MP350 Classical Mechanics

Solutions — Problem set 9

1. (a) The matrices for the two rotations (45° around the x-axis and 45° around the y-axis) are

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{1.1}$$

Multiplying these two together gives

$$A = A_2 A_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}. \tag{1.2}$$

(b) An explicit calculation will show that

$$A^T A = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.3}$$

2. (a) The components of the inertia tensor are

$$\begin{aligned} I_{xx} &= m_1(y_1^2 + z_1^2) + m_2(y_2^2 + z_2^2) + m_3(y_3^2 + z_3^2) \\ &= 3m(0 + a^2) + 4m(a^2 + a^2) + 2m(a^2 + 0) = 13ma^2, && (2.1) \\ I_{yy} &= m_1(x_1^2 + z_1^2) + m_2(x_2^2 + z_2^2) + m_3(x_3^2 + z_3^2) \\ &= 3m(a^2 + a^2) + 4m(a^2 + a^2) + 2m(a^2 + 0) = 16ma^2, && (2.2) \\ I_{zz} &= m_1(y_1^2 + z_1^2) + m_2(y_2^2 + z_2^2) + m_3(y_3^2 + z_3^2) \\ &= 3m(0 + a^2) + 4m(a^2 + a^2) + 2m(a^2 + a^2) = 15ma^2, && (2.3) \\ I_{xy} = I_{yx} &= -m_1x_1y_1 - m_2x_2y_2 - m_3x_3y_3 \\ &= -3m \cdot 0 - 4ma^2 - 2m(-a^2) = -2ma^2, && (2.4) \\ I_{xz} = I_{zx} &= -m_1x_1z_1 - m_2x_2z_2 - m_3x_3z_3 \\ &= -3ma^2 - 4m(-a^2) - 2m \cdot 0 = ma^2, && (2.5) \\ I_{yz} = I_{zy} &= -m_1x_1y_1 - m_2x_2y_2 - m_3x_3y_3 \\ &= -3m \cdot 0 - 4m(-a^2) - 2m \cdot 0 = 4ma^2. && (2.6) \end{aligned}$$

In matrix form, the inertia tensor can be written

$$I = ma^2 \begin{pmatrix} 13 & -2 & 1 \\ -2 & 16 & 4 \\ 1 & 4 & 15 \end{pmatrix}. \tag{2.7}$$

(b) Since the first particle is now at the origin, it does not contribute to the inertia tensor. The inertia tensor now involves the relative positions of the two other particles relative to the first:

$$\begin{aligned}\overrightarrow{r'}_2 &= (x_2 - x_1, y_2 - y_1, z_2 - z_1) = (0, a, -2a)\,, && (2.8)\\ \overrightarrow{r'}_3 &= (x_3 - x_1, y_3 - y_1, z_3 - z_1) = (-2a, a, -a)\,. && (2.9)\end{aligned}$$

The components of the inertia tensor are now

$$\begin{aligned}I'_{xx} &= m_2[(y_2 - y_1)^2 + (z_2 - z_1)^2] + m_3[(y_3 - y_1)^2 + (z_3 - z_1)^2]\\ &= 4m(a^2 + 4a^2) + 2m(a^2 + a^2) = 24ma^2\,, && (2.10)\\ I'_{yy} &= m_2[(x_2 - x_1)^2 + (z_2 - z_1)^2] + m_3[(x_3 - x_1)^2 + (z_3 - z_1)^2]\\ &= 4m(0 + 4a^2) + 2m(4a^2 + a^2) = 26ma^2\,, && (2.11)\\ I'_{zz} &= m_2[(x_2 - x_1)^2 + (y_2 - y_1)^2] + m_3[(x_3 - x_1)^2 + (y_3 - y_1)^2]\\ &= 4m(0 + a^2) + 2m(4a^2 + a^2) = 14ma^2\,, && (2.12)\\ I'_{xy} = I'_{yx} &= -m_2(x_2 - x_1)(y_2 - y_1) - m_3(x_3 - x_1)(y_3 - y_1)\\ &= -4m(0)(a) - 2m(-2a)(a) = 4ma^2\,, && (2.13)\\ I'_{xz} = I'_{zx} &= -m_2(x_2 - x_1)(z_2 - z_1) - m_3(x_3 - x_1)(z_3 - z_1)\\ &= -4m \cdot 0 - 2m(-2a)(-a) = -4ma^2\,, && (2.14)\\ I'_{yz} = I'_{zy} &= -m_2(y_2 - y_1)(z_2 - z_1) - m_3(y_3 - y_1)(z_3 - z_1)\\ &= -4m(a)(-2a) - 2m(a)(-a) = 10ma^2\,. && (2.15)\end{aligned}$$

In matrix form, the inertia tensor can be written

$$I' = ma^2 \begin{pmatrix} 24 & 4 & -4 \\ 4 & 26 & 10 \\ -4 & 10 & 14 \end{pmatrix}\,. \qquad (2.16)$$

