Fluid Dynamics

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§ 1. Introductory remarks: fluids and continua in general

Fluid dynamics is part of the wider subject of *continuum mechanics* which includes *elasticity* as one of its branches. There are also overlaps of the subject matter of continuum mechanics with other branches of physics: for example the subject of *magnetohydrodynamics* which combines the techniques of fluid dynamics with those of electromagnetic theory. The subject matter of the disciplines, *soil mechanics*, *geology*, *polymer physics*, *the low temperature physics of superfluids*, and the *cosmological models* of general relativity are still more examples where continuum mechanics plays a joint role in conjunction with some other branch of condensed matter physics or, in the latter example, the physics of relativity.

However, in these lectures we are only concerned with fluids. We want to emphasise at the beginning that the term *fluid* is a little misleading to some since it may be taken to mean *a liquid*. In fact the term fluid, in these lectures, and in the fluid dynamics literature in general, means either a liquid or *a gas*.

Books on fluid dynamics

There are a huge number of books on such an old and large a subject as fluid dynamics. We have chosen to quote a few titles in the main body of the text together with some comment on their content. In addition we provide a supplementary list of some more titles *without comment*; the reader can readily take matters from there by visiting the fluid mechanics section of the library.

Some selected books

(i) Rutherford D. E., **Fluid dynamics**, *(Oliver and Boyd)*

This is a small concise book which more than covers the material of these lectures; it is clear and well written.

(ii) O' Neill and Chorlton, F., **Ideal and incompressible fluid dynamics** *(Ellis Horwood)*

This is a more verbose and less concise version of Rutherford with a larger number of worked examples.

(iii) McCormack, P. D. and Crane, L., **Physical fluid dynamics** *(Academic Press)*

Finally this text is more advanced than the other two; however in addition to the mathematics it contains good discussions of the physics underlying many fluid phenomena.

Further titles

(i) Lighthill, M. J., **An informal introduction to theoretical fluid mechanics** *(Oxford University Press)*

(ii) Fox, R, W. and McDonald, A. T., **Introduction to fluid mechanics** *(Wiley)*

(iii) Open University, **Looking at fluids in motion. Understanding fluid effects. Modelling fluid and thermodynamic systems** *(Open University Press)*

(iv) Pedlosky, J., **Geophysical fluid dynamics** *(Springer-Verlag)*

(v) Feynman R. P., Leighton R. B. and Sands M. L., **The Feynman Lectures on Physics vol. II** *(Addison–Wesley)*

§ 2. Some basic terms and notions

The main mathematical task in fluid dynamics is to obtain and solve equations for the velocity

$$\mathbf{v}(x, y, z, t) \tag{2.1}$$

where (x, y, z) is a point in the fluid and t is the time. Atomic structure is ignored and the fluid is regarded as a continuum. We shall denote the density of a fluid by

$$\rho = \rho(x, y, z, t) \tag{2.2}$$

and the pressure by

$$p = p(x, y, z, t) \tag{2.3}$$

If a fluid has *constant density*—as is the case to a high degree of approximation for many *liquids*—then it is called *incompressible*.

A fluid whose internal frictional forces are negligible is referred to as being *non-viscous* or *inviscid*. It is usual [1] to take the coefficient of viscosity of a viscous fluid to be a positive constant and denote it by

$$\eta \tag{2.4}$$

[1] Actually we shall see much later that, in some circumstances, there can be *two* coefficients of viscosity for a viscous fluid; however if the fluid is also incompressible then only one of these enters in the equation of motion.

Angular momentum and rotation in a fluid is often studied by focusing on the curl of the velocity vector i.e.

$$\nabla \times \mathbf{v} \tag{2.5}$$

This quantity is known as the *vorticity* and is denoted by $\boldsymbol{\omega}$. In other words the vorticity $\boldsymbol{\omega}$ is defined by writing

$$\boldsymbol{\omega} = \nabla \times \mathbf{v} \tag{2.6}$$

If the flow given by some velocity distribution $\mathbf{v}(x, y, z, t)$ has

$$\boldsymbol{\omega} = 0, \quad \text{everywhere} \tag{2.7}$$

then the flow is called *irrotational*.

A velocity distribution satisfying

$$\frac{\partial \mathbf{v}(x, y, z, t)}{\partial t} = 0, \quad \text{everywhere} \tag{2.8}$$

is called a *steady* flow.

§ 3. The stream derivative D/Dt

In fluid dynamics it is very useful to introduce another rate of change with respect to time: this rate of change is expressible as a derivative involving the time t and the fluid velocity $\mathbf{v}$. It is called the *stream derivative* or simply *differentiation moving with the fluid*. We shall denote it by

$$\frac{D}{Dt} \tag{3.1}$$

but first we must see how the stream derivative arises naturally and then give its definition.

To this end consider a completely arbitrary quantity Q associated with a fluid. Q can be *scalar valued* as it would be if it were temperature, pressure, density etc.; alternatively it could be *vector valued*, for example this would be the case if Q were the velocity itself. In any case the value of Q will depend both on time t and position (x, y, z) within the fluid, i.e. we have

$$Q \equiv Q(x, y, z, t) \tag{3.2}$$

Now we imagine that we select an individual fluid particle located at

$$(x, y, z) \tag{3.3}$$

at this time

$$t \tag{3.4}$$

Then, as time progresses and the fluid flows, this particle traces out a curve in the fluid which we shall denote by

$$(x(t), y(t), z(t)) \tag{3.5}$$

The stream derivative of Q is simply defined to be the rate of change of Q along this curve. In other words one first restricts $Q(x, y, z, t)$ to be on this curve and then differentiates this restricted Q with respect to t; the result then is denoted by DQ/Dt. Let us carry out these two steps and thereby obtain a formula for DQ/Dt: Restricting $Q(x, y, z, t)$ to the curve $(x(t), y(t), z(t))$ gives us the function

$$Q(x(t), y(t), z(t), t) \tag{3.6}$$

and differentiating this function with respect to t gives us the equation

$$\frac{d}{dt}Q(x(t), y(t), z(t), t) = \frac{\partial Q}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial Q}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial Q}{\partial t} \tag{3.7}$$

The RHS of 3.7 is thus the stream derivative of Q so that we can now write

$$\frac{DQ}{Dt} = \frac{\partial Q}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial Q}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial Q}{\partial t} \tag{3.8}$$

It is both useful and usual to abbreviate this equation somewhat by noting that since one has the pair of equations

$$\begin{aligned} \nabla &= \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{j}\frac{\partial}{\partial z} \\ \mathbf{v} &= \frac{\partial x}{\partial t}\mathbf{i} + \frac{\partial y}{\partial t}\mathbf{j} + \frac{\partial z}{\partial t}\mathbf{k} \end{aligned} \tag{3.9}$$

then one sees at once that

$$\mathbf{v}\cdot\nabla = \frac{\partial x}{\partial t}\frac{\partial}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial}{\partial y} + \frac{\partial z}{\partial t}\frac{\partial}{\partial z} \tag{3.10}$$

Hence we now have a much more compact form for the stream derivative namely

$$\frac{DQ}{Dt} = \frac{\partial Q}{\partial t} + (\mathbf{v}\cdot\nabla)Q \tag{3.11}$$

or equivalently, and just as compactly, one can write

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla) \tag{3.12}$$

We shall always assume that matter is conserved when a fluid flows and it turns out that this conservation imposes some restrictions on the possible velocity distributions $\mathbf{v}(x, y, z, t)$ that can represent fluid flows. We explain this in the next section.

The equation of continuity

In general the density ρ varies from point to point in a fluid. However when ρ is a constant the fluid is incompressible—you cannot compress it or you would change the density. This brings us to our first important equation for fluid flow which is known as the equation of continuity. This equation expresses the fact that, as a fluid flows, matter is neither created nor destroyed: it states that

$$\nabla \cdot (\rho \mathbf{v}) + \frac{\partial \rho}{\partial t} = 0 \tag{3.13}$$

We now embark on the proof of 3.13. Take a closed volume V of fluid with surface S out of which fluid is flowing with velocity $\mathbf{v}$, cf. Fig 1.

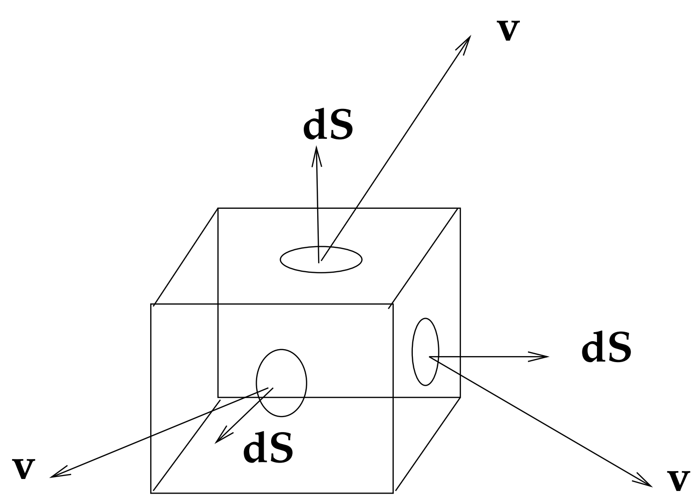


Fig. 1: Fluid flowing out of the closed volume V

The mass of fluid flowing out per unit time through a surface patch labelled by the vector $\mathbf{dS}$ is

$$\rho \mathbf{v} \cdot \mathbf{dS} \tag{3.14}$$

so that the total mass of fluid flowing out of V per unit time is the integral

$$\int_S \rho \mathbf{v} \cdot \mathbf{dS} \tag{3.15}$$

Since matter is conserved this outflow is precisely equal to the corresponding *decrease* of mass of the fluid in V. But this decrease is just

$$-\frac{\partial}{\partial t}\int_V \rho dV \tag{3.16}$$

where the minus sign in the equation above compensates for the fact that the mass of the fluid is V is *decreasing* so that its time derivative is *negative*. In any case matter conservation has now given us the equation

$$\begin{aligned}
\int_S \rho\mathbf{v}\cdot\mathbf{dS} &= -\frac{\partial}{\partial t}\int_V \rho dV \\
\Rightarrow \int_S \rho\mathbf{v}\cdot\mathbf{dS} + \frac{\partial}{\partial t}\int_V \rho dV &= 0 \\
\Rightarrow \int_V \nabla\cdot(\rho\mathbf{v})dV + \frac{\partial}{\partial t}\int_V \rho dV &= 0, \quad \text{using Gauss's divergence theorem} \\
\Rightarrow \int_V \left\{\nabla\cdot(\rho\mathbf{v}) + \frac{\partial\rho}{\partial t}\right\} dV &= 0
\end{aligned} \tag{3.17}$$

But since the volume V is arbitrary the integrand in the last line of 3.17 must be zero and so we have obtained the result that

$$\nabla\cdot(\rho\mathbf{v}) + \frac{\partial\rho}{\partial t} = 0 \tag{3.18}$$

which is the sought for equation of continuity.

Now note that because [2]

$$\nabla\cdot(\rho\mathbf{v}) = \rho\nabla\cdot\mathbf{v} + \mathbf{v}\cdot\nabla\rho \tag{3.20}$$

the equation of continuity can be written as

$$\rho\nabla\cdot\mathbf{v} + \mathbf{v}\cdot\nabla\rho + \frac{\partial\rho}{\partial t} = 0 \tag{3.21}$$

i.e. as

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{v} = 0 \tag{3.22}$$

[2] This fact is a vector identity: i.e. for any vector $\mathbf{A}$ and scalar function f one can check that

$$\nabla\cdot(f\mathbf{A}) = f\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla f \tag{3.19}$$

This means that if we specialise to the case where the fluid density is constant that is the fluid is incompressible then the equation of continuity collapses to just

$$\nabla \cdot \mathbf{v} = 0 \tag{3.23}$$

which is a very useful property to be borne in mind for an incompressible fluid. It should not be forgotten, though, that

$$\nabla \cdot \mathbf{v} = 0 \not\Rightarrow \text{incompressibility} \tag{3.24}$$

rather it only implies that

$$\frac{D\rho}{Dt} = 0 \tag{3.25}$$

which definitely does not require the density to be constant [3].

§ 4. Euler's equation of motion for non-viscous fluids

We are now ready to derive the equation of motion for an inviscid, or non-viscous, fluid. Since we are neglecting the frictional forces due to viscosity then there are only two types of force on the fluid particles and these are

(i) Forces due to pressure differences

(ii) External forces; e.g. gravity or perhaps a pair of magnetic field and electric fields in the case of a charged fluid such as a plasma—a typical plasma is the hot gas found on the Sun or other star.

Let us now consider how these forces act on an infinitesimal cube within the fluid such as that depicted in Fig. 2.

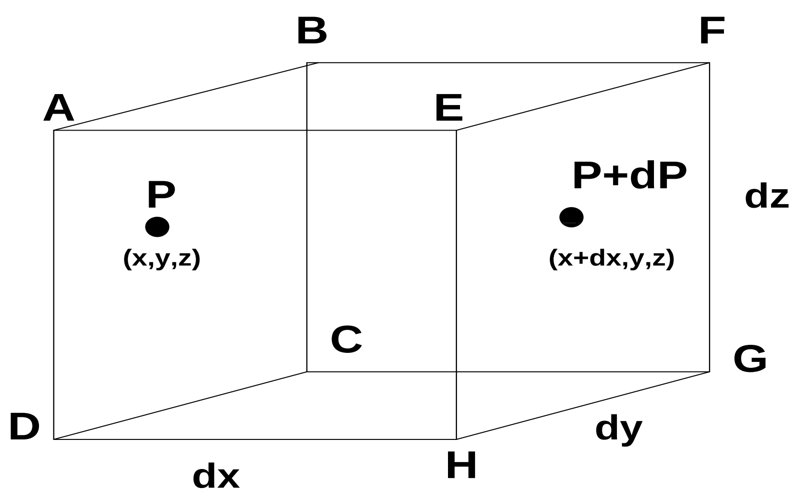


Fig. 2: An infinitesimal cube of fluid of volume $dxdydz$.

[3] A word of caution on terminology: if a flow is such that $\nabla \cdot \mathbf{v} = 0$ then it is sometimes referred to as an incompressible *flow*; by what we have just said this clearly does not necessarily mean that the fluid itself is incompressible.

When talking about external forces on a fluid we shall always work with the external force on a *unit mass* of the fluid; however we shall still denote this "force on a unit mass" by **F** even though it is not a force but an acceleration [4]. Another piece of terminology is that the term *body force* is also sometimes used to denote an external force.

In any case if **F** is the external, or body, force on the infinitesimal cube in Fig. 2 then , since the cube has volume $dxdydz$, the the force exerted by **F** on the cube in the x direction is just

$$\mathbf{F}\cdot\mathbf{i}\,\rho dxdydz \tag{4.1}$$

where ρ is the density of the fluid.

Next we come to the forces on the cube due to pressure differences. As with the external force **F** we just consider the x direction.

Now pressure differences will only produce a force in the x direction if the pressure *varies* in the x direction. It should be clear from Fig. 2 that the forces in the x direction due to pressure differences are given by subtracting the quantity $pressure \times area$ for the two faces $EFGH$ and $ABCD$. So if, as is depicted in Fig 2, **P** denotes the point (x, y, z) on the face $ABCD$ and $\mathbf{P}+\mathbf{dP}$ denotes the point $(x+dx, y, z)$ on the face $EFGH$, the force in the x direction due to pressure differences is

$$p(x,y,z)dydz - p(x+dx,y,z)dydz = \{p(x,y,z) - p(x+dx,y,z)\}\,dydz \tag{4.2}$$

Now by Newton's laws these two contributions 4.1 and 4.2 add up to give the mass times the x component of the acceleration; i.e. we have

$$\rho dxdydz\frac{Dv_x}{Dt} = \mathbf{F}\cdot\mathbf{i}\,\rho dxdydz + \{p(x,y,z) - p(x+dx,y,z)\}\,dydz \tag{4.3}$$

and on dividing by $\rho dxdydz$ we obtain

$$\frac{Dv_x}{Dt} = \mathbf{F}\cdot\mathbf{i} + \frac{1}{\rho}\frac{1}{dx}\{p(x,y,z) - p(x+dx,y,z)\} \tag{4.4}$$

Now Taylor's theorem applied to $p(x+dx, dy, dz)$ gives

$$p(x+dx,dy,dz) = p(x,y,z) + \frac{\partial p}{\partial x}dx + \text{negligible} \tag{4.5}$$

[4] This confusing piece of notation has become a widespread convention in the fluid mechanics literature and so we reluctantly follow it too.

and substituting this into our equation for Dv_x/Dt gives the result

$$\begin{aligned} \frac{Dv_x}{Dt} &= \mathbf{F}\cdot\mathbf{i} + \frac{1}{\rho}\frac{1}{dx}\left\{p(x,y,z) - p(x,y,z) - \frac{\partial p}{\partial x}dx\right\} \\ \Rightarrow \frac{Dv_x}{Dt} &= \mathbf{F}\cdot\mathbf{i} - \frac{1}{\rho}\frac{\partial p}{\partial x} \end{aligned} \tag{4.6}$$

and this latter equation is the equation of motion for the x direction. Exactly similarly the equations of motion in the y and z directions are, respectively, the pair

$$\begin{aligned} \frac{Dv_y}{Dt} &= \mathbf{F}\cdot\mathbf{j} - \frac{1}{\rho}\frac{\partial p}{\partial y} \\ \frac{Dv_z}{Dt} &= \mathbf{F}\cdot\mathbf{k} - \frac{1}{\rho}\frac{\partial p}{\partial z} \end{aligned} \tag{4.7}$$

We now combine these three scalar equations of motion into a single vector equation of motion. To do this we simply note that we have if we decompose the vectors $\mathbf{v}$, $\mathbf{F}$ and ∇p into their components we have

$$\begin{aligned} \mathbf{v} &= v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k} \\ \mathbf{F} &= F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} \\ \nabla p &= \frac{\partial p}{\partial x}\mathbf{i} + \frac{\partial p}{\partial y}\mathbf{j} + \frac{\partial p}{\partial z}\mathbf{k} \end{aligned} \tag{4.8}$$

This allows us to rewrite the x equation of motion as

$$\begin{aligned} \frac{D(\mathbf{v}\cdot\mathbf{i})}{Dt} &= \mathbf{F}\cdot\mathbf{i} - \frac{1}{\rho}(\nabla\mathbf{p}\cdot\mathbf{i}) \\ \text{i.e.} \quad \left(\frac{D\mathbf{v}}{Dt}\right)\cdot\mathbf{i} &= \left(\mathbf{F} - \frac{1}{\rho}\nabla p\right)\cdot\mathbf{i} \end{aligned} \tag{4.9}$$

It is now clear that the vector form of the equation that we are after is therefore simply

$$\frac{D\mathbf{v}}{Dt} = \mathbf{F} - \frac{1}{\rho}\nabla p \tag{4.10}$$

and this equation is called *Euler's equation of motion for an inviscid fluid* (subject to a body force $\mathbf{F}$).

§ 5. Bernoulli's equation

We now come to an important special case of Euler's equation known as Bernoulli's equation.

To obtain Bernoulli's equation we simply start with Euler's equation (so we still have a non-viscous fluid) and assume

(i) The flow is irrotational

(ii) Any body force $\mathbf{F}$ present is *conservative*; this, by definition, means that $\mathbf{F} = -\nabla K$ for some function K.

