Chapter 4

Central forces

If the force between two bodies is directed along the line connecting the (centres of mass of) the two bodies, this force is called a *central force*. Since most of the fundamental forces we know about, including the gravitational, electrostatic and certain nuclear forces, are of this kind, it is clear that studying central force motion is extremely imortant in physics.

Moreover, the motion of a system consisting of only two bodies interacting via a central force is one of the few problems in classical mechanics that can be solved completely (once you add a third body it becomes, in general, unsolveable). Examples of such systems are the motion of planets and comets around a star, or satellites around a planet, or binary stars; and classical scattering of atoms or subatomic particles. The full description of atoms and subatomic particles requires quantum mehcanics, but even here the classical analysis of central forces can yield a great deal of insight.

In the two-body problem we start with a description in terms of 6 coordinates, namely the three (cartesian) coordinates of each of the two bodies. We shall see that it is possible to reduce this to just 2, and for some purposes only 1 effective degree of freedom. This reduction will happen in 3 steps:

1. We can treat the relative motion as a 1-body problem.
2. The relative motion is 2-dimensional (planar).
3. We can use angular momentum conservation to treat the radial motion as 1-dimensional motion in an effective potential.

4.1 One-body reduction, reduced mass

We start with a system of two particles, with coordinates $\vec{r_1}$ and $\vec{r_2}$. We need six coordinates to describe this system, and this is provided by the three components of $\vec{r_1}$ and the three components of $\vec{r_2}$. However, since we know that the potential energy only depends on the combination $r = |\vec{r}| = |\vec{r_1} - \vec{r_2}|$, we may want to describe it instead in terms of the three components of the *relative* coordinate $\vec{r} = \vec{r}_1 - \vec{r}_2$ and a second vector, which we can take to be the centre-of-mass vector

$$\vec{R} = \frac{m_1\vec{r_1} + m_2\vec{r_2}}{m_1 + m_2} \,. \tag{4.1}$$

First we need to express $\vec{r_1}$ and $\vec{r_2}$ in terms of the new coordinates $\vec{R}, \vec{r}$. We have

$$\vec{r} = \vec{r_1} - \vec{r_2} \quad \Longleftrightarrow \quad \vec{r_1} = \vec{r} + \vec{r_2} \tag{4.2}$$

$$\vec{R} = \frac{m_1\vec{r_1} + m_2\vec{r_2}}{m_1 + m_2} = \frac{m_1(\vec{r} + \vec{r_2}) + m_2\vec{r_2}}{m_1 + m_2} = \vec{r_2} + \frac{m_1}{m_1 + m_2}\vec{r} \,, \tag{4.3}$$

which gives

$$\vec{r_2} = \vec{R} - \frac{m_1}{m_1 + m_2}\vec{r} \tag{4.4}$$

$$\vec{r_1} = \vec{r} + \vec{r_2} = \vec{R} + \frac{m_2}{m_1 + m_2}\vec{r} \tag{4.5}$$

We now plug (4.4), (4.5) into the expression for the kinetic energy,

$$\begin{aligned} T &= \frac{1}{2}m_1\dot{\vec{r_1}}^2 + \frac{1}{2}m_2\dot{\vec{r_2}}^2 \\ &= \frac{1}{2}m_1\Big(\dot{\vec{R}} + \frac{m_2}{m_1 + m_2}\dot{\vec{r}}\Big)^2 + \frac{1}{2}m_2\Big(\dot{\vec{R}} - \frac{m_1}{m_1 + m_2}\dot{\vec{r}}\Big)^2 \\ &= \frac{1}{2}m_1\Big(\dot{\vec{R}}^2 + \frac{2m_2}{m_1 + m_2}\dot{\vec{R}}\cdot\dot{\vec{r}} + \frac{m_2^2}{(m_1 + m_2)^2}\dot{\vec{r}}^2\Big) \\ &\quad + \frac{1}{2}m_2\Big(\dot{\vec{R}}^2 - \frac{2m_1}{m_1 + m_2}\dot{\vec{R}}\cdot\dot{\vec{r}} + \frac{m_1^2}{(m_1 + m_2)^2}\dot{\vec{r}}^2\Big) \\ &= \frac{1}{2}(m_1 + m_2)\dot{\vec{R}}^2 + \frac{1}{2}\Big(\frac{m_1 m_2^2}{(m_1 + m_2)^2} + \frac{m_1^2 m_2}{(m_1 + m_2)^2}\Big)\dot{\vec{r}}^2 \\ &= \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 \,. \end{aligned} \tag{4.6}$$

The total lagrangian is therefore

$$L = T - V = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - V(r) \,. \tag{4.7}$$

We see that the lagrangian splits into two separate parts: one describing the motion of the centre of mass, and one describing the relative motion. We can therefore analyse the relative motion without any reference to the overall motion of the centre of mass. Moreover, $\vec{R}$ is cyclic, so its canonical momentum is conserved. The canonical momentum conjugate to $\vec{R}$ is

$$P_i = \frac{\partial L}{\partial \dot{R}_i} = M\dot{R}_i \quad \Longrightarrow \quad \vec{P} = M\dot{\vec{R}} \,. \tag{4.8}$$

This is just the total momentum of the system:

$$\vec{P} = M\dot{\vec{R}} = (m_1 + m_2)\frac{m_1\dot{\vec{r_1}} + m_2\dot{\vec{r_2}}}{m_1 + m_2} = m_1\dot{\vec{r_1}} + m_2\dot{\vec{r_2}} \,. \tag{4.9}$$

Therefore, the absolute motion is merely linear motion with constant total momentum. From now on, we will ignore the absolute motion completely, and focus only on the relative motion — ie, we will drop the first term in (4.7). The lagrangian then becomes

$$L = \frac{1}{2}\mu\dot{\vec{r}}^2 - V(r)\,. \tag{4.10}$$

This looks exactly like the lagrangian for a single particle with position $\vec{r}$ in a potential $V(r)$, but with mass

$$\boxed{\mu = \frac{m_1 m_2}{m_1 + m_2} = \textit{the reduced mass}} \tag{4.11}$$

We have therefore reduced two-body motion to the equivalent motion of a single body with mass μ.

