[y_2u_2'\right] + P\left[y_1u_1' + y_2u_2'\right] + y_1'u_1' + y_2'u_2' \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left[y_1u_1' + y_2u_2'\right] + P\left[y_1u_1' + y_2u_2'\right] + y_1'u_1' + y_2'u_2' = f(x) \end{aligned} \tag{20}$$

We need two equations for two unknown functions $u_1$ and $u_2$. Assuming that these functions satisfy $y_1 u_1' + y_2 u_2' = 0$, the equation above reduces to $y_1' u_1' + y_2' u_2' = f(x)$. By Cramer's rule, the solution of the system

$$\begin{aligned} y_1 u_1' + y_2 u_2' &= 0 \\ y_1' u_1' + y_2' u_2' &= f(x) \end{aligned}$$

can be expressed in terms of determinants:

$$u_1' = \frac{W_1}{W} = -\frac{y_2 f(x)}{W} \quad \text{and} \quad u_2' = \frac{W_2}{W} = \frac{y_1 f(x)}{W} \tag{21}$$

where

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}.$$

The functions $u_1$ and $u_2$ are found by integrating the result in (21). The determinant $W$ is the Wronskian of $y_1$ and $y_2$ whose linear independence ensures that $W \neq 0$.

Example: General solution using variation of parameters

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

From the auxiliary equation $m^2 - 4m + 4 = (m-2)^2 = 0$ we have $y_c = c_1 e^{2x} + c_2 x e^{2x}$. We identify $y_1 = e^{2x}$ and $y_2 = xe^{2x}$ and evaluate the Wronskian

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = e^{4x}$$

The DE above is already in the standard form, so $f(x) = (x+1)e^{2x}$, $W_1$ and $W_2$ are then

$$W_1 = \begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = -(x+1)xe^{4x}, \quad W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix} = (x+1)e^{4x}$$

and so

$$u_1' = -\frac{(x+1)xe^{4x}}{e^{4x}} = -x^2 - x, \qquad u_2' = \frac{(x+1)e^{4x}}{e^{4x}} = x + 1$$

It follows that $u_1 = -\frac{1}{3}x^3 - \frac{1}{2}x^2$ and $u_2 = \frac{1}{2}x^2 + x$, and hence

$$y_p = \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right)e^{2x} + \left(\frac{1}{2}x^2 + x\right)xe^{2x} = \frac{1}{6}x^3e^{2x} + \frac{1}{2}x^2e^{2x}$$

The general solution is then

$$y = y_c + y_p = c_1e^{2x} + c_2xe^{2x} + \frac{1}{6}x^3e^{2x} + \frac{1}{2}x^2e^{2x}$$

Green's function methods
- initial value problems

$2^{nd}$-order ODEs

**Green's functions**

Consider the linear second-order differential equation

$$a_2(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x)$$

In mathematical analysis of physical systems it is often desirable to express the response or output $y(x)$ subject to either *initial conditions* or *boundary conditions* directly in terms of the forcing function or input $g(x)$.

We will start by examining solutions of the initial value problem with the differential equation above in the standard form

$$y'' + P(x)y' + Q(x)y = f(x)$$

and we assume that the functions $P(x)$, $Q(x)$ and $f(x)$ are continuous on some interval $I$.

**Initial value problems**

The solution of the second-order initial value problem

$$y'' + P(x)y' + Q(x)y = f(x), \qquad y(x_0) = y_0, \;\; y'(x_0) = y_1$$

can be expressed as the superposition of two solutions:
(i) the solution $y_h$ of the associated **homogeneous** differential equation with **non-homogeneous** initial conditions

$$y'' + P(x)y' + Q(x)y = 0, \qquad y(x_0) = y_0, \;\; y'(x_0) = y_1,$$

(ii) the solution $y_p$ of the **nonhomogeneous** differential equation with **zero initial conditions**, so called a **rest** solution

$$y'' + P(x)y' + Q(x)y = f(x), \qquad y(x_0) = 0, \;\; y'(x_0) = 0.$$

## Green's function

If $y_1$ and $y_2$ form a fundamental set of solutions on the interval $I$ of the associated homogeneous differential equation, then a particular solution of the non-homogeneous equation on the interval $I$ can be found by variation of parameters:

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where the functions $u_1(x)$ and $u_2(x)$ are determined from the equations

$$u_1'(x) = -\frac{y_2(x)f(x)}{W}, \qquad u_2'(x) = \frac{y_1(x)f(x)}{W}$$

where the linear independence of $y_1(x)$ and $y_2(x)$ on the interval $I$ guarantees that the Wronskian $W \neq 0$ for all $x \in I$.

