MP350 Classical Mechanics

Central forces — summary sheet

1. **Relative motion, reduced mass**
   We can write the kinetic energy of a two-body system as

$$T = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2,$$

   with

$$\begin{aligned} \vec{R} &= \frac{m_1\vec{r_1} + m_2\vec{r_2}}{m_1+m_2} && = \text{centre of mass,} \\ \vec{r} &= \vec{r_1} - \vec{r_2} && = \text{relative coordinate,} \\ \mu &= \frac{m_1 m_2}{m_1+m_2} && = \textbf{reduced mass} \end{aligned}$$

   For an isolated system, $V(\vec{r_1}, \vec{r_2}) = V(\vec{r}) \implies \vec{P} = M\dot{\vec{R}}$ =constant and the CM motion is trivial.

2. **Planar motion**
   For $V = V(r)$, all motion is in the plane spanned by $\vec{r}$ and $\vec{p}$, orthogonal to the constant angular momentum $\vec{L} = \vec{r} \times \vec{p}$.

3. **Angular momentum**

$$L = \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

   Since L does not depend on θ,

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} = \text{constant} = \ell$$

4. **Kepler's second law**
   Area swept out in time dt:

$$dA = \frac{1}{2}r(rd\theta) \quad \implies \quad \frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{\ell}{2\mu} = \text{constant}$$

   Bodies in a central force field move with constant areal velocity.

5. **Effective potential**

$$\dot\theta = \frac{\ell}{\mu r^2} \quad \Longrightarrow \quad \frac{1}{2}\mu r^2 \dot\theta^2 = \frac{\ell^2}{2\mu r^2}$$

$$H = E = T + V = \frac{1}{2}\mu \dot r^2 + \frac{\ell^2}{2\mu r^2} + V(r) = \frac{1}{2}\mu \dot r^2 + V_{\text{eff}}(r)$$

6. **Exploiting energy conservation $E =$ constant**

$$\dot r = \sqrt{\frac{2}{\mu}\Big(E - V_{\text{eff}}(r)\Big)} \quad \Longrightarrow \quad t(r) = \sqrt{\frac{\mu}{2}} \int_{r_0}^{r} \frac{dr'}{\sqrt{E - V_{\text{eff}}(r')}}$$

$$\frac{dr}{dt}\frac{d\theta}{dr} = \frac{d\theta}{dt} = \frac{\ell}{\mu r^2} \quad \Longrightarrow \quad \frac{d\theta}{dr} = \frac{\ell}{\mu r^2} \frac{\sqrt{\mu/2}}{\sqrt{E - V_{\text{eff}}(r')}}$$

7. **Kepler problem: orbit equation**

Integrating $d\theta/dr$ with $V(r) = -k/r$ gives

$$r(\theta) = \frac{\alpha}{1 + e\cos\theta}, \quad \text{with} \quad \alpha = \frac{\ell^2}{\mu k}, \quad e = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}\,.$$

This is the equation for a **conic section**:

$e = 0$	circle
$0 < e < 1$	ellipse
$e = 1$	parabola
$e > 1$	hyperbola

8. **Kepler's first law**

The planets move in elliptic orbits, with the sun at one focus.

Major semiaxis: $$a = \alpha(1 - e^2) = \frac{k}{-2E} = \frac{GmM_\odot}{-2E}$$

Minor semiaxis: $$b = a\sqrt{1 - e^2} = a\sqrt{\frac{\alpha}{a}} = a^{1/2}\frac{\ell}{\sqrt{2\mu}}$$

9. **Kepler's third law**

$$\frac{dA}{dt} = \frac{\ell}{2\mu} = \frac{A}{T} = \frac{\pi ab}{T} \quad \Longrightarrow \quad T = 2\pi\sqrt{\frac{\mu}{k}}a^{3/2}$$

For planets, $\mu/k \approx 1/(GM_\odot) =$ constant.

The square of the orbital period varies like the cube of the major axis.