Using (i) we have

$$\nabla \times \mathbf{v} = \mathbf{0} \Rightarrow \mathbf{v} = -\nabla\phi, \quad \text{for some function } \phi \tag{5.1}$$

This function ϕ is then called the *velocity potential.* Now Euler's equation says that $\mathbf{v}$ obeys

$$\frac{D\mathbf{v}}{Dt} = \mathbf{F} - \frac{1}{\rho}\nabla p \tag{5.2}$$

or, using 3.12 for D/Dt and $\mathbf{F} = -\nabla K$ from (ii), $\mathbf{v}$ obeys

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla K - \frac{1}{\rho}\nabla p \tag{5.3}$$

Next we have the fact (which we quote without proof) that $(\mathbf{v}\cdot\nabla)\mathbf{v}$ satisfies the important vector identity

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = \nabla\left(\frac{\mathbf{v}^2}{2}\right) + (\nabla\times\mathbf{v})\times\mathbf{v} \tag{5.4}$$

Using this identity Euler's equation now takes the form

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\frac{\mathbf{v}^2}{2}\right) + (\nabla\times\mathbf{v})\times\mathbf{v} = -\nabla K - \frac{1}{\rho}\nabla p \tag{5.5}$$

But since, by assumption (i), $\nabla\times\mathbf{v} = \mathbf{0}$ we immediately have

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\frac{\mathbf{v}^2}{2}\right) = -\nabla K - \frac{1}{\rho}\nabla p \tag{5.6}$$

Also if we use the fact that $\mathbf{v} = -\nabla\phi$ in the term $\partial\mathbf{v}/\partial t$ we find that

$$-\frac{\partial}{\partial t}(\nabla\phi) + \nabla\left(\frac{\mathbf{v}^2}{2}\right) = -\nabla K - \frac{1}{\rho}\nabla p \tag{5.7}$$

Interchanging ∇ with $\partial/\partial t$ and rearranging terms gives us

$$\nabla\left(-\frac{\partial\phi}{\partial t}\right) + \nabla\left(\frac{\mathbf{v}^2}{2}\right) + \nabla K + \frac{1}{\rho}\nabla p = 0 \tag{5.8}$$

In a moment we shall integrate this equation along a line but first we need to recall simple piece of calculus which is this: If $f(x, y, z)$ is any function then we know that

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \tag{5.9}$$

Now if we choose an infinitesimal element of length **dl** given by

$$\mathbf{dl} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} \tag{5.10}$$

then taking the dot product of **dl** with the vector ∇f gives precisely df, i.e. we have

$$\begin{aligned} \nabla f \cdot \mathbf{dl} &= (\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}) \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}) \\ &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \\ \text{so} \qquad \nabla f \cdot \mathbf{dl} &= df, \qquad \text{as claimed} \end{aligned} \tag{5.11}$$

Returning to 5.8 we take the dot product of both sides with **dl** and use 5.11 on each of the four terms so that we obtain

$$\begin{aligned} \nabla\left(-\frac{\partial\phi}{\partial t}\right) \cdot \mathbf{dl} + \nabla\left(\frac{\mathbf{v}^2}{2}\right) \cdot \mathbf{dl} + \nabla K \cdot \mathbf{dl} + \frac{1}{\rho}\nabla p \cdot \mathbf{dl} = 0 \\ \Rightarrow d\left(-\frac{\partial\phi}{\partial t}\right) + d\left(\frac{\mathbf{v}^2}{2}\right) + dK + \frac{dp}{\rho} = 0 \end{aligned} \tag{5.12}$$

Finally we integrate both sides of this equation along the line of which **dl** is the line element, and use that fact that $\int df = f$, so that we produce the equation

$$-\frac{\partial\phi}{\partial t} + \frac{\mathbf{v}^2}{2} + K + \int\frac{dp}{\rho} = C(t) \tag{5.13}$$

where $C(t)$ is a constant of integration; also we note that $C(t)$ is allowed to depend on t since t was *not* an integration variable. This last equation, that is to say 5.13, is *Bernoulli's equation.*

Bernoulli's equation for incompressible fluids

If the fluid is *incompressible* so that ρ is constant then we have

$$\int\frac{dp}{\rho} = \frac{1}{\rho}\int dp = \frac{p}{\rho} \tag{5.14}$$

This means that Bernoulli's equation simplifies to

$$-\frac{\partial\phi}{\partial t}+\frac{\mathbf{v}^2}{2}+K+\frac{p}{\rho}=C(t) \tag{5.15}$$

If we further assume that there is no body force and that the flow is steady then $K=0$ and $\partial\phi/\partial t=0$; it will also be the case that the constant $C(t)$ will be t-independent so we have an extremely simple form of Bernoulli's equation namely

$$\frac{\mathbf{v}^2}{2}+\frac{p}{\rho}=C \tag{5.16}$$

(incompressible fluid, steady flow)

This equation 5.16 is simple but very instructive: To see why first note that that since all quantities on the LHS are positive then the LHS itself is positive. This, in turn, means that the constant C is positive. Now we see that neither $\mathbf{v}^2/2$ nor p/ρ can be arbitrarily large since their sum has to be C: in fact neither term can exceed the value C.

The most useful way of looking at this is to note that if $|\mathbf{v}|$ *increases* then p must *decrease* and vice-versa [5]. Reasoning further we can say that, in a fluid, a region of *high pressure* is a region of *low velocity* and vice-versa. Those readers who pay attention to television weather maps may well have noticed this as being property of high and low pressure regions in the Earth's atmosphere

This relationship between p and $\mathbf{v}$ is also the explanation of the curving of the trajectory of spinning golf, tennis or ping-pong balls—the so called Magnus effect—as well as the mechanism of the lift under an airplane's wing; or indeed the method of propulsion of the famous Atlantic crossing boats of the 1920's designed by the German engineer Flettner, cf. Fig. 3.

[5] Remember ρ does not change since, by assumption it, is constant

Taken from Strange Stories, Amazing Facts 1976 published by Reader's digest.

SHIPS WITH ROTATING SAILS

In the 1920's the Germans planned to return to wind propulsion

In 1925 the nautical world was startled by the appearance of ships apparently propelled by gigantic cotton bobbins. Their German inventors believed that these vessels were a major breakthrough in sea transport.

Experiments at the University of Göttingen in 1922 gave Anton Flettner, a German engineer, the idea for the revolutionary rotor ship. He had found that wind pressure on a revolving disk was very much greater than pressure on a stationary disk. Rotor ships operated on this principle of propulsion.

The ships had two rotors, each turned by a small engine at its base. The pressure of the wind on the upended cylinders could be increased by revolving them, and the pressure point could be altered also. A beam wind could be used to push on the rear surface of the cylinders, thus driving the ship forward instead of blowing it sideways.

Long before anyone thought of applying the principle to ships, players of ball games had noticed the phenomenon and had even turned it to their advantage. In baseball a pitcher spins the ball so that the air pressure upon the spinning surface curves the ball's path and deceives the batter. Tennis players and others do the same.

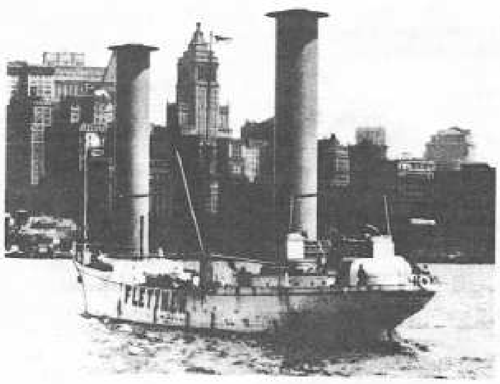
The Flettner rotor ship was said to be faster than sailing ships, cheap to run, and efficient to crew. It could fight its way through storms, using only the power of tiny gas engines, no bigger than those used in small cars. An ordinary sailing vessel is required to take down all canvas in a hurricane, while the rotor ship could continue sailing.

On transatlantic voyages rotor ships reached speeds of 17 knots. They were small vessels—about 600 tons—with their two rotors looking like enormous funnels, 65 feet high and 10 feet in diameter.

Cheap to run

Rotor-ship enthusiasts predicted that all the world's shipping would soon be using rotor power. The advantages the rotor ship had over conventional sailing ships were simplicity and economy in handling. The Flettner ship was 80 percent cheaper to run than sailing ships. Whereas several dozen men would be required to handle the sails, a single man could control the speed of the rotors. The rotor ship could also be turned or reversed simply and quickly.

But less than 20 years later the revolutionary new ships were broken up for scrap. The ships had all developed serious mechanical problems because of the rotor's incessant vibration. And they had proved unreliable because of their dependence on wind power.



ROTOR IN MANHATTAN. *Anton Flettner's rotor ship, the Baden-Baden, weathered several heavy gales on its transatlantic voyage from Hamburg to New York in 1926. Propelled by two rotating yellow towers, the ship caused a sensation in America.*

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Fig. 3: The boats designed by Anton Flettner

Example *Water draining out of a circular tank*

We can use Bernoulli's equation to derive the shape of the trumpet shaped surface created by the draining of water out of the bottom a circular tank—cf. Fig. 4.

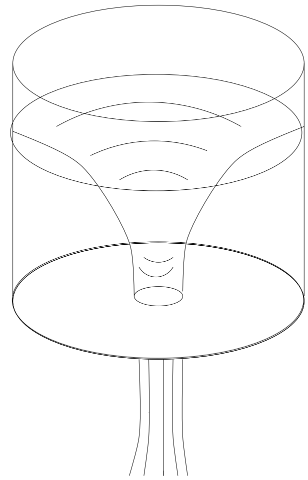

Fig. 4: Fluid flowing out of a circular tank

We can assume that the flow is steady and so Bernoulli's equation takes the form

$$K + \frac{\mathbf{v}^2}{2} + \frac{p}{\rho} = C \tag{5.17}$$

but since the body force is just the downward pull of gravity then

$$\mathbf{F} = -\nabla K, \quad \text{with } K = gz \tag{5.18}$$

so our equation is now

$$gz + \frac{\mathbf{v}^2}{2} + \frac{p}{\rho} = C \tag{5.19}$$

The key idea is now to evaluate this equation on the trumpet shaped surface of the water since, on this surface, since it is the interface between the water and the air the pressure of the water and the air are both equal and hence

$$p = p_A \tag{5.20}$$

where p_A = atmospheric pressure. So now we have *on the trumpet shaped surface*

$$gz + \frac{\mathbf{v}^2}{2} = D, \quad \text{where } D = C - \frac{p_A}{\rho} = \text{a constant} \tag{5.21}$$

next if $r = \sqrt{x^2 + y^2}$ is the distance from a point on the surface to the z-axis, and we simply quote the fact

$$|\mathbf{v}| = \frac{E}{r}, \quad E \text{ a constant} \tag{5.22}$$

then we find that

$$\begin{aligned} gz + \frac{E^2}{2r^2} &= D \\ \Rightarrow z &= \frac{1}{g}\left(D - \frac{E^2}{2r^2}\right) \end{aligned} \tag{5.23}$$

or equivalently

$$r = \frac{E}{\sqrt{2}\sqrt{D - gz}} \tag{5.24}$$

which the reader can easily verify by plotting is the graph of a trumpet shaped surface such as that depicted in Fig. 4.

For example if we choose

$$E = \sqrt{2}, \quad D = 50, \quad g = 9 \tag{5.25}$$

then those readers who use the mathematics computer utility Maple can type in the following lines which create the postscript file *trumpet.eps* which is shown below in Fig. 5—it may help to remember that the r and z of the formulae are the r and z of the r, θ, z *cylindrical coordinates*; the fact that θ is absent means that the equation of the surface is independent of θ i.e. it can be obtained by rotating a curve about an axis (the z axis in this case).

$trumpet := r-> 1/sqrt((50 - 9 * z));$

$plotsetup(ps, plotoutput = `trumpet.eps`, plotoptions = `portrait, noborder`);$

$plot3d(trumpet(r), theta = 0..2 * Pi, z = 0..5, coords = cylindrical);$

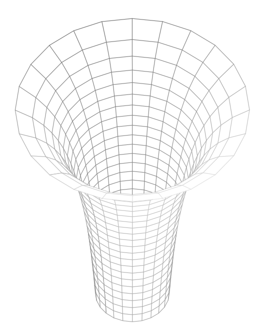

Fig. 5: The trumpet surface for the draining tank

§ 6. Adiabatic flow and the Mach number

In gases one sometimes has to consider what is called *adiabatic flow*: Mathematically speaking[6] this is simply a flow where pressure and density are related by the equation

$$p = k\rho^{\gamma}, \quad \text{with } k \text{ and } \gamma \text{ constant, } \gamma > 1 \tag{6.5}$$

We want to study the effect of this adiabatic relationship between p and ρ using Bernoulli's equation. Hence we start with

$$-\frac{\partial \phi}{\partial t} + \frac{\mathbf{v}^2}{2} + K + \int \frac{dp}{\rho} = C(t) \tag{6.6}$$

but we shall study steady flow so that $\partial\phi/\partial t = 0$ and the constant $C(t)$ is time independent and equal to C, say; we shall also have no body force so that $K = 0$. This leaves us with

$$\frac{\mathbf{v}^2}{2} + \int \frac{dp}{\rho} = C \tag{6.7}$$

But since

$$p = k\rho^{\gamma} \tag{6.8}$$

[6] In physics an *adiabatic change* of a system is one which takes place without exchange of energy between the system and its surroundings. The example here comes from the kinetic theory of gases where one uses perfect gas law

$$\frac{PV}{T} = C, \quad C \text{ a constant} \tag{6.1}$$

to obtain, at constant temperature, the special case

$$PV = \text{const.} \tag{6.2}$$

But if the sample of gas has mass M and density ρ then we find that

$$P = D\rho \quad D \text{ a cosntant} \tag{6.3}$$

All this is for changes which keep the gas in thermodynamic equilibrium and necessitate exchange of energy with its surroundings. For changes that are adiabatic one finds that P and ρ are related by the equation

$$P = k\rho^{\gamma} \tag{6.4}$$

which is what we have here.

then

$$\begin{aligned} dp &= \gamma k \rho^{\gamma-1} d\rho \\ \Rightarrow \int \frac{dp}{\rho} &= \gamma k \int \rho^{\gamma-2} d\rho \\ &= \frac{\gamma}{(\gamma-1)} k \rho^{\gamma-1} \end{aligned} \tag{6.9}$$

Now *sound* is a longitudinal pressure wave travelling through a fluid and, if the *speed of sound* in a fluid is denoted by c then c is given by

$$\begin{aligned} c^2 &= \frac{dp}{d\rho} \\ &= \gamma k \rho^{\gamma-1} \\ &= \gamma \frac{k\rho^{\gamma}}{\rho} \\ \Rightarrow c^2 &= \frac{\gamma p}{\rho} \end{aligned} \tag{6.10}$$

So now Bernoulli's equation gives us

$$\frac{\mathbf{v}^2}{2} + \frac{\gamma}{(\gamma-1)} k \rho^{\gamma-1} = C \tag{6.11}$$

which, if we use the speed of sound c, we can write as

$$\frac{\mathbf{v}^2}{2} + \frac{c^2}{(\gamma-1)} = C \tag{6.12}$$

Now define for convenience the constant c_0 by writing

$$C = \frac{c_0^2}{\gamma - 1} \tag{6.13}$$

then Bernoulli's equation, written as an equation for $\mathbf{v}$, becomes

$$\mathbf{v}^2 = \frac{2}{(\gamma-1)} \left(c_0^2 - c^2\right) \tag{6.14}$$

This equation allows us to find an expression for the *Mach number* M of the flow. M is defined as the ratio of the speed of the flow to the speed of sound, i.e.

$$M = \frac{|\mathbf{v}|}{c} \tag{6.15}$$

Dividing 6.14 by c^2 we get

$$\begin{aligned} M^2 &= \frac{2}{(\gamma-1)}\left(\frac{c_0^2}{c^2}-1\right) \\ \Rightarrow M &= \sqrt{\frac{2}{(\gamma-1)}}\sqrt{\left(\frac{c_0^2}{c^2}-1\right)} \end{aligned} \tag{6.16}$$

so that we now have an equation for the Mach number; incidentally, for air, the constant γ has the value 1.4 approximately.

Finally we have the well known division of the flow into three types: If

$$\begin{cases} |\mathbf{v}| < c & \text{the flow is called } \textit{subsonic} \\ |\mathbf{v}| = c & \text{the flow is called } \textit{sonic} \\ |\mathbf{v}| > c & \text{the flow is called } \textit{supersonic} \end{cases} \tag{6.17}$$

In addition, as long as $|\mathbf{v}| > c$, there is a shock wave produced in the fluid; this is what is usually referred to as a *sonic boom*.

§ 7. Circulation and Kelvin's theorem

For an inviscid fluid there are no 'frictional forces' present to slow down a rotating mass of fluid. This means that rotation, once present will persist forever. Hence, in this case, angular momentum is conserved.

This fact is usually established indirectly by proving what is called Kelvin's theorem; this theorem states, roughly, that the angular momentum per unit mass is conserved. More precisely Kelvin's theorem states that what is called the circulation Γ round any curve C is constant for an inviscid, barotropic fluid subject to a conservative body force $\mathbf{F}$.

We must first define the circulation and its definition goes as follows: Select any closed curve C and integrate the velocity $\mathbf{v}$ around it. This quantity is the circulation round C and is denoted by Γ. In other words we have

$$\Gamma = \int_C \mathbf{v}\cdot\mathbf{dl} \tag{7.1}$$

If S denotes the surface which is surrounded by the curve C, and we use Stokes' theorem, then we have

$$\begin{aligned} \int_C \mathbf{v}\cdot\mathbf{dl} &= \int_S \nabla\times\mathbf{v}\cdot\mathbf{dS} \\ &= \int_S \boldsymbol{\omega}\cdot\mathbf{dS}, \quad \text{where } \boldsymbol{\omega} \text{ is the vorticity} \end{aligned} \tag{7.2}$$

$$\text{Hence} \qquad \Gamma = \int_S \boldsymbol{\omega}\cdot\mathbf{dS}$$

and the relation of Γ to angular momentum is now obvious. The reader can quickly check that Γ has the dimensions of angular momentum per unit mass.

Now we shall formally state and prove Kelvin's theorem.

Theorem (Kelvin's theorem) *The circulation*

$$\Gamma = \int_C \mathbf{v} \cdot \mathbf{dl} \tag{7.3}$$

is a constant of the flow, i.e.

$$\frac{D\Gamma}{Dt} = 0 \tag{7.4}$$

provided the fluid is inviscid, barotropic and the external force is conservative

Proof: First we explain the term *barotropic*: a barotropic fluid is one where the pressure is a function of (only) the density, i.e.

$$p = f(\rho), \quad \text{for some function } f \tag{7.5}$$

The relevance of barotropicity to the proof will emerge shortly.

We now make a start by writing

$$\begin{aligned} \frac{D\Gamma}{Dt} &= \frac{D}{Dt}\int_C \mathbf{v} \cdot \mathbf{dl} \\ &= \int_C \frac{D(\mathbf{v} \cdot \mathbf{dl})}{Dt} \\ &= \int_C \left\{ \frac{D\mathbf{v}}{Dt} \cdot \mathbf{dl} + \mathbf{v} \cdot \frac{D(\mathbf{dl})}{Dt} \right\} \end{aligned} \tag{7.6}$$

Note that the reason that the term

$$\frac{D(\mathbf{dl})}{Dt} \tag{7.7}$$

is not zero and has to be included is that $\mathbf{dl}$ is an element of length of the curve C which is a curve moving with the fluid and hence $\mathbf{dl}$ depends on time; all this matters of course because D/Dt is the stream derivative and not the ordinary partial derivative $\partial/\partial t$. But

$$\frac{D(\mathbf{dl})}{Dt} = d\left\{ \frac{D\mathbf{l}}{Dt} \right\} \tag{7.8}$$

and

$$\frac{D\mathbf{l}}{Dt} = \mathbf{v} \tag{7.9}$$

since a piece $\mathbf{l}$ of the curve moves with the fluid velocity $\mathbf{v}$. Substituting from 7.8 and 7.9 in 7.6 gives

$$\frac{D\Gamma}{Dt} = \int_C \left\{ \frac{D\mathbf{v}}{Dt} \cdot \mathbf{dl} + \mathbf{v} \cdot \mathbf{dv} \right\} \tag{7.10}$$

Now for an inviscid fluid Euler's equation 4.10 says that

$$\frac{D\mathbf{v}}{Dt} = \mathbf{F} - \frac{1}{\rho}\nabla p \tag{7.11}$$

hence we have

$$\frac{D\Gamma}{Dt} = \int_C \left\{ \mathbf{F} \cdot \mathbf{dl} - \frac{1}{\rho}\nabla p \cdot \mathbf{dl} + \mathbf{v} \cdot \mathbf{dv} \right\} \tag{7.12}$$

But the external force $\mathbf{F}$ is conservative so

$$\mathbf{F} = -\nabla K \tag{7.13}$$

and

$$d\left(\frac{\mathbf{v}^2}{2}\right) = \mathbf{v} \cdot \mathbf{dv} \tag{7.14}$$

and we obtain

$$\frac{D\Gamma}{Dt} = \int_C \left\{ -dK - \frac{1}{\rho} dp + d\left(\frac{\mathbf{v}^2}{2}\right) \right\} \tag{7.15}$$

where we have used the fact that $\nabla f \cdot \mathbf{dl} = df$ (cf. 5.11) on K and p.