It is worthwhile looking more closely at the reduced mass and its relation to the masses m_1, m_2. We can assume without any loss of generality that $m_1 \geq m_2$ (since we could just swap the labels if it were the other way around). Then we have

$$\frac{m_1}{m_1 + m_2} \geq \frac{1}{2}\,, \quad \frac{m_2}{m_1 + m_2} \leq \frac{1}{2} \tag{4.12}$$

$$\Longrightarrow \quad \frac{m_1}{2} \geq \frac{m_1 m_2}{m_1 + m_2} \geq \frac{m_2}{2}\,. \tag{4.13}$$

So we see that the reduced mass has a value that lies between *half* the larger mass and *half* the smaller mass.

Two special cases of particular interest are where the two masses are equal, and where one mass is much larger than the other. The first includes scattering of identical particles (for example two α-particles) as well as some binary stars. The second includes the motion of a planet or comet around the sun, or satellites around a planet.

In the first case, $m_1 = m_2 = m$, we get that the reduced mass is $\mu = m/2$, ie the reduced mass is *half* the mass of each body.

In the second case, $m_2 \ll m_1$, we can rewrite the reduced mass as

$$\mu = m_2 \frac{1}{1 + \frac{m_2}{m_1}} = m_2\Big(1 - \frac{m_2}{m_1} + \Big(\frac{m_2}{m_1}\Big)^2 + \ldots\Big)\,. \tag{4.14}$$

If m_2 is small enough compared to m_1 (for example, for the earth–sun system we have $M_\oplus/M_\odot = 3 \cdot 10^{-6}$) we can just set $\mu = m_2$, ie the reduced mass equals the smaller of the two masses.

4.2 Angular momentum and Kepler's second law

Our system is now equivalent to a single particle with mass μ moving in a spherically symmetric potential $V(r)$. Since we have spherical symmetry, the angular momentum $\vec{L} = \vec{r} \times \vec{p}$ is conserved, both in magnitude and in direction. We will use this to simplify the problem further.

First, we note that since the *direction* of $\vec{L}$ is conserved, both the position vector $\vec{r}$ and momentum vector $\vec{p} = \mu\dot{\vec{r}}$ must be in the plane that is orthogonal to this vector, as the cross product between any two vectors is orthogonal to both vectors. We can choose the z-axis of our coordinate system to be pointing in the direction of $\vec{L}$, ie $\vec{L} = \ell\hat{z}$, and in this case both $\vec{r}$ and $\vec{p}$ must be in the xy-plane.

We can get a more physical understanding of this by noting that as long as $\vec{p} = m\dot{\vec{r}}$ remains in the xy-plane, $\vec{r}$ will not move out of this plane, while as long as $\vec{r}$ remains in that plane there is no force that will move $\vec{p}$ our of the plane, since the central force always points towards the centre (or away from it, in the case of a repulsive force), ie along the vector $\vec{r}$.

We have therefore reduced the motion to planar (2-dimensional) motion, and we can use polar coordinates (r, θ) to describe this motion (where θ is the angle with some arbitrarily chosen direction in the plane of motion). The lagrangian for the system, in those coordinates, becomes

$$L = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 - V(r)\,. \tag{4.15}$$

Since L does not depend on the angle θ (this is the remaining rotational symmetry), the angular momentum $\ell = p_\theta$ is conserved,

$$\ell = \mu r^2\dot{\theta} = \text{constant}\,. \tag{4.16}$$

This is exactly equivalent to Kepler's second law for planetary motion, which gives a nice geometrical interpretation of angular momentum conservation. Consider the area dA swept out by the radius vector in a small (infinitesimal) time dt. The angle swept out in that time is $d\theta = \dot{\theta}dt$, and the length of the arc swept out is $ds = rd\theta = r\dot{\theta}dt$ (see fig. 4.1). If $d\theta$ is small we can approximate the area by a triangle with length r and height ds, ie

$$dA = \frac{1}{2}rds = \frac{1}{2}r \cdot r\dot{\theta}dt \quad \Longleftrightarrow \quad \frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{\ell}{2\mu} = \text{constant}\,. \tag{4.17}$$

