1 Lorentz group

The Lorentz group is introduced and discussed in a series of statements and observations below. Hopefully you will find that you already know some of these things.

- A fundamental property of Lorentz transformations is the invariance of $x_\mu x^\mu = c^2t^2 - x^2 - y^2 - z^2$. An equivalent statement is that Lorentz transformation matrices satisfy

$$\Lambda^{\mathrm{T}} g \Lambda = g\,. \tag{1}$$

  Formally, any transformation Λ satisfying Eq. (32) is a Lorentz transformation.

- The set of matrices satisfying Eq. (32) is called the Lorentz group.

  One can show that this set of matrices obey all four aspects of the mathematical definition of a group, under the operation of matrix multiplication.

- Looks like we just defined the Lorentz group as a group of matrices. Physically, of course, the Lorentz group is the group of transformations represented by these matrices. The group operation (matrix multiplication) corresponds to successive application of transformations.

  For example, if Λ_1 and Λ_2 are matrices representing two Lorentz transformations, then the matrix $\Lambda_1\Lambda_2$ represents the following transformation: apply Λ_2 first, and then apply Λ_1. (Note the order.)

- The metric g in Eq. (32) could be either

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \text{or} \quad g = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

  The defining equation (32) is not affected by this choice, since the two g's differ only by a sign.

  The negative signature metric is preferred in particle physics and quantum field theory, while the positive signature metric is preferred in general relativity. We've mostly used the first, in this semester.

- The Lorentz group, as defined, includes also reflections of time and reflections of space. These are not very physical transformations. If we omit these, we obtain the set of PROPER ORTHOCHRONOUS transformations, or the set of physical Lorentz transformations. This restricted set itself forms a group, known as the restricted Lorentz group.

- The sign of the time coordinate gets flipped by an LT if $\Lambda^0{}_0$ is negative.

  A Lorentz transformation that retains the sign of the time coordinate is called orthochronous.

  We will show later that $\left|\Lambda^0{}_0\right| \geq 1$, i.e., values of $\Lambda^0{}_0$ between -1 and $+1$ are excluded.

  Thus a Lorentz transformation Λ is

$$\begin{array}{rll} \text{orthchronous} & \text{if} & \Lambda^0{}_0 > 1, \\ \text{non-orthchronous} & \text{if} & \Lambda^0{}_0 < -1. \end{array}$$

- **Proper and improper:**

  You can show from the definition (32) that the determinant-squared of an LT is 1, so that $\det\Lambda$ is either $+1$ or -1. The LT is proper if $\det\Lambda = 1$ and improper if $\det\Lambda = -1$.

  This terminology is the same as that used for 3×3 rotation matrices $\mathcal{R}$. If $\det\mathcal{R} = -1$, the matrix represents a reflection in addition to a rotation.

  In the case of LT's, $\det\Lambda = -1$ can mean either spatial reflection or temporal reversal. But not both: A Lorentz transformation that involves both time reversal and spatial reflection will have $\det\Lambda = +1$ and hence is 'proper'. Of course, you would probably not regard this transformation as being physical, despite the name proper.

- The matrix

$$T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

  represents the operation of time reversal, as you can see by applying it to a spacetime coordinate (ct, x, y, z). It is a valid Lorentz transformation according to the definition (32). (Please check!)

  The matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

  represents the operation of spatial reflection, also known as a PARITY TRANSFORMATION. This is also a valid Lorentz transformation.

  Note that the matrices T and P look formally identical to the metric tensor g or its negative. This does not have any physical meaning, as g does not represent a transformation.

- The unit matrix is a member of the Lorentz group defined by Eq. (32) and hence represents a valid Lorentz transformation.

  What transformation does the unit matrix represent? The do-nothing transformation, of course. It's the transformation that takes you from the spacetime coordinates measured from frame Σ to the spacetime coordinates measured from the same frame Σ.

  The unit matrix is a proper and orthochronous LT. (Please check.)

- Members of the restricted Lorentz group, i.e., the proper orthochronous Lorentz transformations, are connected continuously to the unity matrix. Those LT's which involve flipped temporal or spatial coordinates are NOT continuously connected to the unit matrix.

  Thus, the Lorentz group can be broken into four disjoint pieces, illustrated in the table:

	$\Lambda^0{}_0 > 1$ ORTHOCHRONOUS	$\Lambda^0{}_0 < 1$ NON-ORTHOCHRONOUS
$\det \Lambda = +1$ PROPER	$\Lambda^{(p.o.)}$	$TP\Lambda^{(p.o.)}$
$\det \Lambda = -1$ IMPROPER	$P\Lambda^{(p.o.)}$	$T\Lambda^{(p.o.)}$

  We can't list all the elements in each of the four blocks; so we have labeled them with representative matrices. Here $\Lambda^{(p.o.)}$ is an arbitrary proper orthogonal Lorentz transformation, i.e., a representative of the the set of physical Lorentz transformations. The top left block represents all transformations continuously connected to this one, i.e., the whole set of physical Lorentz transformations. The lower left block represents all transformations continuously connected to $P\Lambda^{(p.o.)}$, which is the matrix obtained by reflecting the spatial components of $\Lambda^{(p.o.)}$. You should be able to guess the definitions of the other two blocks.