(c) The angular momentum about the origin is $L_i = \sum_j I_{ij}\omega_j$, where I_{ij} is the inertia tensor about the origin. Using (2.7) we find

$$\begin{aligned}L_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z = 13ma^2 \cdot \omega + 0 + ma^2 \cdot (-\omega) = 12ma^2\omega\,,\\ L_y &= I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z = -2ma^2 \cdot \omega + 0 + 4ma^2 \cdot (-\omega) = -6ma^2\omega\\ L_z &= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z = ma^2 \cdot \omega + 0 + 15ma^2 \cdot (-\omega) = -14ma^2\omega\,,\end{aligned}$$

or, in vector form,

$$\overrightarrow{L} = (12, -6, -14)ma^2\omega\,. \qquad (2.17)$$

3. (a)

$$\begin{aligned}\omega_1^2 + \omega_2^2 &= (\dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi)^2 + (\dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi)^2\\ &= \dot\phi^2 \sin^2\theta \sin^2\psi + 2\dot\phi\dot\theta \sin\theta \sin\psi \cos\psi + \dot\theta^2 \cos^2\psi\\ &\quad + \dot\phi^2 \sin^2\theta \cos^2\psi - 2\dot\phi\dot\theta \sin\theta \sin\psi \cos\psi + \dot\theta^2 \sin^2\psi \qquad (3.1)\\ &= \dot\phi^2 \sin^2\theta + \dot\theta^2\,.\end{aligned}$$

The lagrangian is thus

$$\begin{aligned} L = T - V &= \frac{1}{2}I_1(\omega_1^2 + \omega_2^2) + \frac{1}{2}I_3\omega_3^2 - Mgh\cos\theta \\ &= \frac{1}{2}I_1(\dot\phi^2\sin^2\theta + \dot\theta^2) + \frac{1}{2}I_3(\dot\phi\cos\theta + \dot\psi)^2 - Mgh\cos\theta\,. \end{aligned} \tag{3.2}$$

(b) If we write $\dot\phi\cos\theta + \dot\psi = \omega_3(\theta, \dot\phi, \dot\psi)$, we have

$$p_\phi = \frac{\partial L}{\partial\dot\phi} = I_1\dot\phi\sin^2\theta + I_3\omega_3\frac{\partial\omega_3}{\partial\dot\phi} = I_1\dot\phi\sin^2\theta + I_3\cos\theta(\dot\phi\cos\theta + \dot\psi)\,, \tag{3.3}$$

$$p_\theta = \frac{\partial L}{\partial\dot\theta} = I_1\dot\theta \tag{3.4}$$

$$p_\psi = \frac{\partial L}{\partial\dot\psi} = I_3\omega_3\frac{\partial\omega_3}{\partial\dot\psi} = I_3(\dot\phi\cos\theta + \dot\psi)\,. \tag{3.5}$$

Since L does not depend explicitly on either ϕ or ψ, $\partial L/\partial\phi = \partial L/\partial\psi = 0$, and the canonical momenta p_ϕ, p_ψ are conserved.

(c) From (3.5) we see that $\omega_3 = \dot\phi\cos\theta + \dot\psi = p_\psi/I_3$. We can use this to rewrite (3.3),

$$p_\phi = I_1\dot\phi\sin^2\theta + I_3\cos\theta\omega_3 = I_1\dot\phi\sin^2\theta + p_\psi\cos\theta \tag{3.6}$$

$$\Longleftrightarrow \quad p_\phi - p_\psi\cos\theta = I_1\dot\phi\sin^2\theta \quad \Longleftrightarrow \quad \dot\phi = \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta}\,. \tag{3.7}$$

Since there are no time-dependent coordinate transformations or velocity-dependent potentials, the hamiltonian is equal to the total energy,

$$\begin{aligned} H &= T + V \\ &= \frac{1}{2}I_1\left[\left(\frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta}\right)^2\sin^2\theta + \left(\frac{p_\theta}{I_1}\right)^2\right] + \frac{1}{2}I_3\left(\frac{p_\psi}{I_3}\right)^2 + Mgh\cos\theta \\ &= \frac{p_\theta^2}{2I_1} + \frac{p_\psi^2}{2I_3} + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + Mgh\cos\theta\,. \end{aligned} \tag{3.8}$$

Since p_ϕ and p_ψ are both constant, there is effectively only one degree of freedom θ, and the second and third term in H can be combined with the final term $V = Mgh\cos\theta$ to form an effective potential $V_{\text{eff}}(\theta)$. The first term is the usual kinetic energy term.

(d) The first term in V_{eff} is just a constant, so it has no effect on the dynamics of the system, other than increasing the minimum energy. Unless p_ϕ and p_ψ are both precisely zero or $p_\phi = \pm p_\psi$, the second term is nonzero and diverges to $+\infty$ as $\theta \to 0$ and $\theta \to \pi$.[1] Therefore, the motion in θ is bounded for all but very special values of p_ϕ, p_ψ.

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[1] If $p_\phi = p_\psi$ the second term in V_{eff} diverges as $\theta \to \pi$ but goes to 0 as $\theta \to 0$, and conversely if $p_\phi = -p_\psi$. This, however, requires finely balanced initial conditions.