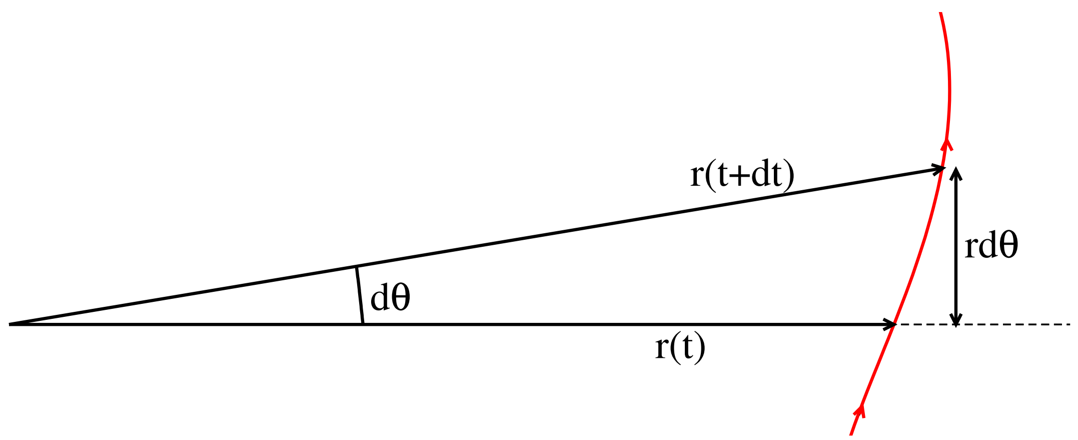

Figure 4.1: The area swept out by a radius vector.

Kepler's second law reads, in words,

> *A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.*

Kepler came to this conclusion through painstaking observation of planetary motions, and the apparatus of newtonian and lagrangian mechanics, which we are using here, was developed long after his time. But from this derivation we can see that this law is valid not just for planetary motion, but for *all* central force motion, whatever the force is, and is equivalent to conservation of angular momentum.

Example 4.1

The planet Mercury orbits the Sun in 87.97 days, in an elliptic orbit with semimajor axis $a = 57.91 \cdot 10^6$km and semiminor axis $b = 56.67 \cdot 10^6$km.

1. What is the areal velocity of Mercury?
2. Use the relation between areal velocity and angular momentum to find the speed of Mercury
   (a) at its perihelion (closest to the sun), $46.00 \cdot 10^6$km from the sun;
   (b) at its aphelion (furthest from the sun), $69.82 \cdot 10^6$km from the sun.

Answer:

1. Since the areal velocity is constant, it is just equal to

$$\frac{dA}{dt} = \frac{A}{T} = \frac{\pi ab}{87.97\text{d}} = \frac{\pi \cdot 57.91 \cdot 56.67 \cdot 10^{12}\text{km}^2}{87.97 \cdot 86400\text{s}} \underline{= 1.3565 \cdot 10^9\,\text{km}^2/\text{s}}\,, \quad (4.18)$$

where we have also used that the area of an ellipse is $A = \pi ab$.

2. From the relation between areal velocity and angular momentum ℓ, we have

$$\frac{dA}{dt} = \frac{\ell}{2\mu} = \frac{|\vec{r} \times \mu\vec{v}|}{2\mu} = \frac{1}{2}|\vec{r} \times \vec{v}|\,. \quad (4.19)$$

At perihelion and aphelion, the radial velocity is zero, so $\vec{v}$ is orthogonal to the radius vector $\vec{r}$. At these points we therefore have

$$\frac{dA}{dt} = \frac{1}{2}rv \quad \Longrightarrow \quad v = \frac{2dA/dt}{r}\,. \quad (4.20)$$

At **perihelion**:

$$v = \frac{2 \cdot 1.3565 \cdot 10^9\text{km}^2/\text{s}}{46.00 \cdot 10^6\text{km}} \underline{= 58.98\text{km/s}}. \quad (4.21)$$

At **aphelion**:

$$v = \frac{2 \cdot 1.3565 \cdot 10^9\text{km}^2/\text{s}}{69.82 \cdot 10^6\text{km}} \underline{= 38.86\text{km/s}}. \quad (4.22)$$

4.3 Effective potential and classification of orbits

Now that we have shown that angular momentum is conserved, we can use this to simplify the problem further. We have already seen how a 2-dimensional system can be treated as one-dimensional motion in an effective potential when one coordinate is cyclic. In our case, we have

$$p_\theta = \ell = \mu r^2 \dot{\theta} \quad \Longleftrightarrow \quad \dot{\theta} = \frac{\ell}{\mu r^2}\,, \tag{4.23}$$

so we can write the hamiltonian or total energy as

$$H = \frac{p_r^2}{2\mu} + \frac{p_\theta^2}{2\mu r^2} + V(r)\,, \tag{4.24}$$

or

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r) = \frac{1}{2}\mu\dot{r}^2 + V_{\text{eff}}(r)\,. \tag{4.25}$$

The term $V_c(r) = \ell^2/2\mu r^2$ is sometimes called the *centrifugal potential*, and can be thought of as giving rise to a (fictitious) "centrifugal force" $F_c \propto r^{-3}$. By investigating the shape of the effective potential V_{eff} we can find out what kinds of motion are possible in the radial direction. Some examples of effective potentials are shown in figure 4.2.

In general, if $\ell \neq 0$, the centrifugal potential $V_c(r) \to +\infty$ as $r \to 0$, providing a barrier against the bodies getting too close. This term will dominate at short distances unless $V(r)$ is strongly attractive, meaning that $V(r) \to -\infty$ fast enough. "Fast enough" here means that

$$V(r) \sim -\frac{1}{r^n}\,, \quad n > 2\,, \tag{4.26}$$

since if $n < 2$ we will always find that $ar^{-n} < br^{-2}$ for any a, b if r is small enough.