We can now integrate the derivatives $u_1'(x)$ and $u_2'(x)$ on the interval $[x_0, x]$ to get

$$
\begin{aligned}
y_p(x) &= y_1(x)\int_{x_0}^{x}\frac{-y_2(t)f(t)}{W(t)}dt + y_2(x)\int_{x_0}^{x}\frac{y_1(t)f(t)}{W(t)}dt \\
&= \int_{x_0}^{x}\frac{-y_1(x)y_2(t)}{W(t)}f(t)dt + \int_{x_0}^{x}\frac{y_1(t)y_2(x)}{W(t)}f(t)dt,
\end{aligned}
$$

where

$$
W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}.
$$

The two integrals

$$y_p(x) = \int_{x_0}^{x} \frac{-y_1(x)y_2(t)}{W(t)} f(t)\, dt + \int_{x_0}^{x} \frac{y_1(t)y_2(x)}{W(t)} f(t)\, dt$$

can be rewritten as a single integral

$$y_p(x) \quad = \quad \int_{x_0}^{x} G(x,t) f(t)\, dt.$$

where the function $G(x,t)$ is called the **Green's function**

$$G(x,t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)}.$$

**Remarks:**

The Green's function depends only on the fundamental solutions $y_1(x)$ and $y_2(x)$ of the associated homogeneous differential equation and **not** on the forcing function $f(x)$.

All linear second-order differential equations with the same left hand side but different forcing functions have the same Green's function.

We can call

$$G(x,t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)}$$

the **Green's function for the second-order differential operator**

$$L = D^2 + P(x)D + Q(x).$$

**Example 1**

Use the Green's function to find a particular solution of $y'' - y = f(x)$.

The solutions of the associated homogeneous equation $y'' - y = 0$ are

$$y_1(x) = e^x, \qquad y_2(x) = e^{-x},$$

and the Wronskian $W = -2$. It follows from the definition of the Green's function above that

$$G(x,t) = \frac{e^t e^{-x} - e^x e^{-t}}{-2} = \frac{e^{x-t} - e^{-(x-t)}}{2} = \sinh(x-t).$$

The particular solution is then

$$y_p(x) = \int_{x_0}^{x} \sinh(x-t) f(t) dt.$$

**Example 2**

Find the general solution of the following nonhomogeneous differential equations

(a) $y'' - y = 1/x,$ (b) $y'' - y = e^{2x}.$

From the Example 1:

(i) both ODEs have the same complementary function $y_c = c_1e^{-x} + c_2e^{x}$, and

(ii) the Green's function for both equations is $G(x, t) = \sinh(x - t)$.

**Case (a):** $y'' - y = 1/x$

With $f(x) = 1/x$ we get the particular solution

$$y_p(x) = \int_{x_0}^{x} \sinh(x-t) f(t) dt = \int_{x_0}^{x} \frac{\sinh(x-t)}{t} dt$$

and the general solution $y = y_c + y_p$ on any interval $[x_0, x]$ not containing the origin is

$$y = c_1 e^{-x} + c_2 e^{x} + \int_{x_0}^{x} \frac{\sinh(x-t)}{t} dt.$$

**Case (b):** With $x_0 = 0$ and $f(x) = e^{2x}$ it follows from

$$y = c_1 e^{-x} + c_2 e^{x} + \int_{x_0}^{x} \sinh(x-t)\, e^{2t}\, dt.$$

that the solution of this IVP is

$$\begin{aligned} y_p(x) &= \int_0^x \sinh(x-t)\, e^{2t}\, dt = \int_0^x \frac{e^{x-t} - e^{-(x-t)}}{2}\, e^{2t}\, dt \\ &= \frac{1}{2}\, e^{x} \int_0^x e^{t}\, dt - \frac{1}{2}\, e^{-x} \int_0^x e^{3t}\, dt \\ &= \frac{1}{3} e^{2x} - \frac{1}{2} e^{x} + \frac{1}{6} e^{-x} \end{aligned}$$

where we have held $x$ constant throughout the integration with respect to $t$.

