MP350 Classical Mechanics

Lagrangian mechanics — summary sheet

1. **Lagrangian** $L = T - V$ = kinetic energy − potential energy.
   $L = L(q, \dot{q}, t)$ is a function of the coordinates q_i, their time derivatives $\dot{q}_i$ and time t.

2. **Generalised coordinates**
   For a system of N particles, we may instead of cartesian coordinates $\overrightarrow{r_i} = (x_i, y_i, z_i)$, $i = 1 \ldots N$, use *any* set of coordinates

   $$q_j = f_j(\overrightarrow{r_1}, \ldots, \overrightarrow{r_N}), \quad j = 1 \ldots M\,.$$

   M is the *number of degrees of freedom* of the system. For an unconstrained system $M = 3N$, but if there are constraints then $M < 3N$.

3. **Principle of least action**
   Nature "chooses" the path $q(t)$ that minimises the action

   $$S = \int_{t_0}^{t_1} L(q(t), \dot{q}(t), t)dt$$

   with $q(t_0) = q_0, q(t_1) = q_1$ kept fixed, or

   $$\delta S = \lim_{\alpha \to 0} \frac{S[q(t)] - S[q(t) + \alpha h(t)]}{\alpha} = 0$$

   for *arbitrary* $h(t)$ with $h(t_0) = h(t_1) = 0$. This leads to

4. **Euler–Lagrange equations**

   $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

5. **Canonical momentum**

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

(a) *Linear momentum*
If $q_i = x_i$ and $L = \frac{1}{2}\sum_i m_i \dot{x}_i^2 - V(x)$, then $p_i = m\dot{x}_i$.

(b) *Angular momentum*
If q_i is a rotational angle ϕ about some axis, then p_i is the angular momentum $[\overrightarrow{L} = \overrightarrow{r} \times (m\overrightarrow{v})]$ about that axis.

6. **Conservation laws**
From the Euler–Lagrange equations we see that if L does not depend explicitly on the coordinate q_i then

$$\frac{dp_i}{dt} = 0 \quad \Longleftrightarrow \quad p_i \text{ is conserved.}$$

7. **Hamiltonian**

$$H = \sum_i p_i \dot{q}_i - L$$

If there are no time-dependent constraints or velocity-dependent forces (or potentials) then $H = T + V =$ total energy.

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t},$$

so if the lagrangian L does not explicitly depend on time, then the hamiltonian H is conserved.