4.4 Integrating the energy equation

We can also use (4.25) to completely solve the motion in r, by using energy conservation and rewriting it to get

$$\frac{1}{2}\mu\dot{r}^2 = E - V_{\text{eff}}(r) \tag{4.27}$$

$$\Longleftrightarrow \quad \dot{r} = \frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}\big(E - V_{\text{eff}}(r)\big)} = \pm\sqrt{\frac{2}{\mu}\Big(E - V(r) - \frac{\ell^2}{2\mu r^2}\Big)} \tag{4.28}$$

$$\Longrightarrow \quad \int \frac{dr}{\sqrt{\frac{2}{\mu}\big(E - V_{\text{eff}}(r)\big)}} = \int dt = t - t_0\,. \tag{4.29}$$

Note that this does not tell us anything directly about the shape of the orbit, since for that we still need to know $\theta(t)$, but once we have found $r(t)$ it is straightforward to obtain θ from (4.23).

However, if we want to obtain the shape of the orbit, and are not particularly interested in the motion in time, we can use a similar trick to obtain r as a function of θ or

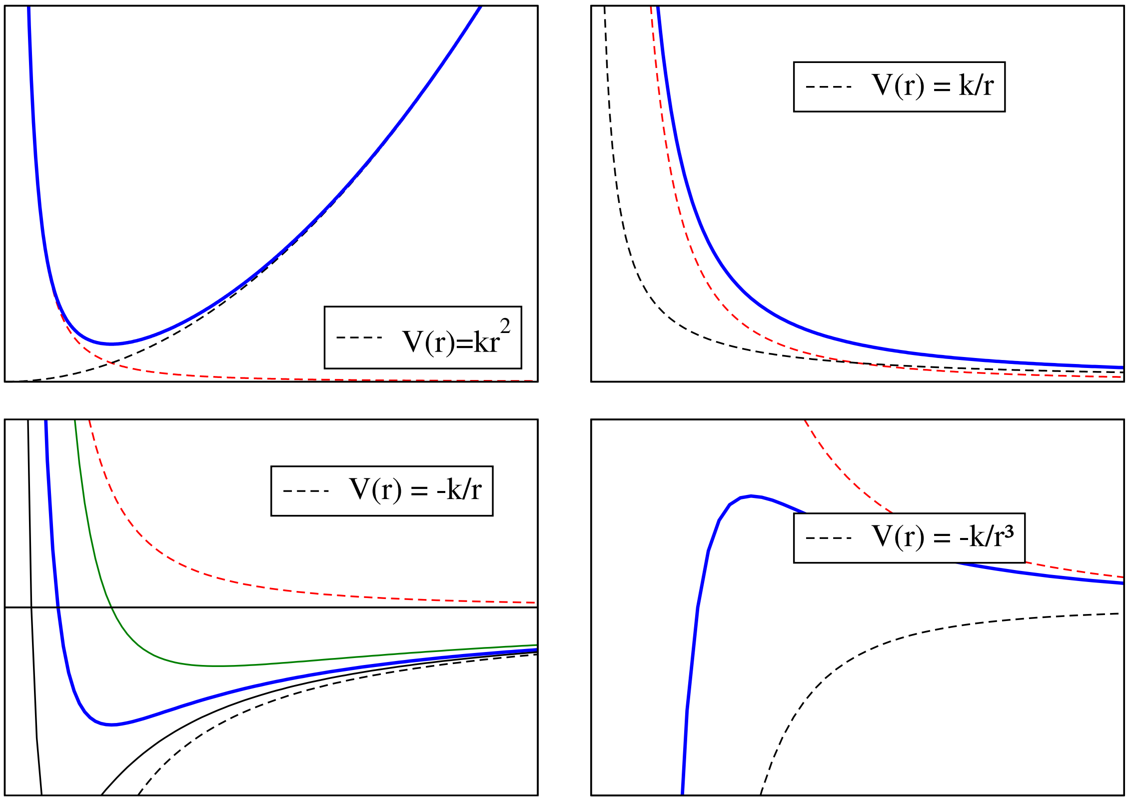

Figure 4.2: The effective potential (4.25) for different types of potential $V(r)$. The dotted red curves denote the centrifugal potential, and the dotted black curves $V(r)$. The thick blue curve is the effective potential $V_{\text{eff}}(r)$. Top left: a quadratic (harmonic oscillator) potential. The motion in r is bounded for all values of $\ell > 0$. Top right: a repulsive, inverse-square force law, $V(r) = k/r$. In this case, only unbounded motion is possible.Bottom left: an attractive, inverse-square force law. The different solid curves correspond to different values of ℓ. Here the motion is always bounded if $E < 0$, and unbounded if $E > 0$. Bottom right: an attractive multipole force with $V(r) = -k/r^3$. Here we can have bounded motion through the origin or unbounded motion.

vice-versa. Using (4.23) and (4.28) together, we can write

$$\frac{d\theta}{dr} = \frac{d\theta}{dt}\frac{dt}{dr} = \frac{\ell}{\mu r^2}\frac{\pm\sqrt{\mu/2}}{\sqrt{E - V_{\text{eff}}(r)}} \tag{4.30}$$

$$\implies \quad \frac{\ell}{\sqrt{2\mu}}\int \frac{dr}{r^2\sqrt{E - V_{\text{eff}}(r)}} = \int d\theta = \theta - \theta_0\,. \tag{4.31}$$

This gives us $\theta(r)$, which we can then invert to find $r(\theta)$, which defines the shape of the orbit.