Now it is time to use the barotropicity of the fluid. To this end we suppose that, for some f,

$$p = f(\rho) \tag{7.16}$$

Then if we define $F(\rho)$ by

$$F(\rho) = \int \frac{dp}{\rho} \tag{7.17}$$

one can see immediately that

$$dF = \frac{dp}{\rho} \equiv \int \frac{1}{\rho}\frac{dp}{d\rho} d\rho \tag{7.18}$$

allowing us to express $D\Gamma/Dt$ as

$$\frac{D\Gamma}{Dt} = \int_C \left\{ -dK - dF + d\left(\frac{\mathbf{v}^2}{2}\right) \right\} \tag{7.19}$$

Finally we note that for *any function* g, say, single valuedness forces dg to have zero integral round any *closed* curve [7]; hence

$$\int_C dg = 0, \quad \text{if } C \text{ is a closed curve} \tag{7.21}$$

So we have

$$\int_C dK = \int_C dF = \int_C d\left(\frac{\mathbf{v}^2}{2}\right) = 0 \tag{7.22}$$

and therefore we do have

$$\frac{D\Gamma}{Dt} = 0 \tag{7.23}$$

and the theorem is proved.

§ 8. Two dimensional flow and complex variable methods

In two dimensional flows the velocity $\mathbf{v}$ only has two components

$$\mathbf{v} = v_x\,\mathbf{i} + v_y\,\mathbf{j} \tag{8.1}$$

and these are functions of the two variables x and y. It turns out that, for some flows, much insight and calculational improvements are obtained by changing variables from x and y to the complex variable

$$z = x + iy \tag{8.2}$$

We now give a short summary of what are called the Cauchy–Riemann equations; these are central to complex variable theory and we need them

[7] It may help to recall that

$$\int_a^b df = \int_a^b \frac{df(x)}{dx} dx = f(b) - f(a) \tag{7.20}$$

and if the line segment $[a, b]$ is bent round to form a curved path with endpoints a and b this result still holds. Hence if the endpoints of the path are joined rendering the path closed then $a = b$ and so the integral is zero.

for our fluid mechanical discussion. The reader who is familiar with them can skip on to the next section.

The Cauchy–Riemann equations

The point of the Cauchy–Riemann equations for a function $f(x, y)$ is that if f is expressed in terms of z and $\bar{z}$ instead of x and y giving $f = f(z, \bar{z})$, then the Cauchy–Riemann equations express the fact that $f(z, \bar{z})$ is independent of $\bar{z}$ so that we can write just $f = f(z)$.

We can easily derive the Cauchy–Riemann equations for f as follows. We write

$$f(x, y) \equiv f(z, \bar{z}) = u(x, y) + iv(x, y) \tag{8.3}$$

where u and v are real-valued functions. The condition we want to impose is [8]

$$\frac{\partial f}{\partial \bar{z}} = 0 \tag{8.4}$$

Expanding this condition more fully gives

$$\begin{aligned} \frac{\partial f}{\partial \bar{z}} &= 0 \\ \Rightarrow \frac{\partial u}{\partial \bar{z}} + i\frac{\partial v}{\partial \bar{z}} &= 0 \\ \Rightarrow \frac{\partial u}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \bar{z}} + i\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \bar{z}}\right) &= 0 \end{aligned} \tag{8.5}$$

But

$$\begin{aligned} z = x + iy, \quad \bar{z} = x - iy, \Rightarrow x &= \frac{1}{2}(z + \bar{z}), \quad y = \frac{1}{2i}(z - \bar{z}) \\ \Rightarrow \frac{\partial x}{\partial \bar{z}} &= \frac{1}{2}, \quad \frac{\partial y}{\partial \bar{z}} = -\frac{1}{2i} = \frac{i}{2} \end{aligned} \tag{8.6}$$

This means that the equation $\partial f/\partial \bar{z} = 0$ now becomes

$$\frac{1}{2}\left\{\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right\} + \frac{i}{2}\left\{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right\} = 0 \tag{8.7}$$

[8] This is only a *formal* calculation which serves to obtain the Cauchy–Riemann equations quickly and to expose the ideas that underly them. In fact $\bar{z}$ cannot be regarded as independent of z because knowing $\bar{z}$ one can also write down z. A rigorous way of obtaining the Cauchy–Riemann equations is to define df/dz appropriately and exploit the fact that one can compute the derivative by taking the limit in any direction in the complex z plane: then one chooses different directions and equates the two derivatives, the resulting equations are the Cauchy–Riemann equations; the two chosen directions being parallel to the x and y-axes respectively.

and, equating real and imaginary parts to zero on both sides, we obtain

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{8.8}$$

which are indeed the Cauchy–Riemann equations for u and v.

§ 9. The complex potential W for a two dimensional flow

Everything centres round a single function $W(z)$ known as the *complex potential* and we now embark on its study.

Let an incompressible fluid (flowing in two dimensions) have a velocity potential $\phi(x, y)$ so that

$$\mathbf{v} = -\nabla\phi = -\frac{\partial \phi}{\partial x}\,\mathbf{i} - \frac{\partial \phi}{\partial y}\,\mathbf{j} = v_x\,\mathbf{i} + v_y\,\mathbf{j} \tag{9.1}$$

Then the complex potential $W(z)$ of this flow is the analytic function of z given by

$$W(z) = \phi(x, y) + i\psi(x, y) \tag{9.2}$$

i.e.

$$Re(W(z)) = \phi(x, y), \quad Im(W(z)) = \psi(x, y) \tag{9.3}$$

The function ψ is called the *stream function* and it is obtained mathematically from ϕ by solving the celebrated Cauchy–Riemann equations for W which are of course

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{9.4}$$

One can also calculate the z-derivative of $W(z)$ by differentiating W along any direction: for example the x-direction. This means that we can write

$$\frac{dW}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} \tag{9.5}$$

which can be useful. This last equation means that $|dW/dz|$ has a simple physical meaning: it turns out to be the *speed* $|\mathbf{v}|$ of the flow. Checking this we have

$$\begin{aligned} \left|\frac{dW}{dz}\right| &= \sqrt{\left\{\frac{\partial \phi}{\partial x}\right\}^2 + \left\{\frac{\partial \psi}{\partial x}\right\}^2} \\ &= \sqrt{(-v_x)^2 + (v_y)^2} \\ &= \sqrt{(v_x)^2 + (v_y)^2} \\ &= |\mathbf{v}| \end{aligned} \tag{9.6}$$

Summarising the properties of the complex potential W, so as to be able to see them all at once, we have

$$\begin{aligned}
W &= \phi + i\psi \\
\frac{\partial \phi}{\partial x} &= \frac{\partial \psi}{\partial y}, \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \\
v_x &= -\frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y} \\
v_y &= -\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x} \\
\frac{dW}{dz} &= \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = -v_x + iv_y \\
\left|\frac{dW}{dz}\right| &= |\mathbf{v}|
\end{aligned} \tag{9.7}$$

One of the most useful properties of W is that the function ψ, which we recall is called the *stream function*, gives the actual lines of flow of the fluid—the so called *streamlines* of the flow. All we have to do is to set ψ equal to a constant; i.e. we simply choose a constant C and write

$$\psi = C \tag{9.8}$$

The point is that ψ is a function of x and y and so equating ψ to a constant fixes x in terms of y and hence defines some curve in the plane. We state without proof the fact that all curves of flow of the fluid—i.e. all *streamlines*—arise in this way by choosing a suitable value for C; conversely all curves of the form $\psi = C$ correspond to streamlines.

All this becomes much easier to understand if we work with some examples and so now proceed to an example.

Example *Uniform flow at an angle to the x-axis*

Choose

$$W = -U\exp(-i\alpha)z, \quad U > 0 \text{ and } \alpha \text{ real constants} \tag{9.9}$$

Now we know that $z = x + iy$ and that $W = \phi + i\psi$ so we can say that

$$\begin{aligned}
W = -U\exp(-i\alpha)z &= \phi + i\psi \\
\Rightarrow -U(\cos(\alpha) - i\sin(\alpha))(x+iy) &= \phi + i\psi \\
\Rightarrow -U\cos(\alpha)x - U\sin(\alpha)y = \phi, &\quad U\sin(\alpha)x - U\cos(\alpha)y = \psi
\end{aligned} \tag{9.10}$$

So we already have isolated the velocity potential ϕ and the stream function ψ. Let us now find the streamlines. Setting ψ equal to a constant we get

$$\begin{aligned} U\sin(\alpha)x - U\cos(\alpha)y &= C \\ \Rightarrow y = \tan(\alpha)x &- \frac{C}{U\cos(\alpha)} \end{aligned} \tag{9.11}$$

But this is instantly recognisable as the equation of a straight line with slope $\tan(\alpha)$ and intercept $-C/U\cos(\alpha)$. Also by varying the constant C all we do is to change the intercept but *not* the slope. Hence the entire flow is a set of *parallel* lines which make an angle α with x-axis—cf. Fig. 6.

Finally we can calculate the speed of the flow by computing $|dW/dz| = |\mathbf{v}|$. To this end we have

$$\begin{aligned} W &= -U\exp(-i\alpha)z \\ \Rightarrow \frac{dW}{dz} &= -U\exp(-i\alpha) \\ \Rightarrow \left|\frac{dW}{dz}\right| &= U = \text{ a constant} \end{aligned} \tag{9.12}$$

Hence the speed $|\mathbf{v}|$ is just given by the constant U. This is why the flow is described as being *uniform*—the phrase *uniform flow* being just another term for a flow of constant speed and direction.

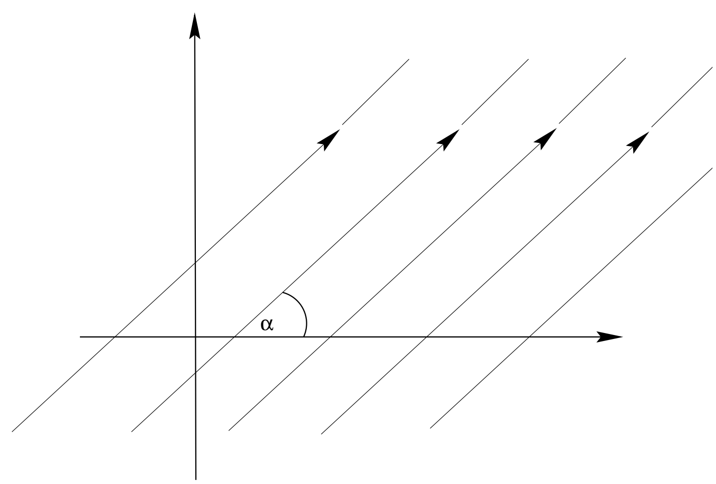

Fig. 6: Streamlines: Uniform parallel flow at an angle α to the x-axis.

Example *A source or sink*

Let W be the potential

$$W = -\frac{m}{2\pi}\ln z, \quad m \text{ a real constant} \tag{9.13}$$

The key trick here is to use the polar representation $z = r\exp(i\theta)$ instead of the Cartesian representation $z = x + iy$. So we start off and obtain

$$\begin{aligned} W &= -\frac{m}{2\pi}\ln z = \phi + i\psi \\ \Rightarrow -\frac{m}{2\pi}\ln(r\exp(i\theta)) &= \phi + i\psi, \quad \text{using } z = r\exp(i\theta) \\ \Rightarrow -\frac{m}{2\pi}(\ln(r) + i\theta) &= \phi + i\psi \\ \Rightarrow \phi = -\frac{m}{2\pi}\ln(r), \quad \psi &= -\frac{m}{2\pi}\theta \end{aligned} \tag{9.14}$$

and so we have very easily found ϕ and ψ. The streamlines $\psi = C$ are therefore given by

$$\begin{aligned} -\frac{m}{2\pi}\theta &= C \\ \Rightarrow \theta &= -\frac{2\pi C}{m} = \text{ a constant} \end{aligned} \tag{9.15}$$

In other words the streamlines are just lines of constant θ i.e. they are radial lines cf. Fig. 7; such a flow is, for obvious reasons, called *a source* (if the flow is outwards from the origin) or *a sink* (if the flow is inwards towards the origin).

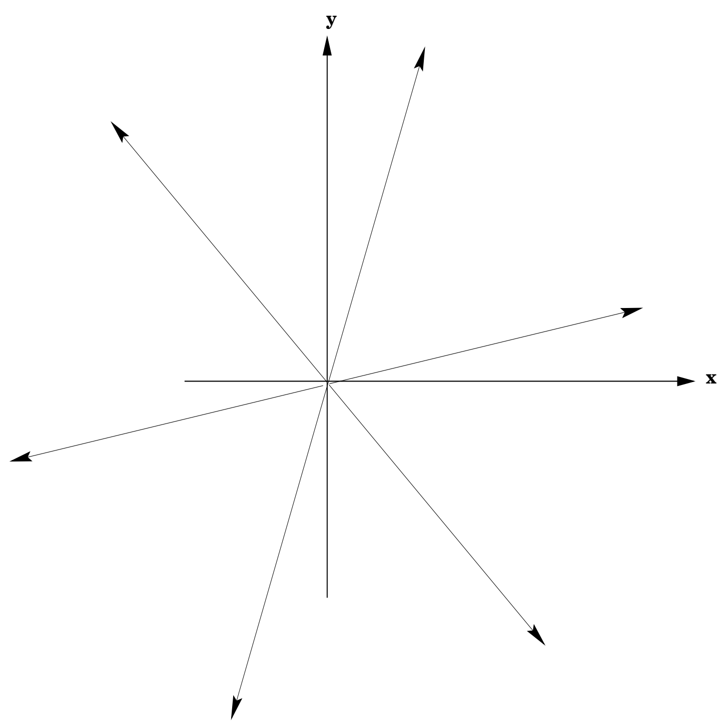

Fig. 7: Streamlines: A source

Example *A vortex*
Consider the potential

$$W = i\frac{\kappa}{2\pi}\ln z \tag{9.16}$$

We notice that $\ln z$ has appeared again and, as we did when working with the previous example, we should write z as $z = r\exp(i\theta)$. Steaming ahead we have

$$\begin{aligned}
W &= i\frac{\kappa}{2\pi}\ln z = \phi + i\psi \\
\Rightarrow i\frac{\kappa}{2\pi}\ln(r\exp(i\theta)) &= \phi + i\psi, \quad (z = r\exp(i\theta)) \\
\Rightarrow i\frac{\kappa}{2\pi}(\ln(r) + i\theta) &= \phi + i\psi \\
\Rightarrow \phi = -\frac{\kappa}{2\pi}\theta, &\quad \psi = \frac{\kappa}{2\pi}\ln(r)
\end{aligned} \tag{9.17}$$

Having found ϕ and ψ the streamlines are clearly given by

$$\begin{aligned}
\frac{\kappa}{2\pi}\ln(r) &= C \\
\Rightarrow \ln(r) &= \frac{2\pi C}{\kappa} = \text{ a constant} \\
\Rightarrow r &= \text{ a constant}
\end{aligned} \tag{9.18}$$

Thus the streamlines have constant r so they are circles centred at the origin i.e.. we have a vortex, cf. Fig. 8.

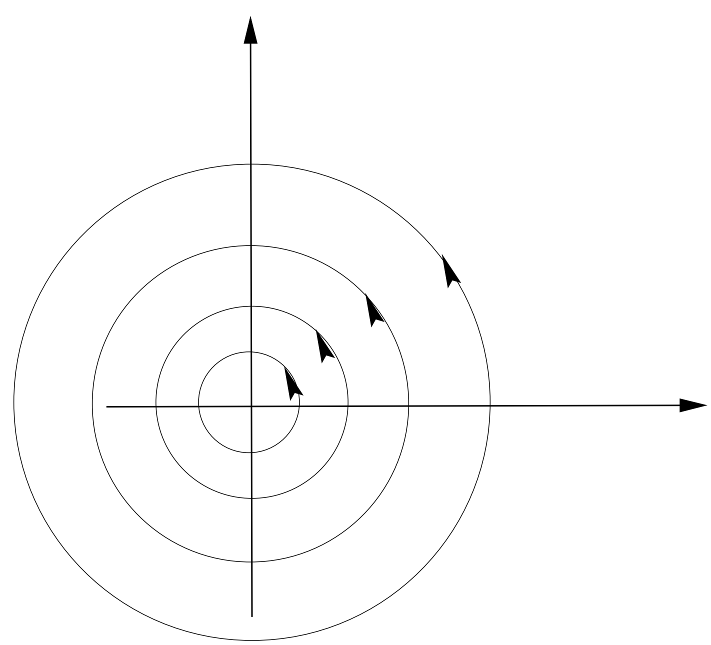

Fig. 8: Streamlines: A vortex

If κ is positive the rotation is anti-clockwise and if it is negative the rotation is clockwise.

Example *The circulation of the vortex*

Finally we shall calculate the circulation or, angular momentum per unit mass, of the vortex round a closed curve C.

We just choose C to be the circle of radius a centred at the origin so that it is given by

$$r = a \tag{9.19}$$

Then we want to calculate

$$\Gamma = \int_C \mathbf{v} \cdot \mathbf{dl} \tag{9.20}$$

Now $\mathbf{dl}$ is tangential to the circle C whose radius is a and so, if $\mathbf{e}_\theta$ is a unit tangent to the circle, then

$$\mathbf{dl} = r\, d\theta\, \mathbf{e}_\theta \tag{9.21}$$

Hence, if $v_\theta = \mathbf{v} \cdot \mathbf{e}_\theta$, we have

$$\mathbf{v} \cdot \mathbf{dl} = r v_\theta\, d\theta \tag{9.22}$$

But in polar coordinates we have

$$\begin{aligned} v_\theta &= -\frac{1}{r}\frac{\partial \phi}{\partial \theta} \\ \Rightarrow v_\theta &= -\frac{1}{r}\left(-\frac{\kappa}{2\pi}\right), \qquad \text{since } \phi = -\frac{\kappa}{2\pi}\theta \\ &= \frac{\kappa}{2\pi r} \end{aligned} \tag{9.23}$$

Putting this collection of modest facts together we find that

$$\begin{aligned} \Gamma &= \int_0^{2\pi} r v_\theta\, d\theta \\ &= \int_0^{2\pi} \frac{\kappa}{2\pi} = \left[\frac{\kappa}{2\pi}\theta\right]_0^{2\pi} \\ \Rightarrow \Gamma &= \kappa\, d\theta \end{aligned} \tag{9.24}$$

and so the circulation Γ round $r = a$ of the vortex $W = i\frac{\kappa}{2\pi}\ln z$ is very simple it is just given by the constant κ. Having spent some time discussing angular momentum of fluids we now move on to discuss another central quantity which is the energy of the flow.

§ 10. Kinetic energy

Let an incompressible fluid undergo potential flow so that its velocity $\mathbf{v}$ is given by

$$\mathbf{v} = -\nabla\phi$$

then we shall derive a simple formula for its kinetic energy.