**Theorem:** Solution of the IVP with nonhomogeneous ODE and zero initial conditions

The function $y_p(x)$ defined by

$$y_p(x) \quad = \quad \int_{x_0}^{x} G(x,t) f(t) \, dt.$$

is the solution of the initial value problem

$$y'' + P(x)y' + Q(x)y = f(x), \qquad y(x_0) = 0, \;\; y'(x_0) = 0.$$

**Proof:**
By construction $y_p(x)$ satisfies the nonhomogeneous differential equation

$$y'' + P(x)y' + Q(x)y = f(x).$$

$$G(x,t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)}$$

Next, the definite integral has the property $\int_a^a = 0$, so

$$y_p(x_0) = \int_{x_0}^{x_0} G(x_0, t)f(t)\, dt = 0$$

To show that $y_p'(x_0) = 0$ we use the Leibniz formula for the derivative of an integral

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{u(x)}^{v(x)} F(x,t)\, dt = F(x, v(x))\, v'(x) - F(x, u(x))\, u'(x) + \int_{u(x)}^{v(x)} \frac{\partial}{\partial x} F(x,t)\, dt$$

to get

$$y_p'(x) = G(x,x)f(x) + \int_{x_0}^{x} \frac{y_1(t)y_2'(x) - y_1'(x)y_2(t)}{W(t)} f(t)\, dt$$

where the first term is zero, and hence

$$y_p'(x_0) = \int_{x_0}^{x_0} \frac{y_1(t)y_2'(x_0) - y_1'(x_0)y_2(t)}{W(t)} f(t)\, dt = 0.$$

**Example 3:** Example 2 revisited

(a) $y'' - y = 1/x, \quad y(1) = 0, y'(1) = 0,$ (b) $y'' - y = e^{2x}, \quad y(0) = 0, y'(0) = 0,$

**Case (a):** With $x_0 = 1$ and $f(x) = 1/x$ it follows from

$$y = c_1 e^{-x} + c_2 e^{x} + \int_{x_0}^{x} \frac{\sinh(x-t)}{t} dt$$

and the theorem that the solution of this IVP is

$$y_p(x) = \int_{1}^{x} \frac{\sinh(x-t)}{t} dt$$

where $[1, x]$, $x > 0$.

**Case (b):** With $x_0 = 0$ and $f(x) = e^{2x}$ it follows from

$$y = c_1 e^{-x} + c_2 e^{x} + \int_{x_0}^{x} \sinh(x-t)\, e^{2t}\, dt.$$

that the solution of this IVP is

$$\begin{aligned} y_p(x) &= \int_0^x \sinh(x-t)\, e^{2t}\, dt = \int_0^x \frac{e^{x-t} - e^{-(x-t)}}{2}\, e^{2t}\, dt \\ &= \frac{1}{2}\, e^{x} \int_0^x e^{t}\, dt - \frac{1}{2}\, e^{-x} \int_0^x e^{3t}\, dt \\ &= \frac{1}{3} e^{2x} - \frac{1}{2} e^{x} + \frac{1}{6} e^{-x} \end{aligned}$$

where we have held $x$ constant throughout the integration with respect to $t$.

**Example 4**

Solve the initial value problem

$$y'' + 4y = x, \quad y(0) = 0, y'(0) = 0.$$

We start with constructing the Green's function:

The two linearly independent solutions of

$$y'' + 4y = 0$$

are $y_1(x) = \cos 2x$ and $y_2(x) = \sin 2x$, and the Wronskian is $W = 2$, so

$$G(x,t) = \frac{\cos 2t \sin 2x - \cos 2x \sin 2t}{2} = \frac{1}{2} \sin 2(x-t).$$

Given the initial condition $x_0 = 0$, the solution of the initial value problem is

$$y_p(x) = \frac{1}{2}\int_0^x t \sin 2(x-t)\, dt.$$

To evaluate the integral, we first write

$$y_p(x) = \frac{1}{2}\sin 2x \int_0^x t \cos 2t\, dt - \frac{1}{2}\cos 2x \int_0^x t \sin 2t\, dt.$$

and integrate by parts:

$$\begin{aligned} y_p(x) &= \frac{1}{2}\sin 2x\left[\frac{1}{2}t\sin 2t + \frac{1}{4}\cos 2t\right]_0^x - \frac{1}{2}\cos 2x\left[-\frac{1}{2}t\cos 2t + \frac{1}{4}\sin 2t\right]_0^x \\ &= \frac{1}{4}x - \frac{1}{8}\sin 2x. \end{aligned}$$