In particular, if the motion is bounded, we can use this to find the *period* T of the radial motion, ie the time it takes to complete one full oscillation in the radial direction, from r_{min} to r_{max} and back again. This is given by

$$T = 2\int_{r_{\text{min}}}^{r_{\text{max}}} \frac{\sqrt{\mu/2}\,dr}{\sqrt{E - V_{\text{eff}}(r)}}\,, \tag{4.32}$$

where the factor 2 accounts for the "return journey" from $r_{\max}$ to $r_{\min}$. Similarly, we can find the *angular period* $\Delta\theta$, which is the angle swept out in the course of one full radial oscillation. It is given by

$$\Delta\theta = \sqrt{\frac{2}{\mu}}\ell \int_{r_{\min}}^{r_{\max}} \frac{dr}{r^2\sqrt{E - V_{\text{eff}}(r)}} \,. \tag{4.33}$$

If $\Delta\theta = 2\pi m/n$, where n and m are integers, then the system will return to the same place after n radial oscillations, having completed m revolutions of an angle of 2π. Such an orbit is *closed*. A remarkable result is that closed (non-circular) orbits are extremely rare: *Bertrand's theorem* (1873) states that the only potentials that give rise to such orbits are the harmonic oscillator $V(r) = kr^2$ and the inverse-square force $V(r) = -k/r$ (see Goldstein, pp. 89–92 for an explanation).

4.5 The inverse square force, Kepler's first law

From now on we will concentrate on the attractive inverse-square force law, ie

$$V(r) = -\frac{k}{r} \qquad \Longrightarrow \qquad V_{\text{eff}}(r) = \frac{\ell^2}{2\mu r^2} - \frac{k}{r} \,. \tag{4.34}$$

This describes the gravitational force between two bodies, with $k = Gm_1m_2$ (where m_1, m_2 are the masses of the two bodies). It could also describe the electrostatic attraction between two opposite charges $Q_1, -Q_2$, in which case we would have $k = Q_1Q_2/4\pi\varepsilon_0$.

Looking at $V_{\text{eff}}(r)$, we find that bounded motion is possible if $E_{\min} \leq E < 0$, where $E_{\min}$ is the minimum value of V_{eff} (which we will derive in a moment). For $E \geq 0$ the radial motion is unbounded, although for any $\ell \neq 0$ there is a minimum distance $r_{\min}$. If $E = E_{\min}$ we have a stable circular orbit.

Let us now find the minimum and maximum distances for a particular energy and angular momentum. They are given by

$$V_{\text{eff}}(r) = \frac{\ell^2}{2\mu r^2} - \frac{k}{r} = E \tag{4.35}$$

$$\Longleftrightarrow \qquad Er^2 + kr - \frac{\ell^2}{2\mu} = 0 \tag{4.36}$$

$$\Longleftrightarrow \qquad r = r_{\min,\max} = \frac{-k \pm \sqrt{k^2 + \frac{2E\ell^2}{\mu}}}{2E} = -\frac{k}{2E}\left(1 \pm \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}\right) = a(1 \pm e)\,, \tag{4.37}$$

with

$$a = -\frac{k}{2E}\,, \quad e = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}\,. \tag{4.38}$$

Inspecting (4.37) we see that

- If $E < E_{\text{min}} = -\mu k^2/2\ell^2$, the expression inside the square root becomes negative, and there is therefore no solution.
- If $E = E_{\text{min}}$ the square root (e) is zero, and we have $r_{\text{min}} = r_{\text{max}} = a = \ell^2/\mu k$. This corresponds to a stable circular orbit with $r = r_o = \ell^2/\mu k$. You can verify that this also corresponds to the minimum of V_{eff}, where $V'_{\text{eff}}(r_o) = 0$.
- If $E_{\text{min}} < E < 0$ there are two solutions, $r_{\text{min}} = a(1-e), r_{\text{max}} = a(1+e)$, and the radial motion is bounded between these two distances.
- If $E = 0$, $a \to \infty$, but (4.35) still has a single solution at $r = r_{\text{min}} = \ell^2/2\mu k$. There is no maximum value for r, so the motion becomes unbounded.
- If $E > 0$ we have $e > 1$ and $a < 0$. Since the distance r must be positive, only the minus sign in (4.37) gives a physically acceptable solution, $r_{\text{min}} = a(1-e) = (e-1)k/2E$. There is again no maximum value for r, and the motion is unbounded.

Example 4.2

The asteroid Pallas orbits the sun in an orbit with perihelion distance $r_{\text{min}} = 3.19 \cdot 10^{11}$m and $e = 0.231$. The speed of Pallas relative to the sun at perihelion is $v = 2.26 \cdot 10^4$m/s. Find the aphelion distance r_{max} of Pallas and its speed at that point.

Answer: From (4.37) we find that the aphelion distance r_{max} is

$$r_{\text{max}} = \frac{1+e}{1-e} r_{\text{min}} = \frac{1.231}{0.769} \cdot 3.19 \cdot 10^{11}\text{m} \underline{= 5.11 \cdot 10^{11}\text{m}\,.} \tag{4.39}$$

To find the speed at aphelion, we need to use either conservation of angular momentum or conservation of energy. Using angular momentum is easier. At perihelion and aphelion we have $\vec{v} \perp \vec{r}$ and therefore $\ell = mvr$ where m is the mass of the asteroid. The aphelion and perihelion speeds v_a, v_p are therefore related by

$$\ell = m v_p r_{\text{min}} = m v_a r_{\text{max}} \tag{4.40}$$

$$\Longrightarrow \quad v_a = \frac{r_{\text{min}}}{r_{\text{max}}} v_a = \frac{3.19}{5.11} \cdot 2.20 \cdot 10^4\text{m/s} \underline{= 1.37 \cdot 10^4\text{m/s}\,.} \tag{4.41}$$