Suppose that the fluid occupies a (finite or infinite) volume V then its kinetic energy E is given by

$$\begin{aligned} E &= \frac{1}{2}\int_V \rho\mathbf{v}^2 dV \\ &= \frac{1}{2}\int_V \rho\nabla\phi^2 dV \end{aligned} \tag{10.1}$$

Now consider the vector

$$\mathbf{A} = \phi\nabla\phi \tag{10.2}$$

Evidently

$$\begin{aligned} \nabla\cdot\mathbf{A} = \nabla\cdot(\phi\nabla\phi) &= (\nabla\phi)\cdot(\nabla\phi) + \phi\nabla^2\phi \\ \Rightarrow (\nabla\phi)^2 &= -\phi\nabla^2\phi + \nabla\cdot(\phi\nabla\phi) \\ \Rightarrow E &= \frac{\rho}{2}\int_V \left\{-\phi\nabla^2\phi + \nabla\cdot(\phi\nabla\phi)\right\} dV \end{aligned} \tag{10.3}$$

But the fluid is incompressible so

$$\nabla\cdot\mathbf{v} = 0 \Rightarrow \nabla^2\phi = 0 \tag{10.4}$$

Hence

$$\begin{aligned} E &= \frac{\rho}{2}\int_V \nabla\cdot(\phi\nabla\phi)\, dV \\ &= \frac{\rho}{2}\int_S \phi\nabla\phi\cdot\mathbf{dS}, \qquad \text{using Gauss' theorem} \end{aligned} \tag{10.5}$$

where the volume V has surface S. The last formula is the one we want so we quote it again by itself giving us

$$E = \frac{\rho}{2}\int_S \phi\nabla\phi\cdot\mathbf{dS} \tag{10.6}$$

We shall now use this in a practical way in a rather novel example which actually looks, at first sight, to be completely outside the range of applicability of this formula.

Example *The kinetic energy of flow outside a solid sphere*

Let a solid sphere of radius r_0 be embedded in an inviscid incompressible fluid. The fluid is at rest at infinity and has a velocity potential ϕ given, in spherical polar coordinates with origin at the centre of the sphere, by

$$\phi = \frac{1}{2} v_0 \frac{r_0^3}{r^2} \cos(\theta), \quad v_0 \text{ constant} \tag{10.7}$$

Hence find the total kinetic energy E of the fluid.

At first it seems that the expression 10.1 we derived above for the kinetic energy is not applicable; however this is not so. Note that our fluid is in three dimensional space which we can approximate (temporarily) by a large hollow ball B_R, say, of radius R.

When we do the fluid occupies the volume inside B_R minus the volume B_{r_0} of the solid sphere; hence the region occupied by the fluid has as boundary the two separate surfaces S_R and S_{r_0} of the two spheres.

Now we *can* use our expression 10.1 for the energy E giving us

$$E = \frac{\rho}{2} \int_{S_R} \phi \nabla \phi \cdot \mathbf{dS} - \frac{\rho}{2} \int_{S_{r_0}} \phi \nabla \phi \cdot \mathbf{dS} \tag{10.8}$$

where the second integral has a minus sign in front of it because $\mathbf{dS}$ always points *outwards* from the volume to which it belongs and this means that $\mathbf{dS}$ is *inwards* on the surface S_{r_0}. However note that the integrand of each integral contains the term $\phi\nabla\phi$ and it is easy to see that this is proportional to r^{-3} on the surface of the sphere. For the large sphere this means that as $R \to \infty$ the entire integral over S_R tends to zero (this also follows from the fact that the fluid is at rest at infinity). The upshot is that, in the limit where $R \to \infty$, which we must take in order to return to the whole of three dimensional space, the integral over S_R tends to zero. Hence E is actually given by the formula

$$E = -\frac{\rho}{2} \int_{S_{r_0}} \phi \nabla \phi \cdot \mathbf{dS} \tag{10.9}$$

We now begin the calculation. If $\mathbf{e_r}$ denotes a unit vector in the radial direction, then on S_{r_0} we have

$$\mathbf{dS} = |\mathbf{dS}|\, \mathbf{e_r} = r_0^2\, d\Omega\, \mathbf{e_r} \tag{10.10}$$

where we used the definition of solid angle. But for the component of $\nabla\phi$ in the radial direction we have

$$\nabla\phi \cdot \mathbf{e_r} = \frac{\partial \phi}{\partial r} \tag{10.11}$$

So on S_{r_0} we find that

$$\nabla\phi \cdot \mathbf{dS} = \left.\frac{\partial \phi}{\partial r}\right|_{r=r_0} r_0^2 d\Omega \tag{10.12}$$

Now

$$\begin{aligned} \phi &= \frac{1}{2}v_0\frac{r_0^3}{r^2}\cos(\theta) \\ \Rightarrow \frac{\partial \phi}{\partial r} &= -v_0\frac{r_0^3}{r^3}\cos(\theta) \\ \Rightarrow \nabla\phi\cdot\mathbf{dS} &= -v_0 \left.\frac{r_0^3}{r^3}\right|_{r=r_0} \cos(\theta)r_0^2 d\Omega \end{aligned} \tag{10.13}$$

But it is standard that $d\Omega$ is given by

$$d\Omega = sin(\theta)d\theta d\phi \tag{10.14}$$

So we find that the entire integrand of the energy expression is just

$$\frac{\rho}{2}\left.\phi\right|_{r=r_0}\nabla\phi\cdot\mathbf{dS} = -\frac{\rho}{2}\left(\frac{1}{2}v_0\frac{r_0^3}{r_0^2}\right)\cos(\theta)^2\sin(\theta)v_0 r_0^2 d\theta d\phi \tag{10.15}$$

Hence E is given by

$$\begin{aligned} E &= \frac{\rho}{4}\int v_0^2 r_0^3\cos^2(\theta)\sin(\theta)d\theta d\phi \\ &= \frac{2\pi\rho v_0^2 r_0^3}{4}\int_0^\pi \cos^2(\theta)\sin(\theta)d\theta \\ &= -\frac{2\pi\rho v_0^2 r_0^3}{4}\left[\frac{\cos^3(\theta)}{3}\right]_0^\pi \\ &= -\frac{\pi\rho v_0^2 r_0^3}{2}\left\{-\frac{1}{3}-\frac{1}{3}\right\} \\ \Rightarrow E &= \frac{\pi\rho v_0^2 r_0^3}{3} \end{aligned} \tag{10.16}$$

and so the energy is found.

The next section marks a return to the subject of vorticity and angular momentum.

§ 11. A vorticity equation and its shortcomings

Consider Euler's equation for an inviscid, incompressible, fluid in the presence of a body force $\mathbf{F}$. Hence we have the equation

$$\frac{D\mathbf{v}}{Dt} = \mathbf{F} - \frac{1}{\rho}\nabla p \tag{11.1}$$

Let $\mathbf{F}$ be conservative so that $\mathbf{F} = -\nabla K$ giving us

$$\frac{D\mathbf{v}}{Dt} = -\nabla K - \frac{1}{\rho}\nabla p \tag{11.2}$$

Using the full form for the stream derivative $D\mathbf{v}/Dt$ we get

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla K - \frac{1}{\rho}\nabla p \tag{11.3}$$

We also recall that

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = \nabla\left(\frac{\mathbf{v}^2}{2}\right) + (\nabla\times\mathbf{v})\times\mathbf{v} \tag{11.4}$$

and using this we obtain

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\frac{\mathbf{v}^2}{2}\right) + (\nabla\times\mathbf{v})\times\mathbf{v} = -\nabla K - \frac{1}{\rho}\nabla p \tag{11.5}$$

Now, in an effort to obtain an equation for the vorticity $\boldsymbol{\omega} = \nabla\times\mathbf{v}$, we take the curl of both sides yielding

$$\nabla\times\frac{\partial \mathbf{v}}{\partial t} + \nabla\times\nabla\left(\frac{\mathbf{v}^2}{2}\right) + \nabla\times(\boldsymbol{\omega}\times\mathbf{v}) = -\nabla\times\nabla K - \frac{1}{\rho}\nabla\times\nabla p \tag{11.6}$$

But since $\nabla\times\nabla f = 0$ for any f this reduces at once to the simpler equation

$$\nabla\times\frac{\partial \mathbf{v}}{\partial t} + \nabla\times(\boldsymbol{\omega}\times\mathbf{v}) = 0 \tag{11.7}$$

and, if we use the fact that $\nabla\times\partial\mathbf{v}/\partial t = \partial(\nabla\times\mathbf{v})/\partial t = \partial\boldsymbol{\omega}/\partial t$, we then get our *unsatisfactory* equation for the vorticity $\boldsymbol{\omega}$, which is

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla\times(\boldsymbol{\omega}\times\mathbf{v}) = 0 \tag{11.8}$$

The reason that this equation is unsatisfactory is the following: In general there is a vector identity for $\nabla\times(\boldsymbol{\omega}\times\mathbf{v})$ giving

$$\nabla\times(\boldsymbol{\omega}\times\mathbf{v}) = (\mathbf{v}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\mathbf{v} \tag{11.9}$$

But in two dimensions (we state this fact without proof but it is easy to check) it is the case that

$$(\boldsymbol{\omega}\cdot\nabla)\mathbf{v} = 0 \tag{11.10}$$

Hence, in two dimensions, our vorticity equation becomes

$$\begin{aligned} \frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{v}\cdot\nabla)\boldsymbol{\omega} &= 0 \\ \text{i.e.} \quad \frac{D\boldsymbol{\omega}}{Dt} &= 0 \\ \Rightarrow \boldsymbol{\omega} \quad &\text{is conserved??} \end{aligned} \tag{11.11}$$

Now $\boldsymbol{\omega}$ *cannot be conserved* because this would mean that vorticity, once created, would never disappear from a fluid, or, once absent could never be created. This state of affairs is in flat contradiction with experiment so there is something amiss.

It turns out that if we include the *viscosity* η of the fluid then the contradiction is avoided and we return to agreement with experiment. This is what we turn to in the next section.

§ 12. Viscous flow, the Navier-Stokes equation and the satisfactory vorticity equation

We shall now simply quote[9] that the viscous forces in an incompressible fluid are given by adding the term

$$\frac{\eta}{\rho}\nabla^2\mathbf{v} \tag{12.1}$$

to the Euler equation yielding the

$$\frac{D\mathbf{v}}{Dt} = \mathbf{F} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v} \tag{12.2}$$

This equation 12.2 is the celebrated *Navier–Stokes equation* for viscous flow.

It is now very simple indeed to check that if we go through the same steps as we did in the previous section to derive an equation for the vorticity $\boldsymbol{\omega}$ the only new feature is the viscous term $(\eta/\rho)\nabla^2\mathbf{v}$ and the effect of this term on the derivation is to add to 11.8 its curl—i.e. the quantity $(\eta/\rho)\nabla\times\nabla^2\mathbf{v}$ so that our vorticity derivation now gives the equation

$$\begin{aligned} \frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla\times(\boldsymbol{\omega}\times\mathbf{v}) &= \frac{\eta}{\rho}\nabla\times\nabla^2\mathbf{v} = \frac{\eta}{\rho}\nabla^2(\nabla\times\mathbf{v}) \\ \Rightarrow \frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla\times(\boldsymbol{\omega}\times\mathbf{v}) &= \frac{\eta}{\rho}\nabla^2\boldsymbol{\omega} \end{aligned} \tag{12.3}$$

[9] In the longer version of this course we also provide a derivation of this fact.

So our new vorticity equation is

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \mathbf{v}) = \frac{\eta}{\rho} \nabla^2 \boldsymbol{\omega} \tag{12.4}$$

and this has no drawbacks.

We want to point out that, for small velocities, the second term on the LHS of 12.4 is smaller than the first one since it is quadratic in $\mathbf{v}$. If we approximate by neglecting this term then the vorticity equation *for small velocities* is

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \frac{\eta}{\rho} \nabla^2 \boldsymbol{\omega} \tag{12.5}$$

and this is exactly like the heat flow equation

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T \tag{12.6}$$

where T is the temperature inside a solid slab, of thermal conductivity κ, through which heat is flowing. Hence we obtain the useful conclusion that, for small velocities, vorticity diffuses through a fluid in the same way as heat does through a solid.

§ 13. The Reynolds number

In this section we apply the technique of *dimensional analysis* to viscous flow and obtain a formula for a very interesting number R known as the *Reynolds number* of the flow. The Reynolds number allows one to understand many features of viscous flow and provides an understanding of why wind tunnel experiments work.

We begin with viscous incompressible flow past an upright cylinder, cf. Fig.9.

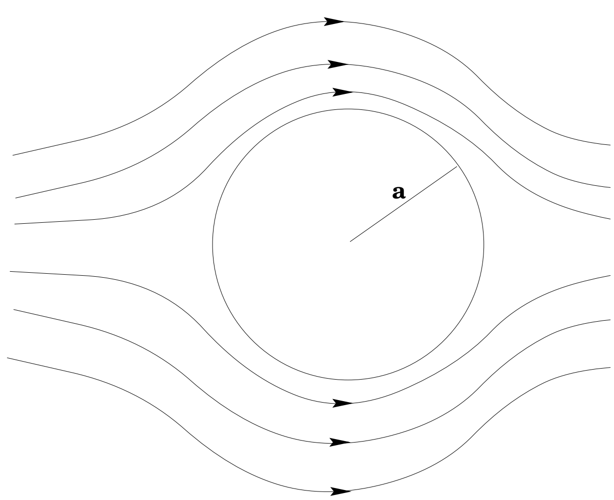

Fig. 9: Viscous flow past an upright cylinder of radius a

Suppose that the cylinder has radius a and that far away from the cylinder the speed of the flow tends to the constant value V. Then the four parameters

$$\eta,\ \rho\,,a \text{ and } V \tag{13.1}$$

characterise the flow and combine to form the number R known as the Reynolds number whose definition is

$$R = \frac{\rho V a}{\eta} \tag{13.2}$$

It is possible to let the Reynolds number emerge naturally from a discussion of the flow which involves a change of variables: First remember that one is interested in factors which particularly depend on *shape* rather than size—think of the wind tunnel where one builds models of the same shape but usually not of the same size as the real life object—one changes to a dimensionless set of variables and, when one does this, it transpires out that R turns up naturally.

To this end let us change the variables x, y, z, t and $\mathbf{v}$ to the *dimensionless set of variables*

$$x', y', z', t' \text{ and } \mathbf{v}' \tag{13.3}$$

where

$$x' = \frac{x}{a}, \quad y' = \frac{y}{a}, \quad z' = \frac{z}{a}, \quad t' = \frac{V}{a}t \quad \text{and} \quad \mathbf{v}' = \frac{\mathbf{v}}{V} \tag{13.4}$$

The next step is to make this change of variables in our vorticity equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \mathbf{v}) = \frac{\eta}{\rho}\nabla^2 \boldsymbol{\omega} \tag{13.5}$$

If we note that, for any function f,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial x} = \frac{1}{a}\frac{\partial f}{\partial x'} \tag{13.6}$$

then it is clear that

$$\nabla = \frac{1}{a}\nabla' \tag{13.7}$$

where ∇' is the same as ∇ except for the substitution of derivatives in x', y' and z' for the derivatives in x, y and z. Now we see that, for the the vorticity

$\boldsymbol{\omega}$, we have

$$\begin{aligned}\boldsymbol{\omega} &= \nabla \times \mathbf{v} \\ &= \frac{1}{a}\nabla' \times \mathbf{v} \\ &= \frac{V}{a}\nabla' \times \mathbf{v}' \\ \Rightarrow \boldsymbol{\omega} &= \frac{V}{a}\boldsymbol{\omega}'\end{aligned} \tag{13.8}$$

where, as should be expected, $\boldsymbol{\omega}'$ denotes $\nabla' \times \mathbf{v}'$. In a similar fashion we can calculate that

$$\frac{\partial f}{\partial t} = \frac{V}{a}\frac{\partial f}{\partial t'} \tag{13.9}$$

Thus

$$\begin{aligned}\frac{\partial \boldsymbol{\omega}}{\partial t} &= \frac{V}{a}\frac{\partial \boldsymbol{\omega}}{\partial t'} \\ &= \frac{V^2}{a^2}\frac{\partial \boldsymbol{\omega}'}{\partial t'}\end{aligned} \tag{13.10}$$

Also

$$\boldsymbol{\omega} \times \mathbf{v} = \left(\frac{V}{a}\boldsymbol{\omega}'\right) \times (V\mathbf{v}') = \frac{V^2}{a}\boldsymbol{\omega}' \times \mathbf{v}' \tag{13.11}$$

from which we readily deduce that

$$\begin{aligned}\nabla \times (\boldsymbol{\omega} \times \mathbf{v}) &= \nabla \times \left(\frac{V^2}{a}\boldsymbol{\omega}' \times \mathbf{v}'\right) \\ &= \frac{1}{a}\nabla' \times \left(\frac{V^2}{a}\boldsymbol{\omega}' \times \mathbf{v}'\right) \\ &= \frac{V^2}{a^2}\nabla' \times (\boldsymbol{\omega}' \times \mathbf{v}')\end{aligned} \tag{13.12}$$

This means that we have dealt with all the terms on the LHS of our vorticity equation; it is completely straightforward to verify that

$$\nabla^2 \boldsymbol{\omega} = \frac{V}{a^3}\left(\nabla'\right)^2 \boldsymbol{\omega}' \tag{13.13}$$

Hence, in the new dimensionless variables, we now have the vorticity equation

$$\frac{V^2}{a^2}\frac{\partial \boldsymbol{\omega}'}{\partial t'} + \frac{V^2}{a^2}\nabla' \times (\boldsymbol{\omega}' \times \mathbf{v}') = \frac{\eta}{\rho}\frac{V}{a^3}\left(\nabla'\right)^2 \boldsymbol{\omega}' \tag{13.14}$$

which we proceed to rewrite in the form

$$\frac{\partial \boldsymbol{\omega}'}{\partial t'} + \nabla' \times (\boldsymbol{\omega}' \times \mathbf{v}') = \frac{\eta}{\rho V a} (\nabla')^2 \boldsymbol{\omega}' \tag{13.15}$$

that is

$$\frac{\partial \boldsymbol{\omega}'}{\partial t'} + \nabla' \times (\boldsymbol{\omega}' \times \mathbf{v}') = \frac{1}{R} (\nabla')^2 \boldsymbol{\omega}', \quad \text{with} \quad R = \frac{\rho V a}{\eta} \tag{13.16}$$

and we see that, as promised above, the Reynolds number R has emerged naturally.

First of all one should think of R loosely as being given by

$$R = \frac{\rho}{\eta} \times \{\text{ a typical length}\} \times \{\text{ a typical velocity}\} \tag{13.17}$$

We now want to explain the significance of the Reynolds number R. Let us consider the example of a wind tunnel: One takes the real full size aircraft with parameters η_1, ρ_1, V_1 and a_1 giving rise to a Reynolds number R_1, say; then one takes the model aircraft in the wind tunnel with parameters η_2, ρ_2, V_2 and a_2 giving rise to a Reynolds number R_2. The *vital* fact about the wind tunnel, though, is that one arranges that its Reynolds number R_2 is the *same* as the original Reynolds number R_1. Hence we always have

$$\begin{aligned} R_1 &= R_2 \\ \text{or} \quad \frac{\rho_1 V_1 a_1}{\eta_1} &= \frac{\rho_2 V_2 a_2}{\eta_2} \end{aligned} \tag{13.18}$$

So the individual values of η_1, ρ_1, V_1, a_1 and η_2, ρ_2, V_2, a_2 can, and *do*, differ but only in a manner which keeps $R_1 = R_2$.

Now we can also see why we can use the phrase *a typical length* and *a typical velocity* in our expression 13.17 above for R. The point is that if we replace the length a by $2a$ then R increases by a factor of 2; but if we now look at the wind tunnel discussion this would have the effect of replacing R_1 and R_2 by $2R_1$ and $2R_2$ respectively, and we see that it doesn't matter whether we require $2R_1$ to equal $2R_2$ or just R_1 to equal R_2 we get the same relation among the two sets of parameters. A similar remark applies to the parameter V.

This method of matching Reynolds numbers is also used in marine technology when designing boats. The model boats are floated in tanks which sometimes contain the liquid *mercury*—a liquid metal—so we see that very different fluids can be used in the two situations as long as the Reynolds numbers match.

The size of the Reynolds number also controls the production of vortices and the consequent onset of turbulence as we shall illustrate diagrammatically in the next section.

§ 14. Turbulence and the Reynolds number

Consider the set of five diagrams shown below in Fig. 10; they show viscous flow past our cylinder for steadily increasing values of the Reynolds number R.