  The identity matrix belongs to the top left segment of the Lorentz group. Thus, the other blocks in the table cannot form groups by themselves — they lack the identity element.

- The Lorentz group is represented as $O(3,1)$ or $O(1,3)$. The numbers show that one of the components is treated specially, i.e., that one of the diagonal elements of the 4×4 metric tensor has opposite sign.

  The subset of the Lorentz group that is *proper* is also a group.

  > Exercise: Show that this subset satisfies closure, i.e., that the product of any two LT matrices having determinant $+1$ is also an LT matrix having determinant $+1$.

  The proper Lorentz group consists of the two upper blocks in the table above. This group is called $SO(3,1)$. The 'S' stands for 'special', meaning positive determinant.

  If we further restrict to transformations that are both proper and orthochronous, we obtain the restricted Lorentz group. This is the top left block in the table. This group is represented by the very fancy name $SO^{\uparrow}(3,1)$. The $\uparrow$ superscript indicates that time is moving forward, in the physically meaningful direction.

  Many people would think of $SO^{\uparrow}(3,1)$ as the class of physical transformations. Sometimes, when people say 'Lorentz transformations', they might mean only this class of transformations, leaving out most of the full Lorentz group. As always, you have to figure out from the context what is meant.

- Showing that $\left|\Lambda^0{}_0\right| \geq 1$

  We've claimed this inequality previously; let's prove it.

  The defining relation (32) can be written in tensor-index notation as

  $$\Lambda^{\mu}{}_{\alpha}\, \Lambda^{\nu}{}_{\beta}\, g_{\mu\nu} = g_{\alpha\beta}$$

  Let's focus on the values $\alpha = \beta = 0$:

  $$\Lambda^{\mu}{}_0 \Lambda^{\nu}{}_0 g_{\mu\nu} = g_{00}$$

  $$\implies \left(\Lambda^0{}_0\right)^2 - \sum_{i=1}^{3} \left(\Lambda^i{}_0\right)^2 = 1$$

  $$\implies \left(\Lambda^0{}_0\right)^2 = 1 + \sum_{i=1}^{3} \left(\Lambda^i{}_0\right)^2 \geq 1$$

2 Boosts and rotations

The restricted Lorentz group contains BOOSTS and ROTATIONS and combinations of the two.

- Boost matrices are SYMMETRIC.

- Two successive boosts result in a pure boost only if they are in the same direction. For example, consider a boost in the x direction followed by another boost in the x direction.

$$\begin{pmatrix} \gamma_{v_2} & -\gamma_{v_2}\left(\frac{v_2}{c}\right) & 0 & 0 \\ -\gamma_{v_2}\left(\frac{v_2}{c}\right) & \gamma_{v_2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_{v_1} & -\gamma_{v_1}\left(\frac{v_1}{c}\right) & 0 & 0 \\ -\gamma_{v_1}\left(\frac{v_1}{c}\right) & \gamma_{v_1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_{w} & -\gamma_{w}\left(\frac{w}{c}\right) & 0 & 0 \\ -\gamma_{w}\left(\frac{w}{c}\right) & \gamma_{w} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{with} \quad w = \frac{v_1 + v_2}{1 + v_1 v_2 / c^2}$$

You know this already, but if you don't remember, you should try multiplying and showing this. Physically, this means considering a transformation from frame Σ to frame Σ' (relative speed v_1), and then from Σ' to $\tilde{\Sigma}$ (relative speed v_2), when all three are in standard configuration, i.e., relative motion in the common x, x', $\tilde{x}$ direction. The net transformation obtained by matrix multiplication is the transformation from Σ to $\tilde{\Sigma}$. The relative speed between frames Σ and $\tilde{\Sigma}$ is of course not $v_1 + v_2$ but rather $(v_1 + v_2)/(1 + v_1 v_2 / c^2)$.

- However, when we apply successively boosts in different directions, we do not obtain a pure boost. For example,

$$\begin{pmatrix} \gamma_{v_2} & 0 & -\gamma_{v_2}\left(\frac{v_2}{c}\right) & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_{v_2}\left(\frac{v_2}{c}\right) & 0 & \gamma_{v_2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_{v_1} & -\gamma_{v_1}\left(\frac{v_1}{c}\right) & 0 & 0 \\ -\gamma_{v_1}\left(\frac{v_1}{c}\right) & \gamma_{v_1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

turns out to be a non-symmetric matrix. Thus a boost applied in the x direction, followed by a boost applied in the y direction, does not result in a pure boost.

This shows that Lorentz boosts do not form a group by themselves. Rotations are needed to complete the group, i.e., to make the set of transformations satisfy closure.

- Symbolically:

$$\text{Restricted Lorentz group} = \text{boosts} + \text{rotations}$$

and also

$$\text{Lorentz group} = \text{boosts} + \text{rotations} + T + P$$