We can now use the methods of Section 4.4 to find an equation for the orbit. For $V(r) = -k/r$ (4.31) becomes

$$\theta(r) = \pm \int \frac{\ell dr}{r^2 \sqrt{2\mu\left(E + \frac{k}{r} - \frac{\ell^2}{2\mu r^2}\right)}} = \pm \int \frac{dr}{r^2 \sqrt{\frac{2E\mu}{\ell^2} + \frac{2\mu k}{\ell^2 r} - \frac{1}{r^2}}}\,. \tag{4.42}$$

We now make the substitution

$$u = \frac{1}{r} \quad \Longrightarrow \quad du = -\frac{dr}{r^2}\,, \tag{4.43}$$

and also introduce the parameter $\alpha = \ell^2/\mu k$. This gives us

$$\theta(r) = \pm \int \frac{du}{\sqrt{\frac{2E\mu}{\ell^2} + 2\frac{u}{\alpha} - u^2}} = \pm \int \frac{dv}{\sqrt{\frac{e^2}{\alpha^2} - v^2}}\,, \tag{4.44}$$

where in the second step we have made the further substitution $v = u - 1/\alpha$ and introduced the parameter e from (4.38). We can look up this integral in a table, or solve it using additional clever substitutions. We can introduce the angle ϕ given by

$$v = \frac{e}{\alpha}\cos\phi \Longrightarrow \quad dv = -\frac{e}{\alpha}\sin\phi\, d\phi\,, \quad \sqrt{\frac{e^2}{\alpha^2} - v^2} = \frac{e}{\alpha}\sqrt{1-\cos^2\phi} = \pm\frac{e}{\alpha}\sin\phi\,, \tag{4.45}$$

and therefore

$$\theta = \int \frac{dv}{\sqrt{\frac{e^2}{\alpha^2} - v^2}} = -\int d\phi = -\phi\,, \tag{4.46}$$

where we have chosen the integration constant to be zero. We can do this because the rotational symmetry of the problem means that we can choose $\theta = 0$ to be any direction in the plane. Working our way back, we then find

$$v = \frac{e}{\alpha}\cos\theta = \frac{1}{r} - \frac{1}{\alpha} \quad \Longleftrightarrow \quad \frac{\alpha}{r} = e\cos\theta + 1 \tag{4.47}$$

$$\Longleftrightarrow \quad \boxed{r = \frac{\alpha}{1 + e\cos\theta} \qquad \text{with} \qquad \alpha = \frac{\ell^2}{\mu k}\,, \quad e = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}\,.} \tag{4.48}$$

This is the equation for a *conic section*, where e is the *eccentricity*, and governs the shape of the orbit (while α governs the size).

- If $e = 0$, r is constant and the orbit is a circle, as can be easily seen from (4.48).
- If $0 < e < 1$, the orbit is closed, with $1/(1+e) \leq r \leq 1/(1-e)$. In this case, the orbit is an *ellipse*.
- If $e = 1$, $r \to \infty$ as $\theta \to \pi$, so the orbit is open (just). In this case the orbit is a *parabola*.
- If $e > 1$, the orbit is again open (unbounded), but there is a limit to the possible angles, $\cos\theta < 1/e$. In this case the orbit is a *hyperbola*.

4.5.1 The shapes of the orbits

Starting from equation (4.48) we may now write the shape of the orbit in Cartesian coordinates. We remember that $\cos\theta = \frac{x}{r}$ and insert that into (4.48), yielding

$$r = \frac{\alpha}{1 + \frac{ex}{r}}$$

This equation can be rewritten as $r = \alpha - ex$. By squaring the equation and substituting $r^2 = x^2 + y^2$ we get

$$y^2 = \alpha^2 - 2\alpha e x + (e^2 - 1)x^2.$$

This is the equation for a conic section, which has a solutions ellipses, parabolas and hyperbolas. We note that the sign in front of x^2 depends on whether $e > 1$ or $e < 1$. Completing the square over x then reads

$$
\begin{aligned}
y^2 &= \alpha^2 + \left(e^2 - 1\right)\left(x^2 - 2x\frac{\alpha e}{e^2-1}\right) \\
&= \alpha^2 + \left(e^2 - 1\right)\left(x^2 - 2x\frac{\alpha e}{e^2-1} + \frac{\alpha^2 e^2}{\left(e^2-1\right)^2} - \frac{\alpha^2 e^2}{\left(e^2-1\right)^2}\right) \\
&= \alpha^2 - \frac{\alpha^2 e^2}{e^2-1} + \left(e^2 - 1\right)\left(x - \frac{\alpha e}{\left(e^2-1\right)}\right)^2 \\
&= -\frac{\alpha^2}{e^2-1} + \left(e^2 - 1\right)\left(x - \frac{\alpha e}{e^2-1}\right)^2
\end{aligned}
$$

Now if we identify $a = \frac{\alpha}{1-e^2}$ and $b^2 = \frac{\alpha^2}{(1-e^2)^2}$ we can reformulate the equations as

$$
\boxed{\frac{\left(x-ea\right)^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{with} \qquad e^2 - 1 = \frac{2E\ell^2}{\mu k^2}} \tag{4.49}
$$

We can now see that when:

- $E < 0 \Rightarrow 1 - e^2 > 0$. This equation describes an ellipse with semi-major axis a and semi-minor axis b. See Figure 4.3.
- $E = 0 \Rightarrow e = 1$. Then (4.49) reduces to $y^2 = \alpha^2 - 2\alpha x$, which is a parabolic curve.
- $E > 0 \Rightarrow e > 1 \Rightarrow b^2 < 0$, and the above equation reads

$$
\frac{\left(x-ea\right)^2}{a^2} - \frac{y^2}{|b|^2} = 1.
$$

  This equation describes a hyperbolic curve with the closet passage to the origin being ea and the asymptotic lines being $y = x\frac{|b|}{a}$. See Figure 4.4.