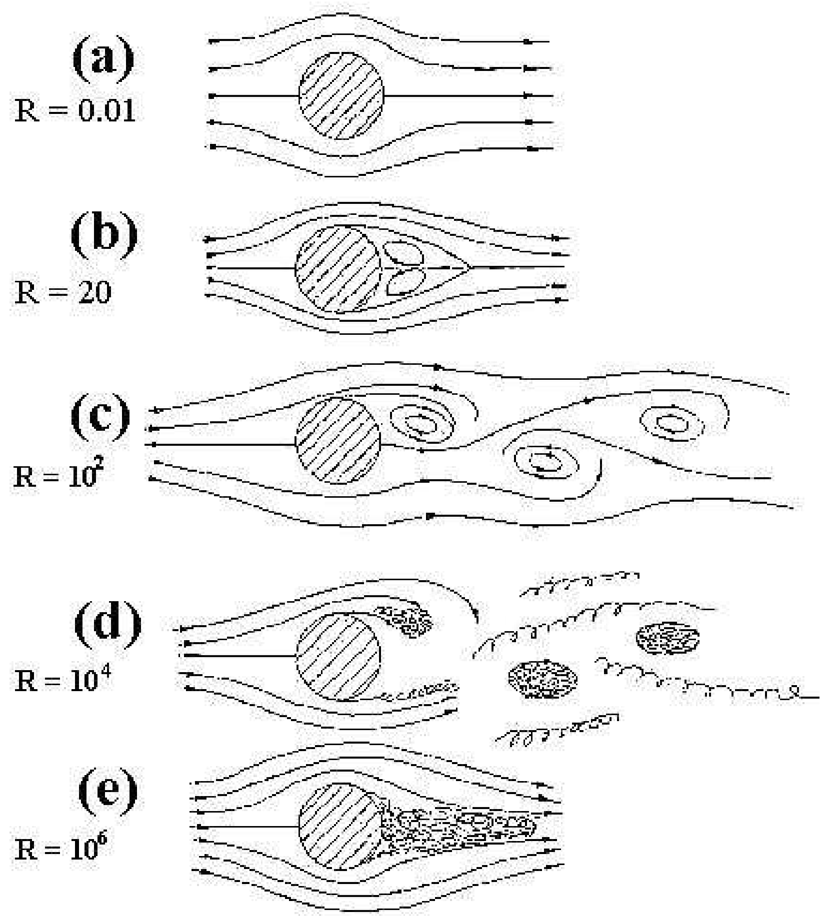

Fig. 10: Viscous flow past a cylinder as R increases

We note that in Fig. 10 (a) where $R = 10^{-2}$, there is no vorticity but in Fig. 10 (b) where R has been increased to 20 there is now a pair of vortices behind the cylinder but staying close to the solid boundary. Then, as R increases, the vortices break away from the cylinder and move off down stream cf. Fig. 10 (c) where $R = 100$. The final stages are attained when R goes through the values 10^4–10^6 which we see in the last two figures. In Fig. 10 (d) the vortices are huge in number and move together in clumps that look a bit like egg shells; then when $R = 10^6$ there is a continuous wake of turbulent vortices starting at the solid boundary (i.e. the cylinder) and stretching indefinitely far downstream. This flow is no longer steady and indeed has been unsteady since R increased above the value 40 or so.

We are now ready to look at some solutions to the Navier-Stokes equations.

§ 15. Some exact solutions to the Navier–Stokes equations

The first problem in viscous flow that we shall solve is the flow down a circular pipe.

Example *Viscous flow in a pipe or Poiseuille flow*

We shall take a horizontal circular pipe of radius a and length L down which *incompressible viscous* fluid is flowing. The assumptions that we make to simplify the calculation somewhat are

(i) The flow is steady.
(ii) The flow is parallel to the axis of the pipe.
(iii) There is no body force.
(iv) There is a constant pressure difference across the ends of the pipe.

A vital extra *experimental fact* about viscous flow past a solid object is that the fluid velocity dies away to zero on the surface of the object. This is called the *no slip boundary condition* and must not be forgotten.

We begin with the full Navier–Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{F} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v} \tag{15.1}$$

Now since there is no body force and the flow is also steady we have at once that

$$\mathbf{F} = \mathbf{0}, \quad \frac{\partial \mathbf{v}}{\partial t} = 0 \tag{15.2}$$

so we are left with the equation

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v} \tag{15.3}$$

Next we *choose* to make the x-axis coincide with the axis of the pipe and, having done this, the fact that the flow is *parallel* means that $\mathbf{v}$ only has an x-component i.e.

$$\mathbf{v} = v_x(x, y, z)\mathbf{i} \tag{15.4}$$

Steaming on we note that incompressibility means that

$$
\begin{aligned}
\nabla \cdot \mathbf{v} &= 0 \\
\Rightarrow \frac{\partial v_x}{\partial x} &= 0, \qquad \text{since } v_y = v_z = 0 \\
\Rightarrow v_x &= v_x(y, z)
\end{aligned}
\tag{15.5}
$$

so v_x is independent of x. This has the very useful consequence that $(\mathbf{v}\cdot\nabla)\mathbf{v}$ which in full is given by

$$
(\mathbf{v} \cdot \nabla)\mathbf{v} = (v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z})(v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}) \tag{15.6}
$$

reduces to

$$
\begin{aligned}
(\mathbf{v} \cdot \nabla)\mathbf{v} &= v_x \frac{\partial v_x}{\partial x} \mathbf{i} \\
&= 0
\end{aligned}
\tag{15.7}
$$

Hence the Navier–Stokes equation has now become just

$$
\begin{aligned}
0 &= -\frac{1}{\rho} \nabla p + \frac{\eta}{\rho} \nabla^2 \mathbf{v} \\
\Rightarrow \nabla^2 \mathbf{v} &= \frac{1}{\eta} \nabla p
\end{aligned}
\tag{15.8}
$$

We also know that there is a constant pressure difference across the ends of the pipe and we shall further assume [10] that p only depends linearly on x and so is given by

$$
p = Ax + B, \quad A \text{ and } B \text{ constants} \tag{15.9}
$$

Now if we let the beginning of the pipe be at $x = 0$ and label the pressure there by P_0 then the end of the pipe must be at $x = L$ and we denote the pressure there by P_1. Thus we have

$$
p(x = 0) = P_0, \quad p(x = L) = P_1 \tag{15.10}
$$

[10] We don't have to do this but it simplifies things. In fact the Navier Stokes equation for the components v_y and v_z, which we know are zero, show that $\partial p/\partial y = \partial p/\partial z = 0$ so that p only depends on x. To show that this dependence is linear we take the divergence of the Navier–Stokes equation for v_x which incompressibility reduces for us to the statement $\nabla^2 p = 0$; this then immediately gives $p = Ax + B$ since ∇^2 reduces to d^2/dx^2 in this case.

from which we deduce that

$$A = \frac{(P_1 - P_0)}{L}, \quad B = P_0 \tag{15.11}$$

giving

$$\begin{aligned} p &= \frac{(P_1 - P_0)}{L} x + P_0 \\ \Rightarrow \nabla p &= \frac{(P_1 - P_0)}{L} \mathbf{i} \end{aligned} \tag{15.12}$$

Now we turn to the Navier–Stokes equation for v_x which is

$$\begin{aligned} \nabla^2 v_x(y, z) &= \frac{(P_1 - P_0)}{\eta L} \\ \Rightarrow \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) v_x(y, z) &= \frac{(P_1 - P_0)}{\eta L} \end{aligned} \tag{15.13}$$

This equation is simple enough to guess its solution. Simply note that, if C and D are constants, then a solution is

$$v_x = C - D(y^2 + z^2) \tag{15.14}$$

if we choose D appropriately. Substituting in to the Navier–Stokes equation gives us the requirement that

$$4D = -\frac{(P_1 - P_0)}{\eta L} \tag{15.15}$$

so D is found. This must be *the unique solution* if we can make it satisfy the no slip boundary condition

$$\mathbf{v} = \mathbf{0}, \quad \text{on the surface of the pipe} \tag{15.16}$$

But the pipe has radius a and so its surface has equation

$$y^2 + z^2 = a^2 \tag{15.17}$$

so setting $y^2 + z^2 = a^2$ in the equation $v_x = C - D(y^2 + z^2)$ and requiring v_x to vanish gives

$$0 = v_x = C - Da^2 \tag{15.18}$$

so that $C = Da^2$ and our solution is complete. Summarising we have found that

$$\begin{aligned} &\mathbf{v} = v_x \mathbf{i} \\ \text{with} \qquad &v_x = \frac{(P_1 - P_0)}{4\eta L}(y^2 + z^2 - a^2) \end{aligned} \tag{15.19}$$

Poiseuille's law *The quantity of fluid delivered by a pipe of radius a.*

It is very interesting to calculate the total mass Q of fluid delivered by the flow down the pipe per second.

For convenience we set

$$r^2 = y^2 + z^2 \tag{15.20}$$

and now we consider a thin annulus of radius r $(r < a)$ and thickness dr inside the pipe. This annulus flows a distance precisely $v_x(r)$ in one second and thereby traces out a volume $2\pi r dr v_x(r)$. The mass of fluid in this volume is just

$$\rho\, 2\pi r dr v_x(r) \tag{15.21}$$

and so the total mass Q flowing through the pipe per second is given by integrating over r. We have

$$Q = \int_0^a \rho\, 2\pi r v_x(r) dr$$

Using

$$v_x(r) = \frac{(P_1 - P_0)}{4\eta L}(r^2 - a^2) \tag{15.22}$$

we find that

$$\begin{aligned} Q &= 2\pi\rho \frac{(P_1 - P_0)}{4\eta L} \int_0^a (r^2 - a^2) r dr \\ &= 2\pi\rho \frac{(P_1 - P_0)}{4\eta L} \left[\frac{r^4}{4} - \frac{r^2 a^2}{2} \right]_0^a \\ &= 2\pi\rho \frac{(P_1 - P_0)}{4\eta L} \left(-\frac{a^4}{4} \right) \\ \Rightarrow Q &= \pi\rho \frac{(P_0 - P_1)}{8\eta L} a^4 \end{aligned} \tag{15.23}$$

This formula

$$Q = \pi\rho \frac{(P_0 - P_1)}{8\eta L} a^4 \tag{15.24}$$

for Q is known as *Poiseuille's law.*

We note that Q has a very strong dependence on the radius a of the pipe: we see that

$$Q \propto a^4 \tag{15.25}$$

This law applies to the flow of gases and liquids in all kinds of different circumstances. The a^4 dependence has some dramatic consequences—we provide one illustration.

Let us consider blood flow in an artery of radius a. If cholesterol deposition has narrowed the artery by 10%—this is a conservative supposition, in individuals with severe heart disease an artery can be blocked—then a is reduced to $0.9a$. This 10% narrowing means that Q is reduced by a factor

$$(0.9a)^4 = 0.6561a^4 \tag{15.26}$$

which amounts to a reduction in blood flow of approximately 35%. So we have found that a loss of more than a third of the blood throughput comes from a reduction in the artery radius of 10%. This is somewhat idealised as, actually, arteries have a certain amount of elasticity. The reader can readily construct other examples of this formula in action.

Example *Viscous flow in an infinitely deep straight channel*

Our next example models flow in a deep straight canal. We shall take the flow of an incompressible viscous fluid down a straight channel. Our assumptions this time are

(i) The flow is steady.
(ii) The channel is infinitely long, infinitely deep and has parallel flat walls.
(iii) The flow is parallel to the walls.

We also recall that, since the flow is viscous, we automatically have the no slip boundary condition which asserts that

$$\mathbf{v} = \mathbf{0}, \qquad \text{on the walls} \tag{15.27}$$

We take the width of the channel to be $2b$ and show its orientation relative to the $x - y$ plane in Fig. 11 below.

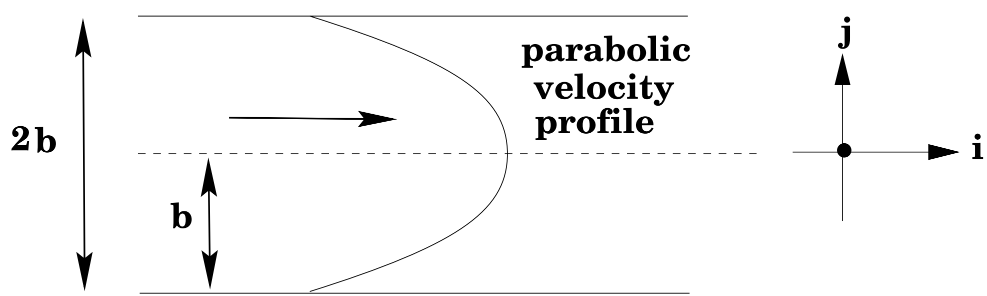

Fig. 11: Viscous flow through a straight channel

Now, since the flow is parallel, we have

$$\mathbf{v} = v_x \mathbf{i} \tag{15.28}$$

Now just as we had in the previous example incompressibility gives

$$\begin{aligned} \nabla \cdot \mathbf{v} &= 0 \\ \Rightarrow \frac{\partial v_x}{\partial x} &= 0, \qquad \text{since } v_y = v_z = 0 \\ \Rightarrow v_x &= v_x(y, z) \end{aligned} \tag{15.29}$$

This also gives again

$$(\mathbf{v} \cdot \nabla)\mathbf{v} = 0 \tag{15.30}$$

When we take account of the fact that the flow is steady, and that there is no body force, the Navier–Stokes equation becomes

$$0 = -\frac{\nabla p}{\rho} + \frac{\eta}{\rho} \nabla^2 \mathbf{v} \tag{15.31}$$

So far this is very close to the previous example; a further simplification here is that, although we have,

$$v_x = v_x(y, z) \tag{15.32}$$

in fact v_x cannot depend on z since the channel has infinite depth. Hence all we have for v_x is

$$v_x = v_x(y) \tag{15.33}$$

The resulting Navier–Stokes equation is now very simple being

$$\begin{aligned} \nabla^2 v_x &= \frac{1}{\eta}\frac{dp}{dx} \\ \Rightarrow \frac{d^2 v_x}{dy^2} &= \frac{1}{\eta}\frac{dp}{dx} = \text{ a constant} \end{aligned} \tag{15.34}$$

where we note that dp/dx is constant as it was in the Poiseuille example. All this means that we can integrate the Navier–Stokes equation twice and we have its solution which is

$$v_x = \frac{1}{2\eta}\frac{dp}{dx} y^2 + Ay + B, \quad A \text{ and } B \text{ constants} \tag{15.35}$$

We find A and B by imposing the no slip boundary condition on the walls, i.e.

$$v_x = 0, \qquad \text{when } y = \mp b \tag{15.36}$$

So when $y = b$ we have $v_x = 0$ yielding

$$\frac{1}{2\eta}\frac{dp}{dx}b^2 + Ab + B = 0 \tag{15.37}$$

and when $y = -b$ we also have $v_x = 0$ so that

$$\frac{1}{2\eta}\frac{dp}{dx}b^2 - Ab + B = 0 \tag{15.38}$$

We easily solve these two equations for A and B obtaining the result

$$A = 0, \qquad B = -\frac{1}{2\eta}\frac{dp}{dx}b^2 \tag{15.39}$$

and so our final solution for the velocity $\mathbf{v}$ down the channel is

$$\begin{aligned} &\mathbf{v} = v_x\mathbf{i} \\ \text{where} \qquad &v_x = \frac{1}{2\eta}\frac{dp}{dx}(y^2 - b^2) \end{aligned} \tag{15.40}$$

We note that $\mathbf{v}$ has a *parabolic* velocity profile as indicated in Fig. 11.

Example *Viscous flow through a channel with one moving wall*

Our last example is a refinement of the previous one where we allow one of the walls to move while the other remains stationary—cf. Fig. 12. This situation models the flow between the side of a super tanker as it comes in to dock.

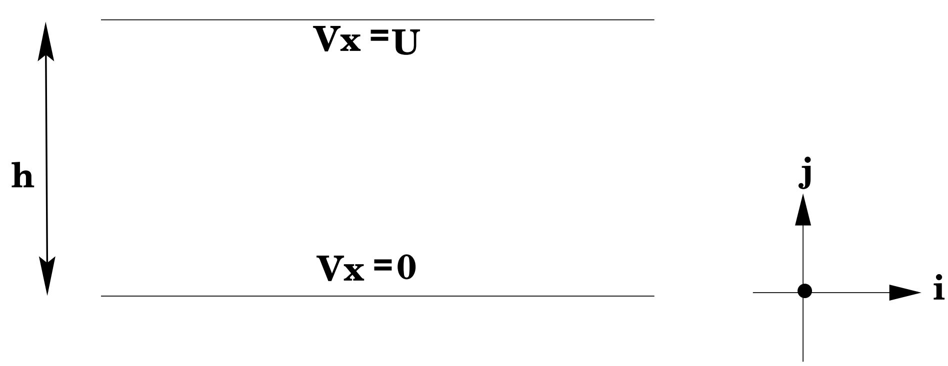

Fig. 12: Viscous flow past a moving wall

We see from Fig. 12 that the upper wall moves parallel to the lower one with speed U. We note, too, that for convenience we have moved the position of the $x - y$ axes and also renamed the width of the channel to be h.

Mathematically the only new feature, as compared with the previous example, is that the boundary conditions change. They are now

$$\begin{aligned} v_x &= U, \quad \text{at } y = h \\ v_x &= 0, \quad \text{at } y = 0 \end{aligned} \tag{15.41}$$

The boundary condition at the upper wall ensures that the fluid does not *slip* when it touches this wall. Again we have a solution of the form

$$v_x = \frac{1}{2\eta}\frac{dp}{dx}y^2 + Ay + B \tag{15.42}$$

and the boundary conditions give the simultaneous equations

$$\begin{aligned} \frac{1}{2\eta}\frac{dp}{dx}h^2 + Ah + B &= U, \quad \text{at } y = h \\ B &= 0, \quad \text{at } y = 0 \end{aligned} \tag{15.43}$$

from which we deduce that

$$A = \frac{U}{h} - \frac{1}{2\eta}\frac{dp}{dx}h, \quad B = 0 \tag{15.44}$$

so that our final solution is

$$\begin{aligned} \mathbf{v} &= v_x\mathbf{i} \\ \text{where} \qquad v_x &= \frac{1}{2\eta}\frac{dp}{dx}(y^2 - hy) + \frac{U}{h}y \end{aligned} \tag{15.45}$$

The shape of the velocity profile, in this super tanker case, is determined by the quantity S where

$$S = -\frac{h^2}{2\eta}\frac{dp}{dx} \tag{15.46}$$

If we use S we can write v_x as

$$v_x = S\frac{y}{h}\left(1 - \frac{y}{h}\right) + \frac{U}{h}y \tag{15.47}$$

Then we can distinguish three distinct cases, namely,

(i) $S > 0$
(ii) $S = 0$
(iii) $S < 0$

If $S > 0$ then $dp/dx < 0$ and p decreases as we move to the right along the channel giving forward flow.

If $S = 0$ than $dp/dx = 0$ and we have what is called *simple Couette flow* giving a straight velocity profile.

Finally, if $S < 0$, then $dp/dx > 0$ and p *increases* as we move to the right along the channel giving the possibility of backward flow cf. Fig. 13.

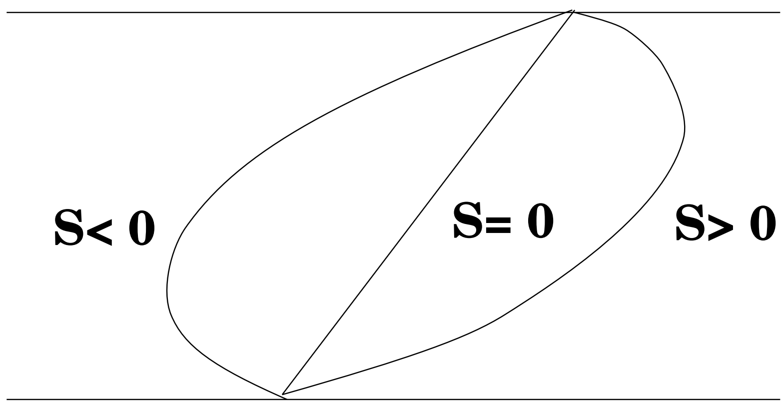

Fig. 13: The velocity profile and the parameter S

§ 16. 'Derivation' of the Navier–Stokes equation

We start with the Euler equation with an unknown viscous term $\mathbf{F}$ added

$$\rho\frac{D\mathbf{v}}{Dt} = -\nabla p + \mathbf{F} \tag{16.1}$$

and the idea is that we want to derive the form of this term $\mathbf{F}$.