**Theorem**

If $y_h$ is the solution of the initial value problem

$$y'' + P(x)y' + Q(x)y = 0, \qquad y(x_0) = y_0, \;\; y'(x_0) = y_1,$$

and $y_p$ is the solution of the initial value problem

$$y'' + P(x)y' + Q(x)y = f(x), \qquad y(x_0) = 0, \;\; y'(x_0) = 0,$$

on the interval I, then

$$y = y_h + y_p$$

is the solution of the initial value problem

$$y'' + P(x)y' + Q(x)y = f(x), \qquad y(x_0) = y_0, \;\; y'(x_0) = y_1.$$

**Proof:** Since $y_h$ is a linear combination of the fundamental solutions, it follows that $y = y_h + y_p$ is a solution of the nonhomogeneous differential equation.

Moreover, since $y_h$ satisfies the initial conditions in

$$y'' + P(x)y' + Q(x)y = 0, \qquad y(x_0) = y_0, \ \ y'(x_0) = y_1,$$

and $y_p$ satisfies the initial conditions in

$$y'' + P(x)y' + Q(x)y = f(x), \qquad y(x_0) = 0, \ \ y'(x_0) = 0,$$

we have

$$\begin{aligned} y(x_0) &= y_h(x_0) + y_p(x_0) = y_0 + 0 = y_0 \\ y'(x_0) &= y_h'(x_0) + y_p'(x_0) = y_1 + 0 = y_1. \end{aligned}$$

We see that the response $y(x) = y_h(x) + y_p(x)$ described by the full initial value problem can be separated into response $y_h$ due to the nontrivial initial conditions $y(x_0) = y_0, y'(x_0) = y_1$ and the response $y_p$ due to the forcing function.

**Example 5**

Solve the initial value problem

$$y'' + 4y = \sin 2x, \quad y(0) = 1, y'(0) = -2.$$

We solve two initial value problems:

(i) $y'' + 4y = 0, y(0) = 1, y'(0) = -2$

by applying the initial conditions to the general solution $y(x) = c_1 \cos 2x + c_2 \sin 2x$ of the associated homogeneous differential equation, we find $c_1 = 1, c_2 = -1$ and thus

$$y_h(x) = \cos 2x - \sin 2x.$$

(ii) $y'' + 4y = \sin 2x, y(0) = 0, y'(0) = 0$

Since the l.h.s. of the ODE is the same as in Example 4, we know the Green's function:

$$G(x, t) = \frac{1}{2} \sin 2(x - t).$$

With $f(t) = \sin 2t$, the solution of this IVP (see Problem 2) is

$$y_p(x) = \frac{1}{2} \int_0^x \sin 2(x - t) \sin 2t \, dt.$$

Using the trigonometric identity

$$\sin A \sin B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

with $A = 2(x - t)$ and $B = 2t$:

$$\begin{aligned}
y_p(x) &= \frac{1}{2}\int_0^x \sin 2(x - t)\sin 2t \, dt \\
&= \frac{1}{4}\int_0^x \left[\cos(2x - 4t) - \cos 2x\right] \, dt \\
&= \frac{1}{4}\left[-\frac{1}{4}\sin(2x - 4t) - t\cos 2x\right]_0^x \\
&= \frac{1}{8}\sin 2x - \frac{1}{4}x\cos 2x.
\end{aligned}$$

The full solution can now be written as

$$y(x) = y_h(x) + y_p(x) \quad = \quad \cos 2x - \sin 2x + \left(\frac{1}{8}\sin 2x - \frac{1}{4}x\cos 2x\right)$$

where the first two terms on the r.h.s. correspond to the response of the system due to the initial conditions $y(0) = 0, y'(0) = 0$ and the last term to the response of the system to the forcing function or input $f(x) = \sin 2x$.