4.6 More on conic sections

Conic sections are all solutions of equations of the type

$$
Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0\,.
$$

They are called conic sections because this type of equation appears when you intersect a cone, described by the equation $x^2+y^2 = kz^2$, with a plane, described by the equation $\alpha x + \beta y + \gamma z = \delta$. The type of curve described by these equations depends on the parameters A, B, C:

- If $B^2 - 4AC < 0$, we get an *ellipse*. After a change of variables, this curve can be written on the form

$$
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,. \tag{4.50}
$$

- If $B^2 - 4AC = 0$, we get a *parabola*. The equation can then be written, after a change of variables, as
$$y = ax^2\,. \tag{4.51}$$

- If $B^2 - 4AC > 0$, we get a *hyperbola*. We can then make a change of variables to write the resulting equation as
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{or} \qquad xy = a^2\,. \tag{4.52}$$

That this definition is equivalent to (4.48) is by no means obvious. In the following we will look in some more detail at the different definitions of conic sections, and establish that they are indeed equivalent. In the process, we will also find some important relations between the various parameters of the conic sections, in particular the ellipse.

There is yet another general definition of a conic section, namely the curve defined by $r = ep$, where p is the distance to a straight line, called the *directrix*. It is straightforward to prove the equivalence of this and (4.48). If we place a straight line parallel to the y-axis at $x = \alpha/e$, then the distance p from a point on the curve (4.48) to this line is

$$\begin{aligned} p = \frac{\alpha}{e} - x = \frac{\alpha}{e} - r\cos\theta = \frac{\alpha}{e} - \frac{\alpha\cos\theta}{1+e\cos\theta} \\ = \frac{\alpha(1+e\cos\theta) - e\alpha\cos\theta}{e(1+e\cos\theta)} = \frac{\alpha}{e(1+e\cos\theta)} = \frac{r}{e}\,. \end{aligned} \tag{4.53}$$

4.6.1 Ellipse

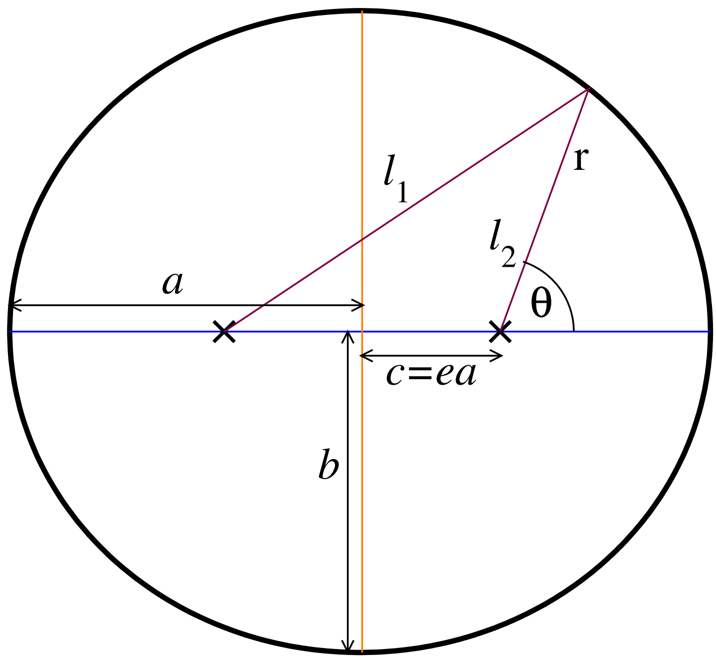

Figure 4.3: An ellipse with major semiaxis a, minor semiaxis b and foci at a distance c from the centre of the ellipse.

An ellipse, pictured in Figure 4.3, can be defined by first identifying two points, called the *foci* (denoted by crosses in Fig. 4.3). If ℓ_1 and ℓ_2 are the distances of a point from each of the two foci, then an ellipse is any set of points where the sum of ℓ_1 and ℓ_2 is constant,

$$\ell_1 + \ell_2 = 2a\,. \tag{4.54}$$

This definition has an optical interpretation: if you place a light source at one focus (and a screen between to the two foci), then light that is reflected off any point of the ellipse will gather at the second focus — which is indeed why it is called a focus. This definition can also be used to draw an ellipse, using two pins and a piece of string. You pin the ends of a string of length $2a$ at each focus, pull the string tight with the tip of your pen, and move the pen around while keeping the string tight. This will trace out an ellipse.

We will now show that (4.54) is equivalent to (4.50), if we take the origin to be at the centre of the ellipse. The two foci are then located at $(-c, 0)$ and $(c, 0)$, and the distances ℓ_1, ℓ_2 from a point (x, y) to each focus are

$$\ell_1^2 = (x+c)^2 + y^2\,, \qquad \ell_2^2 = (x-c)^2 + y^2\,. \tag{4.55}$$

The ellipse is given by

$$\begin{aligned}
& \ell_1 + \ell_2 = \sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a & (4.56)\\
\iff\quad & \sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2} & (4.57)\\
\iff\quad & (x+c)^2 + y^2 = 4a^2 + (x-c)^2 + y^2 - 4a\sqrt{(x-c)^2 + y^2} & (4.58)\\
\iff\quad & 2cx = 4a^2 - 2cx - 4a\sqrt{(x-c)^2 + y^2} & (4.59)\\
\iff\quad & \sqrt{(x-c)^2 + y^2} = a - \frac{c}{a}x & (4.60)\\
\iff\quad & x^2 - 2cx + c^2 + y^2 = a^2 - 2cx + \frac{c^2}{a^2}x^2 & (4.61)\\
\iff\quad & y^2 + \frac{a^2 - c^2}{a^2}x^2 = a^2 - c^2 & (4.62)\\
\iff\quad & \boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{with} \quad b^2 = a^2 - c^2\,.} & (4.63)
\end{aligned}$$

The two main ("long" and "short") diameters of the ellipse, with lengths $2a$ and $2b$, are called the *major axis* and the *minor axis*, respectively. The parameters a and b, being half of the major and minor axes, are called the *major semiaxis* and *minor semiaxis*.