From now on we use the summation convention for indices. To make any further progress we have to assume that the fluid is what is called *Newtonian* (almost all fluids are Newtonian) and this means that the viscous term $\mathbf{F}$ is given in terms of a derivative as follows

$$\begin{aligned} \mathbf{F} &= F_i\mathbf{e}_i \\ \text{where} \quad F_i &= -\partial_j Q_{ij}, \quad \text{for some } Q_{ij} \end{aligned} \tag{16.2}$$

Q_{ij} is called the *shear stress*. We keep this form for $\mathbf{F}$ in storage for a few lines and turn to do a little work on the other terms in the equation of motion. Using the definition of $D\mathbf{v}/Dt$ this equation can be written as

$$\rho\frac{\partial\mathbf{v}}{\partial t} = -\rho(\mathbf{v}\cdot\nabla)\mathbf{v} - \nabla p + \mathbf{F} \tag{16.3}$$

But

$$\begin{aligned}\frac{\partial}{\partial t}(\rho\mathbf{v}) &= \frac{\partial\rho}{\partial t}\mathbf{v}+\rho\frac{\partial\mathbf{v}}{\partial t}\\ \Rightarrow \rho\frac{\partial\mathbf{v}}{\partial t} &= \frac{\partial}{\partial t}(\rho\mathbf{v})-\frac{\partial\rho}{\partial t}\mathbf{v}\end{aligned}\tag{16.4}$$

the continuity equation says that

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0\tag{16.5}$$

and substituting for $\partial\rho/\partial t$ in the equation above we find that

$$\rho\frac{\partial\mathbf{v}}{\partial t}=\frac{\partial}{\partial t}(\rho\mathbf{v})+\nabla\cdot(\rho\mathbf{v})\mathbf{v}\tag{16.6}$$

Now we substitute for $\rho\partial\mathbf{v}/\partial t$ in 16.3 obtaining thereby the equation

$$\begin{aligned}\frac{\partial}{\partial t}(\rho\mathbf{v})+\nabla\cdot(\rho\mathbf{v})\mathbf{v} &= -\rho(\mathbf{v}\cdot\nabla)\mathbf{v}-\nabla p+\mathbf{F}\\ \Rightarrow\frac{\partial}{\partial t}(\rho\mathbf{v}) &= -\nabla\cdot(\rho\mathbf{v})\mathbf{v}-\rho(\mathbf{v}\cdot\nabla)\mathbf{v}-\nabla p+\mathbf{F}\\ \Rightarrow\frac{\partial}{\partial t}(\rho\mathbf{v}) &= -\left\{\nabla\cdot(\rho\mathbf{v})+\rho(\mathbf{v}\cdot\nabla)\right\}\mathbf{v}-\nabla p+\mathbf{F}\end{aligned}\tag{16.7}$$

But $\nabla\cdot(\rho\mathbf{v})$ expands according to the identity

$$\nabla\cdot(\rho\mathbf{v})=\rho\nabla\cdot\mathbf{v}+\mathbf{v}\cdot\nabla\rho\tag{16.8}$$

and using this we obtain the equation

$$\frac{\partial}{\partial t}(\rho\mathbf{v})=-\left\{\rho\nabla\cdot\mathbf{v}+\mathbf{v}\cdot\nabla\rho+\rho(\mathbf{v}\cdot\nabla)\right\}\mathbf{v}-\nabla p+\mathbf{F}\tag{16.9}$$

However if we adopt the summation convention we can simply check by differentiation that the following is true

$$\mathbf{e}_i\partial_j(\rho v_i v_j)=\left\{\rho\nabla\cdot\mathbf{v}+\mathbf{v}\cdot\nabla\rho+\rho(\mathbf{v}\cdot\nabla)\right\}\mathbf{v}\tag{16.10}$$

and thus we obtain

$$\frac{\partial}{\partial t}(\rho\mathbf{v})=-\nabla p-\mathbf{e}_i\partial_j(\rho v_i v_j)+\mathbf{F}\tag{16.11}$$

But recall that we had, for $\mathbf{F} = F_i \mathbf{e}_i$, the Newtonian fluid property $F_i = -\partial_j Q_{ij}$; if we add to this the fact that

$$\mathbf{v} = v_i\,\mathbf{e}_i, \qquad \text{and} \qquad \nabla p = \partial_i p\,\mathbf{e}_i \tag{16.12}$$

then we can write the equation of motion in component form as

$$\frac{\partial}{\partial t}(\rho v_i) = -\partial_j(p\,\delta_{ij} + \rho v_i v_j + Q_{ij}) \tag{16.13}$$

where we used the easily verified property that $\partial_i p = \partial_j p\,\delta_{ij}$

We point out in passing (we shall not actually make any use of this fact here) that we can combine together the first and third terms on the RHS of this equation by introducing what is called the *stress tensor* S_{ij} whose definition is simply that [11]

$$S_{ij} = -p\delta_{ij} - Q_{ij} \tag{16.16}$$

For a Newtonian fluid we make the assumption that Q_{ij} is symmetric in i and j and constructed entirely from derivatives of the velocity components. To this end we write [12]

$$Q_{ij} = \frac{a}{2}(\partial_i v_j + \partial_j v_i) + b(\partial_k v_k)\delta_{ij}, \qquad a \text{ and } b \text{ constants} \tag{16.19}$$

[11] The key property of S_{ij} being, as one can check in one differentiation, that

$$\partial_j S_{ij} = -\partial_i p - \partial_j Q_{ij} \tag{16.14}$$

so the Navier–Stokes equation 16.13 takes on the compact form

$$\frac{\partial}{\partial t}(\rho v_i) = -\partial_j(\rho v_i v_j) + \partial_j S_{ij} \tag{16.15}$$

[12] This somewhat arbitrary looking assumption can be elaborated on though we do not have the space to give much more detail here. Nevertheless we add that if one thinks of the evolution of the velocity vector from $\mathbf{v}$ at the point (x, y, z) to $\mathbf{v} + \mathbf{dv}$ at the point $(x + dx, y + dy, z + dz)$, then the chain rule for partial differentiation says that

$$\begin{aligned} dv_i &= \frac{\partial v_i}{\partial x^j} dx^j \\ &= \frac{1}{2}\left\{\left(\frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i}\right) + \left(\frac{\partial v_i}{\partial x^j} - \frac{\partial v_j}{\partial x^i}\right)\right\} dx^j \\ &= M^S_{ij} dx^j + M^A_{ij} dx^j \\ \text{where } M^S_{ij} &= \frac{1}{2}\left(\frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i}\right), \quad M^A_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x^j} - \frac{\partial v_j}{\partial x^i}\right) \end{aligned} \tag{16.17}$$

Now the constants a and b, as defined, are not in general positive, so, to get rid of this inconvenience we replace them by the two constants

$$\eta, \quad \text{and} \quad \zeta \tag{16.20}$$

which are always positive. The relation between the two sets of constants is just that

$$a = -2\eta \quad \text{and} \quad b = \frac{2}{3}\eta - \zeta \tag{16.21}$$

The constants η and ζ are known as *coefficients of viscosity*—of course we have already met η but ζ is a new one. Using η and ζ we find that

$$Q_{ij} = -\eta\left(\partial_i v_j + \partial_j v_i\right) + \left(\frac{2}{3}\eta - \zeta\right)(\partial_k v_k)\delta_{ij} \tag{16.22}$$

Now we compute $\partial_j Q_{ij}$ and find that

$$\begin{aligned}
\partial_j Q_{ij} &= -\eta\partial_j\left(\partial_i v_j + \partial_j v_i\right) + \left(\frac{2}{3}\eta - \zeta\right)\partial_j(\partial_k v_k)\delta_{ij} \\
&= -\eta\partial_j\partial_j v_i - \eta\partial_i(\partial_j v_j) + \left(\frac{2}{3}\eta - \zeta\right)\partial_i(\partial_k v_k) \\
&= -\eta\partial_j\partial_j v_i - \frac{\eta}{3}\partial_i(\partial_k v_k) - \zeta\partial_i(\partial_k v_k), \quad \text{since } \partial_j v_j = \partial_k v_k \\
&= -\eta\partial_j\partial_j v_i - \left(\zeta + \frac{\eta}{3}\right)\partial_i(\partial_k v_k)
\end{aligned} \tag{16.23}$$

Now if we insert this expression for Q_{ij} back where it came from, i.e. into 16.13, it yields the equation

$$\frac{\partial}{\partial t}(\rho v_i) = -\partial_i p - \partial_j(\rho v_i v_j) + \eta\partial_j\partial_j v_i + \left(\zeta + \frac{\eta}{3}\right)\partial_i(\partial_k v_k) \tag{16.24}$$

It turns out that $M^A_{ij}dx^j$ only *rotates* an infinitesimal cube of fluid since it is clear that

$$\mathbf{e}_i M^A_{ij}dx^j = -\frac{1}{2}\boldsymbol{\omega} \times \mathbf{dr} \tag{16.18}$$

but the term $M^S_{ij}dx^j$ *strains* the cube of fluid—i.e. changes its volume or shape and it is this action which is resisted by viscous forces. This makes it plausible that M^S_{ij} should be part of the tensor Q_{ij} and this is one of the main parts of the assumption that a fluid is Newtonian. This footnote is not an explanation but just a pointer to where one can go for further material.

The last step in this calculation is to put 16.24 back into the usual vector form. To accomplish this multiply both sides of 16.24 by $\mathbf{e}_i$ and note that, from our earlier manipulations, we know that

$$\begin{aligned} \mathbf{e}_j\partial_j &= \nabla, \quad \partial_j\partial_j = \nabla^2 \\ \mathbf{e}_i\frac{\partial}{\partial t}(\rho v_i) + \mathbf{e}_i\partial_j(\rho v_i v_j) &= \rho\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} = \rho\frac{D\mathbf{v}}{Dt} \end{aligned} \tag{16.25}$$

The result of this is to give us the equation

$$\rho\frac{D\mathbf{v}}{Dt} = -\nabla p + \eta\nabla^2\mathbf{v} + \left(\zeta + \frac{\eta}{3}\right)\nabla(\nabla\cdot\mathbf{v}) \tag{16.26}$$

This equation is the *full Navier–Stokes equation* for a viscous Newtonian fluid and we see that it involves the *two* coefficients of viscosity η and ζ. However we shall only do detailed calculations for a viscous *incompressible fluid* so that $\nabla\cdot\mathbf{v} = 0$; and we see at once that this makes the term containing ζ in 16.26 disappear leaving us with the simpler version of the Navier–Stokes equation that we have met before namely

$$\rho\frac{D\mathbf{v}}{Dt} = -\nabla p + \eta\nabla^2\mathbf{v} \tag{16.27}$$

and so we are back to having only one coefficient of viscosity which is, of course, η.

As an illustration of the viscosity in action we now look at the drag created as a viscous fluid flows past a solid boundary.

Example *The viscous drag on a flat plate*

We return to the problem of the flow between two flat plates one of which is moving cf. Fig. 12.

We already know that the solution of this problem is

$$\mathbf{v} = v_x\,\mathbf{i}, \quad v_x = S\frac{y}{h}\left(1 - \frac{y}{h}\right) + \frac{U}{h}y \tag{16.28}$$

where

$$S = -\frac{h^2}{2\eta}\frac{dp}{dx} = \text{ a constant} \tag{16.29}$$

The shear stress Q_{ij} has only one non-zero component namely Q_{12} and if we use 16.22 we have

$$Q_{12} = -\eta\left(\frac{\partial v_1}{\partial x^2} + \frac{\partial v_2}{\partial x^1}\right) \tag{16.30}$$

Now clearly we have the correspondences

$$(v_x, v_y, v_z) = (v_1, v_2, v_3), \quad (x, y, z) = (x^1, x^2, x^3) \tag{16.31}$$

So we know at once that

$$v_1 = S\frac{x^2}{h}\left(1 - \frac{x^2}{h}\right) + \frac{U}{h}x^2, \quad v_2 = v_3 = 0 \tag{16.32}$$

and so we easily compute that

$$\begin{aligned} Q_{12} &= -\eta\left(\frac{\partial v_1}{\partial x^2} + 0\right) \\ &= -\eta\left\{S\frac{1}{h}\left(1 - \frac{x^2}{h}\right) + S\frac{x^2}{h}\left(-\frac{1}{h}\right) + \frac{U}{h}\right\} \\ &= -\eta\left\{-\frac{h^2}{2\eta}\frac{dp}{dx^1}\frac{1}{h}\left(1 - \frac{x^2}{h}\right) - \frac{h^2}{2\eta}\frac{dp}{dx^1}\frac{x^2}{h}\left(-\frac{1}{h}\right) + \frac{U}{h}\right\} \\ &= \frac{h^2}{2}\frac{dp}{dx^1}\frac{1}{h}\left(1 - \frac{x^2}{h}\right) + \frac{h^2}{2}\frac{dp}{dx^1}\frac{x^2}{h}\left(-\frac{1}{h}\right) - \eta\frac{U}{h} \end{aligned} \tag{16.33}$$

Now the *viscous drag* is given by the contribution to Q_{12} which is left if we set the pressure gradient dp/dx^1 to zero: doing this gives just the term

$$\eta\frac{U}{h} = \text{ the viscous drag on the upper plate} \tag{16.34}$$

which we also note is the *only* part of Q_{12} which depends on the viscosity η.

§ 17. Dissipation of energy in a viscous incompressible fluid

We are already familiar with the expression for the kinetic energy T of an incompressible fluid since we met it in 10.1; in any case we know that

$$T = \frac{\rho}{2}\int_V \mathbf{v}^2\, d^3\mathbf{r} \tag{17.1}$$

The rate of dissipation of energy on the fluid is just $\dot{T}$ so our task is to calculate

$$\begin{aligned} \dot{T} &= \frac{\partial}{\partial t}\frac{\rho}{2}\int_V \mathbf{v}^2\, d^3\mathbf{r} \\ &= \frac{\rho}{2}\int_V \frac{\partial \mathbf{v}^2}{\partial t}\, d^3\mathbf{r}, \qquad \text{since } \rho \text{ is constant} \end{aligned} \tag{17.2}$$

We see that the main task is to calculate $\partial \mathbf{v}^2/\partial t$ and we set about doing this right away. We have

$$\begin{aligned}\frac{\rho}{2}\frac{\partial \mathbf{v}^2}{\partial t} &= \rho \mathbf{v}\cdot\frac{\partial \mathbf{v}}{\partial t} \\ &= -\rho\mathbf{v}\cdot\{(\mathbf{v}\cdot\nabla)\mathbf{v}\} - \mathbf{v}\cdot\nabla p + \eta\mathbf{v}\cdot\nabla^2\mathbf{v}, \quad \text{using 16.27}\end{aligned} \tag{17.3}$$

But

$$\begin{aligned}(\mathbf{v}\cdot\nabla)\mathbf{v} &= \nabla\left(\frac{\mathbf{v}^2}{2}\right) + (\nabla\times\mathbf{v})\times\mathbf{v} \\ \Rightarrow \mathbf{v}\cdot\{(\mathbf{v}\cdot\nabla)\mathbf{v}\} &= \mathbf{v}\cdot\nabla\left\{\frac{\mathbf{v}^2}{2}\right\}, \quad \text{since } \mathbf{v}\cdot\{(\nabla\times\mathbf{v})\times\mathbf{v}\} = 0\end{aligned} \tag{17.4}$$

Thus we can now write

$$\frac{\rho}{2}\frac{\partial \mathbf{v}^2}{\partial t} = -\rho\mathbf{v}\cdot\nabla\left\{\frac{\mathbf{v}^2}{2} + \frac{p}{\rho}\right\} + \eta\mathbf{v}\cdot\nabla^2\mathbf{v} \tag{17.5}$$

The next step requires us to do some manipulations on the viscous term $\eta\mathbf{v}\cdot\nabla^2\mathbf{v}$. First note that

$$\nabla^2 v_i = \partial_j\partial_j v_i \tag{17.6}$$

and also that since we have assumed that the fluid is incompressible we have $\nabla\cdot\mathbf{v} = \partial_j v_j = 0$ and so we can write

$$\begin{aligned}\nabla^2 v_i &= \partial_j\partial_j v_i = \partial_j(\partial_j v_i + \partial_i v_j) \\ &= \partial_j\tau_{ij}, \quad \text{where } \tau_{ij} = \partial_j v_i + \partial_i v_j\end{aligned} \tag{17.7}$$

Hence the viscous term $\eta\mathbf{v}\cdot\nabla^2\mathbf{v}$ is given by

$$\eta v_i\partial_j\tau_{ij} \tag{17.8}$$

With this information 17.5 becomes

$$\frac{\rho}{2}\frac{\partial \mathbf{v}^2}{\partial t} = -\rho\mathbf{v}\cdot\nabla\left\{\frac{\mathbf{v}^2}{2} + \frac{p}{\rho}\right\} + \eta v_i\partial_j\tau_{ij} \tag{17.9}$$

Now note that

$$\begin{aligned}\eta\partial_j(v_i\tau_{ij}) &= \eta\partial_j v_i\tau_{ij} + \eta v_i\partial_j\tau_{ij} \\ \Rightarrow \eta v_i\partial_j\tau_{ij} &= \eta\partial_j(v_i\tau_{ij}) - \eta\partial_j v_i\tau_{ij} \\ &= \eta\nabla\cdot(\theta_i\mathbf{e}_i) - \eta\partial_j v_i\tau_{ij}, \quad \text{where } \theta_i = v_i\tau_{ij}\end{aligned} \tag{17.10}$$

Now the latest version of our equation for $(\rho/2)\partial(\mathbf{v}^2)/\partial t$ is

$$
\begin{aligned}
\frac{\rho}{2}\frac{\partial \mathbf{v}^2}{\partial t} &= -\rho\mathbf{v}\cdot\nabla\left\{\frac{\mathbf{v}^2}{2}+\frac{p}{\rho}\right\}+\eta\nabla\cdot(\theta_i\mathbf{e}_i)-\eta\partial_j v_i\tau_{ij}\\
&= -\mathbf{v}\cdot\nabla\left\{\rho\left(\frac{\mathbf{v}^2}{2}+\frac{p}{\rho}\right)\right\}+\eta\nabla\cdot(\theta_i\mathbf{e}_i)-\eta\partial_j v_i\tau_{ij},\ \text{since } \rho \text{ is constant}
\end{aligned}
\tag{17.11}
$$

The last step in our workings is to use the vector identity

$$
\nabla\cdot(f\mathbf{A}) = \mathbf{A}\cdot\nabla f + f\nabla\cdot\mathbf{A} \tag{17.12}
$$

with

$$
\mathbf{A}=\mathbf{v}\quad\text{and}\quad f=\rho\left\{\frac{\mathbf{v}^2}{2}+\frac{p}{\rho}\right\} \tag{17.13}
$$

With this choice of f and $\mathbf{A}$ we find that

$$
\begin{aligned}
\mathbf{v}\cdot\nabla\left\{\rho\left(\frac{\mathbf{v}^2}{2}+\frac{p}{\rho}\right)\right\} &= \nabla\cdot\left\{\rho\left(\frac{\mathbf{v}^2}{2}+\frac{p}{\rho}\right)\mathbf{v}\right\}-rho\left(\frac{\mathbf{v}^2}{2}+\frac{p}{\rho}\right)\nabla\cdot\mathbf{v}\\
&= \nabla\cdot\left\{\rho\left(\frac{\mathbf{v}^2}{2}+\frac{p}{\rho}\right)\mathbf{v}\right\},\quad \text{since } \nabla\cdot\mathbf{v}=0
\end{aligned}
\tag{17.14}
$$

Substituting into our last equation for $(\rho/2)\partial(\mathbf{v}^2)/\partial t$ and combining all the 'div' terms into one we obtain the result that we were after, namely

$$
\frac{\rho}{2}\frac{\partial \mathbf{v}^2}{\partial t} = -\nabla\cdot\left\{\rho\left(\frac{\mathbf{v}^2}{2}+\frac{p}{\rho}\right)\mathbf{v}+\eta\theta_i\mathbf{e}_i\right\}-\eta\partial_j v_i\tau_{ij} \tag{17.15}
$$

Finally substituting 17.15 into 17.2 yields the equation

$$
\begin{aligned}
\dot{T} &= -\int_V \nabla\cdot\left\{\rho\left(\frac{\mathbf{v}^2}{2}+\frac{p}{\rho}\right)\mathbf{v}+\eta\theta_i\mathbf{e}_i\right\}dV-\eta\int_V \partial_j v_i\tau_{ij}dV\\
&= -\int_S \left\{\rho\left(\frac{\mathbf{v}^2}{2}+\frac{p}{\rho}\right)\mathbf{v}+\eta\theta_i\mathbf{e}_i\right\}\cdot\mathbf{dS}-\eta\int_V \partial_j v_i\tau_{ij}dV
\end{aligned}
\tag{17.16}
$$

where S is the surface of the volume V and we have used Gauss's divergence theorem. However we now impose the no slip boundary condition on the surface S which means that the entire integrand of the surface integral vanishes when $\mathbf{v}=0$ so we are left with the dissipation formula

$$
\dot{T} = -\eta\int_V \partial_j v_i\tau_{ij}dV \tag{17.17}
$$

and this formula was our goal.