By combining the similar terms this physical significance is lost

$$y(x) = y_h(x) + y_p(x) \quad = \quad \cos 2x - \frac{7}{8}\sin 2x - \frac{1}{4}x\cos 2x.$$

The beauty of the solution written in the form

$$y(x) = y_h(x) + y_p(x) = \cos 2x - \sin 2x + \frac{1}{2}\int_0^x \sin 2(x-t)\sin 2t\, dt$$

is that we can immediately write down the response of a system if the initial conditions remain the same but the forcing function is changed. For example, if the problem is changed to

$$y'' + 4y = x, \quad y(0) = 1, y'(0) = -2.$$

we replace $\sin 2t$ in the integral by $t$ and the solution becomes

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) = \cos 2x - \sin 2x + \frac{1}{2}\int_0^x t\sin 2(x-t)\, dt \\ &= \frac{1}{4}x + \cos 2x - \frac{9}{8}\sin 2x. \end{aligned}$$

Physically relevant example to Problem 5 is offered by undamped forced motion:

The initial value problem

$$\frac{d^2x}{dt^2} + \omega_0^2 x = F_0 \sin \omega t, \qquad x(0) = 0, \quad x'(0) = 0$$

has the solution of the form

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{\left(\omega_0^2 - \omega^2\right)} \sin \omega t.$$

with the constants $c_1 = 0$ and $c_2 = -\omega F_0/\omega_0 \left(\omega_0^2 - \omega^2\right)$ given by the initial conditions.

The solution of the initial value problem can be written as

$$x(t) = \frac{F_0}{\omega_0 \left(\omega_0^2 - \omega^2\right)} \left(-\omega \sin \omega_0 t + \omega_0 \sin \omega t\right).$$

**Example 6**

Solve the initial value problem

$$y'' + 4y = f(x), \quad y(0) = 1, y'(0) = -2,$$

where the forcing function $f$ is piecewise defined:

$$f(x) = \begin{cases} 0, & x < 0 \\ \sin 2x, & 0 \le x \le 2\pi \\ 0, & x > 2\pi. \end{cases}$$

**Solution:**
Recall the solution of Example 5 and replace $\sin 2t$ by the forcing function $f(t)$:

$$y(x) \quad = \quad \cos 2x - \sin 2x + \frac{1}{2}\int_0^x \sin 2(x-t) f(t)\, dt.$$

Since $f(x)$ is piecewise defined on three intervals, we have to consider the evaluation of the definite integral accordingly:

(i) $x < 0$

$$y_p(x) = \frac{1}{2}\int_0^x \sin 2(x-t) 0 \, dt = 0.$$

(ii) $0 \le x \le 2\pi$

$$\begin{aligned} y_p(x) &= \frac{1}{2}\int_0^x \sin 2(x-t) \sin 2t \, dt \\ &= \frac{1}{8}\sin 2x - \frac{1}{4} x \cos 2x, \end{aligned}$$

where we used the integration detailed in Example 5.

(iii) $x > 2\pi$

$$
\begin{aligned}
y_p(x) &= \frac{1}{2}\int_0^{2\pi} \sin 2(x-t)\sin 2t\, dt + \frac{1}{2}\int_{2\pi}^{x} \sin 2(x-t)0\, dt \\
&= \frac{1}{2}\int_0^{2\pi} \sin 2(x-t)\sin 2t\, dt \\
&= \frac{1}{4}\left[-\frac{1}{4}\sin(2x-4t) - t\cos 2x\right]_0^{2\pi} \\
&= -\frac{1}{16}\sin(2x-8\pi) - \frac{1}{2}\pi\cos 2x + \frac{1}{16}\sin 2x \\
&= -\frac{1}{2}\pi\cos 2x.
\end{aligned}
$$

Consequently, $y_p(x)$ is

$$y_p(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{8}\sin 2x - \frac{1}{4}x\cos 2x, & 0 \le x \le 2\pi \\ -\frac{1}{2}\pi\cos 2x, & x > 2\pi. \end{cases}$$

and the complete solution is

$$y_p(x) = y_h(x) + y_p(x) = \begin{cases} \cos 2x - \sin 2x, & x < 0 \\ \left(1 - \frac{1}{4}x\right)\cos 2x - \frac{7}{8}\sin 2x, & 0 \le x \le 2\pi \\ \left(1 - \frac{1}{2}\pi\right)\cos 2x - \sin 2x, & x > 2\pi. \end{cases}$$