MP350 Classical Mechanics

Problem set 9

1. (a) Find the rotation matrix for a rotation of 45 degrees around the x-axis followed by a rotation of 45 degrees around the y-axis.

   (b) Check by explicit calculation that the matrix is orthogonal.

2. A body consists of three particles with masses $3m, 4m$ and $2m$, located at $(a, 0, a), (a, a, -a)$ and $(-a, a, 0)$ respectively and connected with light rigid rods. Determine the inertia tensor of the body,

   (a) about the origin,

   (b) in a coordinate system with the origin at the location of the first particle.

   (c) The body is set to rotate with angular velocity $\vec{\omega} = (\omega, 0, -\omega)$ about the origin. Find the angular momentum $\vec{L}$ of the body.

*3. Consider a symmetric top with mass M and $I_1 = I_2 \neq I_3$, rotating under the influence of gravity, and with one point fixed. The kinetic energy of the top is (in the body coordinate system x_1, x_2, x_3 defined by the principal axes of inertia)

$$T = \frac{1}{2}\sum_i I_i \omega_i^2 = \frac{1}{2} I_1 (\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3 \omega_3^2 \,,$$

and the angular velocities in the body coordinate system can be written in terms of the Euler angles as

$$\omega_1 = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi \,,$$
$$\omega_2 = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi \,,$$
$$\omega_3 = \dot{\phi} \cos\theta + \dot{\psi} \,.$$

Note that the angle θ is the angle between the x_3-axis and the fixed vertical (z-)axis, and the potential energy is thus $V = Mgh\cos\theta$, where h is the distance from the fixed point of rotation to the centre of mass (along the x_3-axis).

(a) Show that $\omega_1^2 + \omega_2^2 = \dot{\phi}^2 \sin^2\theta + \dot{\theta}^2$, and hence that the lagrangian for the top can be written as

$$L = \frac{1}{2} I_1 (\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{2} I_3 (\dot{\phi}\cos\theta + \dot{\psi})^2 - Mgh\cos\theta\,.$$

(b) Determine the canonical momenta p_ϕ, p_θ, p_ψ, and show that p_ϕ and p_ψ are conserved.

(c) Show that the hamiltonian (or total energy) of the system can be written as

$$H = \frac{p_\theta^2}{2I_1} + \frac{p_\psi^2}{2I_3} + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + Mgh\cos\theta\,,$$

and hence that the motion of the system can be treated as one-dimensional motion in an effective potential

$$V_{\text{eff}}(\theta) = \frac{p_\psi^2}{2I_3} + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + Mgh\cos\theta\,.$$

(d) Explain why this motion is bounded for generic values of p_ϕ, p_ψ, so that the angle θ with the vertical will be confined to the region $\theta_{\min} \le \theta \le \theta_{\max}$. If $\theta_{\max} < \pi/2$ this ensures that the top does not fall over.