We finish by remarking that we would expect, on physical grounds, that T should decrease with time as a viscous liquid flows. This means that $\dot{T}$ should be negative. In fact we can easily see that this is indeed correct for note that we can write $\partial_j v_i \tau_{ij}$ as

$$\begin{aligned} \partial_j v_i \tau_{ij} &= (\partial_j v_i)(\partial_j v_i + \partial_i v_j) \\ &= \frac{1}{2}(\partial_j v_i + \partial_i v_j)^2 \end{aligned} \tag{17.18}$$

where multiplying out the last expression explicitly and dividing by the 1/2 will soon convince the reader that the last equality is correct. This means that

$$\dot{T} = -\eta \int_V \frac{1}{2}(\partial_j v_i + \partial_i v_j)^2 dV \tag{17.19}$$

and since η is positive, and the integrand is now explicitly positive, one sees immediately that $\dot{T} < 0$ as conjectured.

§ 18. Assorted problems on inviscid and viscous fluids

Problem 1. *An infinite mass of inviscid fluid of constant density ρ is initially at rest and has a spherical cavity of radius*

$$a(t) = a + \cos(bt)$$

embedded in it where a and b are constants. The fluid moves radially outwards, the pressure at infinity is zero, and the velocity potential ϕ at a point a distance r from the centre of the cavity is given by

$$\phi(t) = \frac{f(t)}{r}$$

Calculate the pressure p on the surface of the cavity and find when p attains its maximum.

Solution sketch: The fluid is inviscid and there is a velocity potential and this suggests that we use Bernoulli's equation to calculate the pressure.

Hence we start with

$$-\frac{\partial \phi}{\partial t} + \frac{\mathbf{v}^2}{2} + K + \int \frac{dp}{\rho} = C(t) \tag{18.1}$$

Since $\phi(t) = f(t)/r$ and the pressure at ∞ is zero then taking $r \to \infty$ in Bernoulli's equation shows that $C(t) = 0$. Also here, $K = 0$, and the fluid is incompressible so we obtain

$$\begin{aligned} -\frac{\partial \phi}{\partial t} + \frac{\mathbf{v}^2}{2} + \frac{p}{\rho} = 0 \\ \Rightarrow p = \rho \left\{ \frac{\partial \phi}{\partial t} - \frac{\mathbf{v}^2}{2} \right\} \end{aligned} \tag{18.2}$$

The key idea of the solution is to realise that, *on the surface of the cavity* one can equate $da(t)/dt$ and the radial component of velocity $v_r = -\partial\phi/\partial r$. More explicitly we have

$$\begin{aligned} \frac{da(t)}{dt} &= - \left.\frac{\partial \phi}{\partial r}\right|_{r=a(t)}, \quad \text{with } a(t) = a + \cos(bt) \\ \Rightarrow -b\sin(bt) &= \frac{f(t)}{a(t)^2}, \quad \text{using } \phi(t) = \frac{f(t)}{r} \\ \Rightarrow f(t) &= -b\sin(bt)(a+\cos(bt))^2 \end{aligned} \tag{18.3}$$

Now all the ingredients are in place to calculate the pressure p since we know $f(t)$ and so can calculate $\partial\phi/\partial t$, in addition $\mathbf{v}^2 = \dot{a}^2(t)$; substituting all this into our expression above for p we find that

$$\begin{aligned} p = \rho \Bigg\{ & -\frac{b^2\sin^2(bt)(a+\cos(bt))^4}{2((a+\cos(bt))^4} - \frac{b^2\cos(bt)}{(a+\cos(bt))}(a+\cos(bt))^2 \\ & + \frac{2b^2\sin^2(bt)(a+\cos(bt))}{(a+\cos(bt))} \Bigg\} \end{aligned} \tag{18.4}$$

which simplifies to

$$p = \rho \left\{ \frac{3b^2}{2}\sin^2(bt) - b^2\cos(bt)(a+\cos(bt)) \right\} \tag{18.5}$$

and so we have now calculated the pressure on the surface of the cavity. We are also asked to find when p attains its maximum: to do this we simply compute dp/dt and equate it to zero. The reader will easily find that this gives the equation

$$\cos(bt) = -\frac{a}{5} \tag{18.6}$$

so that p attains its maximum when

$$t = \frac{1}{b}\cos^{-1}\left(-\frac{a}{5}\right) \tag{18.7}$$

Problem 2. *Incompressible viscous fluid flows steadily parallel to the axis in the space between two fixed coaxial cylinders of radii a and b, $a < b$, under a constant pressure gradient P. Show that the velocity* $\mathbf{v}$ *is given by*

$$\mathbf{v} = \frac{P}{4\eta}\left[r^2 - \frac{(a^2-b^2)}{\ln(a/b)}\ln r - a^2 + \frac{(a^2-b^2)}{\ln(a/b)}\ln(a)\right]\mathbf{e}$$

where $\mathbf{e}$ *is a unit vector in the direction of the axis.*

Solution sketch: This time we have a viscous fluid so we must use the Navier–Stokes equation which, since there is no body force and the fluid is also incompressible, takes the form

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v} \tag{18.8}$$

We shall take the common axis of the cylinders to coincide with the z-axis so that $\mathbf{e} = \mathbf{k}$. But the flow steady and parallel to the axis so

$$\frac{\partial \mathbf{v}}{\partial t} = 0, \qquad \text{and } \mathbf{v} = v_z\,\mathbf{k} \tag{18.9}$$

Incompressibility plus parallel flow means that

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{0} \tag{18.10}$$

for exactly the same reasons as it did in some earlier works, cf. eq. 15.7 for example. So we are left with the task of solving

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) v_z(x,y) = \frac{P}{\eta} \tag{18.11}$$

To do this we use cylindrical coordinates (r,θ,z) in which we have

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) \tag{18.12}$$

So we have

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) v_z(r,\theta) = \frac{P}{\eta} \tag{18.13}$$

Circular symmetry means that $v_z(r,\theta)$ is independent of θ so that

$$\frac{\partial v_z}{\partial \theta} = 0 \tag{18.14}$$

and we are left with

$$\begin{aligned}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)v_z &= \frac{P}{\eta}\\ \Rightarrow \left(r\frac{\partial^2}{\partial r^2}+\frac{\partial}{\partial r}\right)v_z &= \frac{Pr}{\eta}\end{aligned} \tag{18.15}$$

The obvious thing to do is to set $u = \partial v_z/\partial r$ thereby obtaining the equation

$$\begin{aligned} r\frac{\partial u}{\partial r}+u &= \frac{Pr}{\eta}\\ \Rightarrow \frac{\partial (ru)}{\partial r} &= \frac{Pr}{\eta}\\ \Rightarrow ru &= \frac{Pr^2}{2\eta}+A\\ \Rightarrow u &= \frac{Pr}{2\eta}+\frac{A}{r}\\ \Rightarrow v_z &= \frac{Pr^2}{4\eta}+A\ln r+B \end{aligned} \tag{18.16}$$

where A and B are constants. To find A and B we just implement the no slip boundary conditions on the surfaces of the inner and outer cylinders. In other words we demand that v_z vanish at $r = a$ and at $r = b$; this easily determines A and B and one finds that

$$\mathbf{v} = \frac{P}{4\eta}\left[r^2 - \frac{(a^2-b^2)}{\ln(a/b)}\ln r - a^2 + \frac{(a^2-b^2)}{\ln(a/b)}\ln(a)\right]\mathbf{e}$$

as required.

Problem 3. *A compressible inviscid fluid with $p = k\rho^\gamma$ flows steadily round a solid core of radius b with $|\mathbf{v}|$ inversely proportional to the distance from the centre of the core. Obtain an equation for $\mathbf{v}$ of the form*

$$\frac{2c^2}{(\gamma-1)} + \mathbf{v}^2 = B$$

where B is a constant and $c^2 = dp/d\rho$. Show also that there is a minimum value b_0 that can be specified for the core radius.

Solution sketch: This is a two dimensional calculation where we treat the flow of a compressible fluid round a solid core whose radius we take to be

$$b \tag{18.17}$$

We want to show that the velocity satisfies the equation

$$\frac{2c^2}{(\gamma-1)} + \mathbf{v}^2 = \frac{A^2}{b_0^2} \tag{18.18}$$

where b_0 is the minimum value that can be taken for the core radius b, and γ and c have their usual meanings for the case of adiabatic compressible flow.

The appropriate equation to use is Bernoulli's equation so that we have

$$-\frac{\partial\phi}{\partial t} + K + \frac{\mathbf{v}^2}{2} + \int \frac{dp}{\rho} = C(t) \tag{18.19}$$

Now we know that the flow is *steady* and that it is circulating round the core with velocity components given by

$$\begin{aligned} v_\theta &= \frac{A}{r}, \quad (A \text{ a constant}),\ v_r = 0 \\ \text{and } \mathbf{v} &= v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta \end{aligned} \tag{18.20}$$

where r is the distance from the centre of the core. The fact that the flow is steady means that

$$\frac{\partial\phi}{\partial t} = 0 \tag{18.21}$$

and that $C(t) = C$ a constant independent of t. We also take $K = 0$ since there is no body force. Using these facts Bernoulli's equations gives us

$$\frac{1}{2}\frac{A^2}{r^2} + \int \frac{dp}{\rho} = C \tag{18.22}$$

at a distance r from the centre of the core.

Now the flow is compressible and

$$p = k\rho^\gamma \tag{18.23}$$

so that we now obtain

$$\frac{A^2}{2r^2} + \frac{\gamma k}{(\gamma-1)}\rho^{\gamma-1} = C \tag{18.24}$$

Recalling that the speed of sound c (which of course is not constant) is given by

$$c^2 = \gamma k \rho^{\gamma-1} \tag{18.25}$$

Hence we can now write

$$\frac{2c^2}{(\gamma-1)} + \frac{A^2}{r^2} = 2C \tag{18.26}$$

If r shrinks to b, the radius of the core this equation becomes

$$\frac{2c_b^2}{(\gamma-1)} + \frac{A^2}{b^2} = 2C \tag{18.27}$$

where we write c as c_b to emphasize that this the value of c at $r = b$. Now the *maximum* value that the term A^2/b^2 can have is attained when $c_b = 0$ i.e. when

$$2C = \frac{A^2}{b^2} \tag{18.28}$$

but this is also the *minimum value* of the core radius b. If we denote this value of b by b_0, then we have

$$2C = \frac{A^2}{b_0^2} \tag{18.29}$$

and, at a general distance r from the core centre, we have the equation

$$\frac{2c^2}{(\gamma-1)} + \mathbf{v}^2 = \frac{A^2}{b_0^2} \tag{18.30}$$

which is what we wished to derive.

Problem 4. *A stationary solid sphere of radius a is placed in an incompressible fluid. The velocity potential Φ is given, in spherical polar coordinates, with origin at the centre of the sphere by*

$$\Phi = U\cos\theta\left(r + \frac{1}{2}\frac{a^3}{r^2}\right), \quad r \geq a.$$

By using Bernoulli's equation find the pressure p on the sphere's surface. Show that on the equator $r = a$, $\theta = \pi/2$ p is a minimum, show further that this minimum pressure is zero if

$$U = \sqrt{\frac{8p_\infty}{5\rho}}$$

Where p_∞ is the pressure at infinity.

Solution sketch: For this problem, as indicated by their appearance in the wording of the problem itself, we must use spherical polar coordinates (r, θ, ϕ). Doing this we have three velocity components v_r, v_θ and v_ϕ and these are given by

$$\begin{aligned} v_r &= -\frac{\partial \Phi}{\partial r} = -U\cos\theta\left(1 - \frac{a^3}{r^3}\right) \\ v_\theta &= -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} = U\sin\theta\left(1 + \frac{1}{2}\frac{a^3}{r^3}\right) \\ v_\phi &= 0 \end{aligned} \tag{18.31}$$

We use Bernoulli's equation to find p. Bernoulli's equation gives us

$$\frac{p}{\rho} + \frac{\mathbf{v}^2}{2} = C \tag{18.32}$$

where C is a time independent constant. But $\mathbf{v} = v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta$ so

$$\mathbf{v}^2 = v_r^2 + v_\theta^2 = U^2\cos^2\theta\left(1 - \frac{a^3}{r^3}\right)^2 + U^2\sin^2\theta\left(1 + \frac{1}{2}\frac{a^3}{r^3}\right)^2 \tag{18.33}$$

Hence, on the sphere's surface where $r = a$, we have

$$\mathbf{v}^2 = \left(\frac{3}{2}U\sin\theta\right)^2 \tag{18.34}$$

while at ∞, where $r \to \infty$, we have $\mathbf{v}^2 = U^2$ and $p = p_\infty$.

So using Bernoulli's equation to equate things at $r = a$ and $r = \infty$ we get

$$\begin{aligned} \frac{p}{\rho} + \frac{1}{2}\left(\frac{3}{2}U\sin\theta\right)^2 &= \frac{p_\infty}{\rho} + \frac{1}{2}U^2 \\ \Rightarrow p = p_\infty &+ \frac{1}{2}\rho U^2\left(1 - \frac{9}{4}\sin^2\theta\right) \end{aligned} \tag{18.35}$$

so the pressure p has been found.

To find the minimum pressure we solve

$$\frac{dp}{d\theta} = 0 \tag{18.36}$$

which yields the result that

$$\theta = \frac{\pi}{2} \tag{18.37}$$

and we readily calculate that, at this minimum, p takes the value p_{min} where

$$p_{min} = p_\infty - \frac{5}{8}\rho U^2 \tag{18.38}$$

and it is clear immediately that $p_{min} = 0$ if

$$U = \sqrt{\frac{8p_\infty}{5\rho}} \tag{18.39}$$

as required.

Problem 5. *Viscous fluid flows steadily under no forces along a pipe of length l in the direction of the z-axis. The cross-section of the pipe is the ellipse $x^2/4 + y^2 = 1$ and a pressure difference P is maintained across the ends. If the velocity $\mathbf{v}$ is such that*

$$\mathbf{v} = w\mathbf{k}$$

Find a sol$\underline{^n}$ in the form

$$w = \left(x^2 + 4y^2 - 4\right)\ f(x, y)$$

and show that the total mass of fluid Q delivered through the pipe/second is given by

$$Q = \frac{2\pi P\rho}{5\eta l}$$

Solution sketch: This is the same as the example we dealt with when considering Poiseuille flow the only difference is that the pipe now has an elliptical cross section. This means that we can skip ahead to the equation for w which is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) w(x, y) = \frac{1}{\eta}\frac{\partial p}{\partial z} = C, \qquad \text{a constant} \tag{18.40}$$

But

$$\begin{aligned} \frac{1}{\eta}\frac{\partial p}{\partial z} &= C \\ \Rightarrow p &= \eta C z + B \end{aligned} \tag{18.41}$$

Now, using $w = (x^2 + 4y^2 - 4)f$, we find that f obeys the equation

$$(x^2 + 4y^2 - 4)(f_{xx} + f_{yy}) + 4xf_x + 16yf_y + 10f = C \tag{18.42}$$

But this is clearly a solution if f is just a constant, say

$$f = D \tag{18.43}$$

where we must choose

$$D = \frac{C}{10} \tag{18.44}$$

So we now have our solution in the form

$$\mathbf{v} = (x^2 + 4y^2 - 4)\frac{C}{10}\,\mathbf{k} \tag{18.45}$$

we also know that the pressure difference across the pipe is P and

$$P = p(z = 0) - p(z = l) = -\eta Cl \tag{18.46}$$

so we have found C: it is given by

$$C = -\frac{P}{\eta l} \tag{18.47}$$

and the final expression for the velocity $\mathbf{v}$ is

$$\mathbf{v} = -(x^2 + 4y^2 - 4)\frac{P}{10\eta l}\,\mathbf{k} \tag{18.48}$$

The tricky part is to calculate Q because the elliptical cross section makes things difficult. However the basic form for Q is still an integral over shells, it is just that they are now elliptical shells instead of circular ones. hence we can write, in analogy with what we wrote in 15.22

$$Q = \int_{\text{elliptical shells}} \rho|\mathbf{v}|dxdy \tag{18.49}$$

It is easier to consider a general ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{18.50}$$

and then specialise to the one that we have which has $a = 2$ and $b = 1$ so we shall do things that way. The idea is to change variables so that the ellipse becomes a circle: we accomplish this by changing from (x, y) to (x', y') where

$$x = ax', \quad y = by' \tag{18.51}$$

The equation of the ellipse then becomes

$$\begin{aligned} &\frac{(ax')^2}{a^2} + \frac{(by')^2}{b^2} = 1 \\ &\Rightarrow (x')^2 + (y')^2 = 1 \end{aligned} \tag{18.52}$$

so the ellipse has become a circle of radius 1. Hence all we have to do, is to implement the change of variable in the integral and then integrate over circular shells whose radius varies in size from 0 to 1. Implementing the change of variable in the integral is easy: one knows, in this simple case, that

$$dxdy = abdx'dy' \tag{18.53}$$

and $dx'dy'$ for a circular shell of radius λ is given by

$$dx'dy' = 2\pi\lambda d\lambda \tag{18.54}$$

So the integral for Q is given by

$$Q = \rho \int_0^1 |\mathbf{v}| ab 2\pi\lambda d\lambda \tag{18.55}$$

But, now we set $a = 2$ and $b = 1$, note that

$$|\mathbf{v}| = -(x^2 + 4y^2 - 4)\frac{P}{10\eta l} \tag{18.56}$$

and in terms of x', y' which satisfy

$$(x')^2 + (y')^2 = \lambda^2 \tag{18.57}$$

we get

$$Q = -\int_0^1 \rho(4\lambda^2 - 4)\frac{P}{10\eta l} 4\pi\lambda d\lambda \tag{18.58}$$

which one easily checks gives

$$Q = \frac{2\pi P\rho}{5l\eta} \tag{18.59}$$

as claimed.

Problem 6. *A complex potential W is given by*

$$W = Uz + \frac{Ua^2}{z} + ik\ln z, \quad U, a > 0.$$

and represents the 2-dim flow of an incompressible fluid of density ρ. Show that

(a) The flow is due to a cylinder of radius a, centre the origin placed in a uniform stream.
(b) The circulation about the cylinder is $2\pi k$.
(c) The cylinder experiences a lifting force/unit length of $2\pi\rho kU$.

Solution sketch: For (a) we just need to show that the circle $r = a$ is a streamline. Put

$$z = re^{i\theta} \tag{18.60}$$

and we readily compute that

$$\begin{aligned} W &= Ure^{i\theta} + \frac{Ua^2}{r}e^{-i\theta} + ik(\ln r + i\theta) \\ &= \phi + i\psi \end{aligned} \tag{18.61}$$

So we see that

$$\psi = Ur\sin\theta - \frac{Ua^2}{r}\sin\theta + k\ln r \tag{18.62}$$

Now if we set $r = a$ in the expression for ψ we find that

$$\begin{aligned} \psi &= Ua\sin\theta - Ua\sin\theta + k\ln a \\ &= k\ln a = \text{ a constant} \end{aligned} \tag{18.63}$$

so $r = a$ is a streamline as required.

For (b) one simply has to calculate

$$\int_C \mathbf{v}\cdot\mathbf{dl} \tag{18.64}$$

where C is a circle of radius b with $b > a$. Now, on C

$$\begin{aligned} \mathbf{dl} &= b\,d\theta\,\mathbf{e}_\theta \\ \mathbf{v} &= v_r\,\mathbf{e}_r + v_\theta\,\mathbf{e}_\theta = -\nabla\phi \\ &= -\frac{\partial\phi}{\partial r}\,\mathbf{e}_r - \frac{1}{r}\frac{\partial\phi}{\partial\theta}\,\mathbf{e}_\theta \end{aligned} \tag{18.65}$$

This gives

$$
\begin{aligned}
\mathbf{v}\cdot\mathbf{dl} &= -\frac{1}{b}\left.\frac{\partial\phi}{\partial\theta}\right|_{r=b} b\,d\theta = -\frac{\partial\phi}{\partial\theta}d\theta \\
\Rightarrow \Gamma &= -\int_0^{2\pi}\frac{\partial\phi}{\partial\theta}d\theta \\
&= -\phi(2\pi)+\phi(0)
\end{aligned}
\tag{18.66}
$$

But it is simple to compute ϕ from W obtaining the result that

$$
\phi = Ur\cos\theta + \frac{Ua^2}{r}\cos\theta - k\theta \tag{18.67}
$$

from which we find that

$$
\Gamma = 2\pi k \tag{18.68}
$$

which is what we were supposed to get.

Finally for (c) and the calculation of the force per unit length we use Bernoulli's equation

$$
\frac{p}{\rho} + \frac{\mathbf{v}^2}{2} = C \tag{18.69}
$$

and on a streamline[13]

$$\mathbf{v}^2 d\bar{z} = \left(\frac{dW}{dz}\right)^2 dz \tag{18.71}$$

so we obtain

$$\left(\frac{dW}{dz}\right)^2 dz = \left(2C - \frac{2p}{\rho}\right) d\bar{z} \tag{18.72}$$

Then we integrate this expression around the cylinder, i.e. around the circle C above. We get

$$\begin{aligned}\int_C \left(\frac{dW}{dz}\right)^2 dz &= \int_C \left(2C - \frac{2p}{\rho}\right) d\bar{z} \\ &= 0 - \frac{2}{\rho}\int_C p\, d\bar{z} \\ &= -\frac{2}{\rho}\int p\, dx + \frac{2i}{\rho}\int p\, dy\end{aligned} \tag{18.73}$$

[13] Here we use the fact that the velocity is tangential on a streamline C so that $(v_x + iv_y)/|\mathbf{v}|$ is a unit tangent vector to C. Thus, if dl is an element of length along C, we have

$$dz = dl\frac{(v_x + iv_y)}{|\mathbf{v}|}$$

so that also

$$d\bar{z} = dl\frac{(v_x - iv_y)}{|\mathbf{v}|}$$

from which we deduce that

$$\begin{aligned}d\bar{z} &= \frac{d\bar{z}}{dz}dz \\ &= \frac{(v_x - iv_y)}{(v_x + iv_y)}dz\end{aligned}$$

However, recall that $|\mathbf{v}|^2 = v_x^2 + v_y^2 = (v_x - iv_y)(v_x - iv_y)$, and

$$\begin{aligned}\frac{dW}{dz} &= -v_x + iv_y \\ \Rightarrow |\mathbf{v}|^2 d\bar{z} &= (v_x + iv_y)(v_x - iv_y)\frac{(v_x - iv_y)}{(v_x + iv_y)}dz \\ \Rightarrow |\mathbf{v}|^2 d\bar{z} &= (v_x - iv_y)^2 dz \\ &= \left(\frac{dW}{dz}\right)^2 dz\end{aligned} \tag{18.70}$$

as claimed.

Let us define F_x and F_y by

$$F_x = -\int p\,dy, \quad \text{and} \quad F_y = \int p\,dx \tag{18.74}$$

then these are known as the *lifts or thrusts per unit length*. So now we have

$$\int_C \left(\frac{dW}{dz}\right)^2 dz = -\frac{2}{\rho}F_y - \frac{2i}{\rho}F_x \tag{18.75}$$

The last step is to calculate the integral on the LHS using Cauchy's residue theorem. We have

$$\begin{aligned} W &= Uz + \frac{Ua^2}{z} + ik\ln z \\ \Rightarrow \frac{dW}{dz} &= U - \frac{Ua^2}{z^2} + \frac{ik}{z} \\ \Rightarrow \left(\frac{dW}{dz}\right)^2 &= U^2 - \frac{2U^2a^2}{z^2} + \frac{U^2a^4}{z^4} + \frac{2ikU}{z} - \frac{2ikUa^2}{z^3} - \frac{k^2}{z^2} \end{aligned} \tag{18.76}$$

Cauchy's theorem says that the only term in the integrand which contributes is the $1/z$ pole term

$$\frac{2ikU}{z} \tag{18.77}$$

and its contribution is $(2\pi i)(2ikU)$. Hence we have done the integral around C and deduced that

$$\begin{aligned} \int_C \left(\frac{dW}{dz}\right)^2 dz &= (2\pi i)(2ikU) = -4\pi kU \\ \Rightarrow -4\pi kU &= -\frac{2}{\rho}F_y - \frac{2i}{\rho}F_x \\ \Rightarrow F_x &= 0, \quad F_y = 2\pi k\rho U \end{aligned} \tag{18.78}$$

and this is the required lifting force per unit length exerted on the cylinder by the flow.

Problem 7. *Incompressible viscous fluid rotates in the space between two infinite coxial rotating cylinders of radii a and b and angular velocities Ω_1 and Ω_2 respectively ($a < b$). Find the resulting fluid velocity* $\mathbf{v}$. *Also, if $b = 2a$, $\Omega_1 = -2\Omega$ and $\Omega_2 = \Omega$, show that the fluid is at rest at a distance $\sqrt{2}a$ from the axis of the two cylinders.*

Solution sketch: This is a viscous fluid so we must use the Navier–Stokes equation again. The geometry of the situation is crying out for us to use cylindrical coordinates and so we shall do so.

Let (r, θ, z) be the cylindrical coordinates and $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ be their associated orthonormal basis vectors. Note first that, in these cylindrical coordinates, we have

$$\begin{aligned}(\mathbf{v}\cdot\nabla)\mathbf{v} &= \left(v_r\partial_r + \frac{v_\theta}{r}\partial_\theta + v_z\partial_z\right)(v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta + v_z\mathbf{e}_z) \\ &= \frac{v_\theta}{r}\partial_\theta(v_\theta\mathbf{e}_\theta), \qquad \text{since fluid is rotating so } v_r = v_z{=}0\end{aligned} \tag{18.79}$$

Further, by cylindrical symmetry, and the fact that the cylinders are infinitely long, we can infer that v_θ is independent of θ and z respectively. Hence we write

$$v_\theta = v_\theta(r) \tag{18.80}$$

So now we have [14]

$$\begin{aligned}(\mathbf{v}\cdot\nabla)\mathbf{v} &= \frac{v_\theta}{r}\partial_\theta(v_\theta\mathbf{e}_\theta) \\ &= \frac{v_\theta^2(r)}{r}\partial_\theta\mathbf{e}_\theta \\ \Rightarrow (\mathbf{v}\cdot\nabla)\mathbf{v} &= -\frac{v_\theta^2(r)}{r}\mathbf{e}_r, \qquad \text{since } \partial_\theta\mathbf{e}_\theta = -\mathbf{e}_r\end{aligned} \tag{18.82}$$

Consider next the pressure p which also only depends on r so that $p = p(r)$. Hence ∇p reduces to just

$$\nabla p = \frac{\partial p}{\partial r}\mathbf{e}_r \tag{18.83}$$

Thus the Navier–Stokes equation

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v} \tag{18.84}$$

[14] To see why $\partial_\theta\mathbf{e}_\theta = -\mathbf{e}_r$ we reason as follows: $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ are an orthonormal triad with $\mathbf{e}_z$ (which is identical to $\mathbf{k}$) constant then, denoting ∂_θ by a dot, we have

$$\begin{aligned}\mathbf{e}_z\times\mathbf{e}_r &= \mathbf{e}_\theta \\ \Rightarrow \mathbf{e}_z\times\dot{\mathbf{e}}_r &= \dot{\mathbf{e}}_\theta \\ \text{but}\quad \dot{\mathbf{e}}_r &= \mathbf{e}_\theta \\ \Rightarrow \dot{\mathbf{e}}_\theta &= \mathbf{e}_z\times\mathbf{e}_\theta \\ &= -\mathbf{e}_r\end{aligned} \tag{18.81}$$

becomes

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{v_\theta^2}{r} \tag{18.85}$$

in the $\mathbf{e}_r$ direction and

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)(v_\theta(r)\mathbf{e}_\theta) = 0 \tag{18.86}$$

in the $\mathbf{e}_\theta$ direction. We now note that since $\dot{\mathbf{e}}_\theta = -\mathbf{e}_r$ then

$$\ddot{\mathbf{e}}_\theta = -\mathbf{e}_\theta \tag{18.87}$$

so the equation of motion in the $\mathbf{e}_\theta$ direction becomes

$$\begin{aligned} \left(\frac{\partial^2 v_\theta(r)}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta(r)}{\partial r} - \frac{1}{r^2}v_\theta(r)\right)\mathbf{e}_\theta = 0 \\ \Rightarrow r^2\frac{\partial^2 v_\theta(r)}{\partial r^2} + r\frac{\partial v_\theta(r)}{\partial r} - v_\theta(r) = 0 \end{aligned} \tag{18.88}$$

and this is recognisable as an equation of Euler type with solutions of the form r^n for some n. The reader can easily check that the allowed values of n are $n = \mp 1$. Thus the general solution is

$$v_\theta = Ar + \frac{B}{r}, \quad A \text{ and } B \text{ constants} \tag{18.89}$$

The no slip boundary conditions that fix A and B are that

$$\begin{aligned} v_\theta = a\Omega_1, &\quad \text{on inner cylinder, i.e. when } r = a \\ v_\theta = b\Omega_2, &\quad \text{on outer cylinder, i.e. when } r = b \end{aligned} \tag{18.90}$$

It is routine to verify that this gives

$$A = \frac{(b^2\Omega_2 - a^2\Omega_1)}{(b^2 - a^2)}, \quad B = \frac{(\Omega_1 - \Omega_2)}{(b^2 - a^2)}a^2 b^2 \tag{18.91}$$

So $\mathbf{v}$ is now completely determined.

To see when $\mathbf{v}$ can vanish we simply demand that

$$\begin{aligned} v_\theta &= 0 \\ \Rightarrow Ar &= -\frac{B}{r} \\ \Rightarrow r^2 &= -\frac{B}{A} \end{aligned} \tag{18.92}$$

and now if we set $b = 2a$ and $\Omega_2 = \Omega_1$, $\Omega_1 = -2\Omega$ we find at once that

$$r = \sqrt{2}a \tag{18.93}$$

as it should.

Problem 8. *Calculate the pressure difference between the inner and outer cylinders in the preceding example*

Solution sketch: To calculate the pressure difference in the previous problem we go to the equation of motion in the $\mathbf{e}_r$ direction which we recall is

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{v_\theta^2}{r} \tag{18.94}$$

But now we know v_θ so we have

$$\begin{aligned}
\frac{1}{\rho}\frac{\partial p}{\partial r} &= \frac{1}{r}\left(Ar + \frac{B}{r}\right)^2 \\
\Rightarrow \frac{1}{\rho}\frac{\partial p}{\partial r} &= A^2 r + \frac{2AB}{r} + \frac{B^2}{r^3} \\
\Rightarrow p(r) &= \rho\left(\frac{A^2 r^2}{2} + 2AB\ln r - \frac{B^2}{2r^2} + C\right), \quad C \text{ a constant}
\end{aligned} \tag{18.95}$$

So the desired pressure difference is simply the difference

$$p(b) - p(a) \tag{18.96}$$

which is a piece of arithmetic we leave to the reader; note that the value of C is immaterial since it will cancel in this difference.

Problem 9. *Examine again the problem of the first example and show that the pressure can be written in the form*

$$p = p_0 + \frac{\rho}{2}(3\dot{R}^2 + 2R\ddot{R})$$

where $R(t)$ is the radius of the bubble.

Solution sketch: Recall from the first problem that, if $R(t)$ is the radius of the bubble, we had

$$\dot{R}(t) = -\left.\frac{\partial \phi}{\partial r}\right|_{r=R(t)} \tag{18.97}$$

But

$$\frac{\partial\phi}{\partial r} = -\frac{f(t)}{r^2}$$
$$\Rightarrow \dot{R}(t) = \frac{f(t)}{R^2(t)} \tag{18.98}$$
$$\Rightarrow f = R^2(t)\dot{R}(t)$$

Thus Bernoulli's equation gives

$$-\frac{1}{R}\frac{d}{dt}(R^2\dot{R}) + \frac{\mathbf{v}^2}{2} + \frac{p}{\rho} = \frac{p_o}{\rho} \tag{18.99}$$

where note we have written the constant as p_0/ρ, p_0 being the pressure at infinity. Remember that on the surface of the bubble $|\mathbf{v}|$ equals $\dot{R}$ so Bernoulli gives

$$-\frac{1}{R}\frac{d}{dt}(R^2\dot{R}) + \frac{\dot{R}^2}{2} + \frac{p}{\rho} = \frac{p_o}{\rho}$$
$$\Rightarrow p = p_0 + \rho\left(\frac{1}{R}\frac{d}{dt}(R^2\dot{R}) - \frac{\dot{R}^2}{2}\right) \tag{18.100}$$
$$\Rightarrow p = p_0 + \frac{\rho}{2}\left(3\dot{R}^2 + 2R\ddot{R}\right)$$

and we are finished.

§ 19. A simple example of waves in a liquid

The complex potential method we studied earlier can be used to study wave motion in a liquid. In particular it can be employed to describe *surface waves* in a tank of liquid. The bulk of the treatment is in the example that now follows.

Example *A wave travelling from left to right*

We take an infinitely long rectangular tank filled with liquid to a depth h. Note carefully that the y-axis coincides with the *vertical direction* (we cannot use z as is usual since we are using z as the complex variable $z = x + iy$).

The complex potential W that describes the waves is given by

$$W = A\cos\left\{\frac{2\pi}{\lambda}(z + ih - ct)\right\}, \quad (z = x + iy) \tag{19.1}$$

and we now decompose W into real and imaginary parts in the usual manner giving

$$W = \phi + i\psi \tag{19.2}$$

So

$$\begin{aligned}\phi &= \text{Re}\, W \\ &= A\cos\left\{\frac{2\pi}{\lambda}(x - ct)\right\}\cosh\left\{\frac{2\pi}{\lambda}(y + h)\right\}\end{aligned} \tag{19.3}$$

where we used the fact that

$$\begin{aligned}\cos(a + ib) &= \frac{1}{2}(e^{i(a+ib)} + e^{-i(a+ib)}) \\ &= \frac{1}{2}(e^{ia}e^{-b} + e^{-ia}e^{b}) \\ &= \frac{1}{2}\left\{(\cos a + i\sin a)e^{-b} + (\cos a - i\sin a)e^{b}\right\} \\ &= \cos a\frac{e^{b} + e^{-b}}{2} - i\sin a\frac{e^{b} - e^{-b}}{2} \\ &= \cos a\cosh b - i\sin a\sinh b\end{aligned} \tag{19.4}$$

and we took

$$a + ib = \frac{2\pi}{\lambda}(z + ih - ct) = \frac{2\pi}{\lambda}(x + iy + ih - ct) \tag{19.5}$$

and chose

$$a = x - ct \quad \text{and} \quad b = y + h \tag{19.6}$$

Similarly the stream function ψ is given by

$$\begin{aligned} \psi &= \operatorname{Im} W \\ &= -A \sin\left\{\frac{2\pi}{\lambda}(x-ct)\right\} \sinh\left\{\frac{2\pi}{\lambda}(y+h)\right\} \end{aligned} \tag{19.7}$$

Now we resort to Bernoulli's equation. We also note that, because the complex potential W depends on t, the flow will not be steady. In any case Bernoulli's equation gives us

$$-\frac{\partial \phi}{\partial t} + gy + \frac{\mathbf{v}^2}{2} + \frac{p}{\rho} = C \quad (C \text{ a constant}) \tag{19.8}$$

where g is the acceleration due to gravity. Now we assume that we have waves on the surface of an almost static liquid so that we can neglect $\mathbf{v}^2$ giving

$$-\frac{\partial \phi}{\partial t} + gy + + \frac{p}{\rho} = C \tag{19.9}$$

The wave-shaped surface of the liquid is a surface of constant (atmospheric) pressure p_A so that on this wave surface y satisfies

$$\begin{aligned} &-\frac{\partial \phi}{\partial t} + gy + + \frac{p_A}{\rho} = C \\ \Rightarrow &-\frac{\partial \phi}{\partial t} + gy = C - \frac{p_A}{\rho} = D, \text{ say} \end{aligned} \tag{19.10}$$

Using our expression for ϕ above this immediately gives us a formula for y, namely

$$y = \frac{2\pi c A}{g\lambda} \sin\left\{\frac{2\pi}{\lambda}(x-ct)\right\} \cosh\left\{\frac{2\pi}{\lambda}(y+h)\right\} + \frac{D}{g} \tag{19.11}$$

Next we *assume* that the wave is small and that y varies by a small amount so that we can neglect the variation of the quantity

$$\cosh\left\{\frac{2\pi}{\lambda}(y+h)\right\} \tag{19.12}$$

and approximate it by its value at $y = 0$, i.e. by

$$\cosh\frac{2\pi h}{\lambda} \tag{19.13}$$

Now if we take the undisturbed surface of the liquid to correspond to $y = 0$, then setting D to zero makes y coincide with the *displacement* of the free surface of the liquid from $y = 0$. So, with our approximation, the equation

$$y = \frac{2\pi cA}{g\lambda} \cosh \frac{2\pi h}{\lambda} \sin \frac{2\pi}{\lambda}(x - ct) \tag{19.14}$$

represents a wave of wavelength λ, velocity c, and amplitude

$$\frac{2\pi cA}{g\lambda} \tag{19.15}$$

It is clear, too, that the wave travels from left to right along the x-axis.

Example *A standing wave*

If we modify W slightly, but otherwise retain the facts and assumptions above, we can describe a standing wave.

To this end choose

$$W = A\cos\left\{\frac{2\pi}{\lambda}(z + ih)\right\}\cos\frac{2\pi ct}{\lambda}, \quad (z = x + iy) \tag{19.16}$$

Now if we use

$$\cos a \cos b = \frac{1}{2}\left\{\cos(a + b) + \cos(a - b)\right\} \tag{19.17}$$

we can write W as

$$W = \frac{A}{2}\cos\left\{\frac{2\pi}{\lambda}(z + ih + ct)\right\} + \frac{A}{2}\cos\left\{\frac{2\pi}{\lambda}(z + ih - ct)\right\} \tag{19.18}$$

and this leads, by exactly the same sequence of steps in the previous example, to an expression for y which is

$$y = \frac{\pi cA}{g\lambda}\cosh\frac{2\pi h}{\lambda}\sin\frac{2\pi}{\lambda}(x + ct) + \frac{\pi cA}{g\lambda}\cosh\frac{2\pi h}{\lambda}\sin\frac{2\pi}{\lambda}(x - ct) \tag{19.19}$$

This is a superposition of a left-right moving and a right-left moving wave of identical amplitudes, wavelengths and velocities. Hence it is a standing wave.

§ 20. The wind chill factor

The formula, sometimes attributed to Hill, for heat loss from human skin at temperature into *dry air* is

$$H = \left(10.45 + 10\sqrt{v} - v\right)(33 - T) \tag{20.1}$$

where v is the wind speed, T is the temperature of the air, and H denotes the heat loss per unit area per unit time. The units used for these quantities are as follows: H is measured in $kilo{-}cal/m^2/hour$, v is measured in m/sec and T is in ^0C; the significance of the number 33 is that 33^0C is the normal temperature for bare skin (37^0C is body temperature).