To show that this is also equivalent to (4.48), we will have to put the origin at one of the foci. Let us choose the right one to be specific. Instead of (4.50) we now have

$$\frac{(x+c)^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{4.64}$$

$$\begin{aligned}\iff \quad a^2b^2 &= b^2x^2 + 2b^2cx + b^2c^2 + a^2y^2 \\ &= (b^2 - a^2)x^2 + a^2(x^2 + y^2) + 2b^2cx + b^2c^2 \\ &= -c^2x^2 + a^2r^2 + 2b^2cx + b^2c^2\end{aligned} \tag{4.65}$$

$$\iff \quad a^2r^2 = b^2(a^2 - c^2) - 2b^2cx + c^2x^2 = b^4 - 2b^2cx + c^2x^2 = (b^2 - cx)^2 \tag{4.66}$$

$$\iff \quad r = \frac{b^2 - cx}{a} = \frac{b^2 - cr\cos\theta}{a} \tag{4.67}$$

$$\iff \quad r(1 + \frac{c}{a}\cos\theta) = \frac{b^2}{a} \tag{4.68}$$

$$\iff \quad \boxed{r = \frac{\alpha}{1 + e\cos\theta} \qquad \text{with} \qquad e = \frac{c}{a}, \quad \alpha = \frac{b^2}{a} = a(1 - e^2)} \tag{4.69}$$

So we see that (4.48) indeed corresponds to an ellipse, with the origin at one of the foci, if $e < 1$. This corresponds to Kepler's first law:

> *The planets move in elliptical orbits, with the sun at one focus.*

4.6.2 Parabola

If one of the foci is taken away to infinity, the resulting curve becomes a parabola. The optical interpretation of this is that parallel rays (corresponding to rays coming in from a source at infinity) are focused in a single point. This property is widely used in telescopes and satellite dishes, which all tend to have a parabolic shape.

It will be left as an exercise to prove that (4.48), with $e = 1$, is equivalent to the usual $y = Ax^2$.

4.6.3 Hyperbola

If the eccentricity $e > 1$, the resulting curve is a hyperbola. On inspection of (4.48) you will see that $r \to \infty$ as $\theta \to \pm\cos^{-1}(1/e) = \pm\theta_{\text{max}}$. The curve will therefore approach, but never touch, the two straight lines defined by $\theta = \pm\theta_{\text{max}}$. One may also construct the mirror image of this curve,

$$r = \frac{\alpha}{1 - e\cos\theta} = \frac{\alpha}{1 + e\cos(\pi - \theta)} \,. \tag{4.70}$$

The two mirror images, called the two *branches* of the hyperbola, can be shown to be equivalent to the expressions (4.52). In the second case, we see that $\theta_{\text{max}} = \frac{\pi}{4}$ or 45°.

Example 4.3

A satellite in orbit around the earth has speed $v = 7400$m/s at its apogee, 630km above the surface of the earth. What is its distance from the surface of the earth at perigee, and what is its speed at that point?

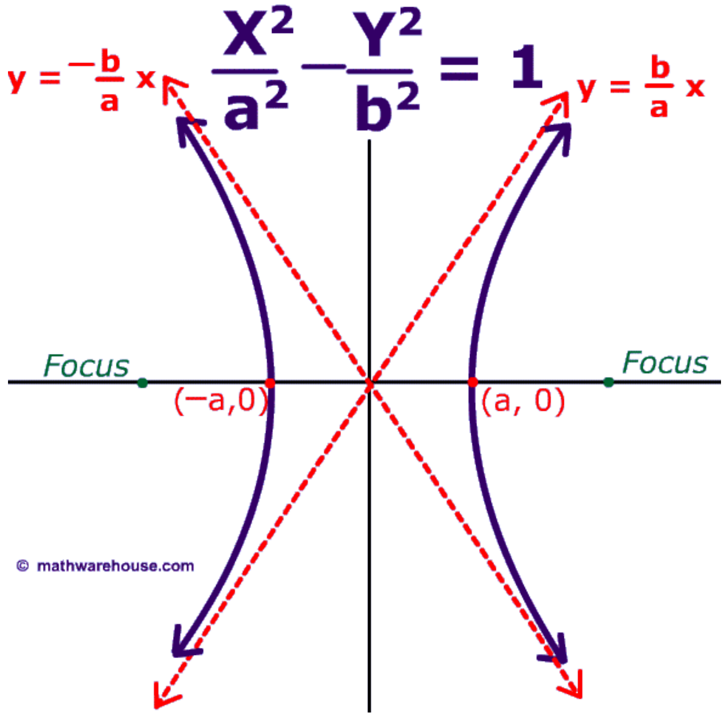

Figure 4.4: A hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.The two branches stay within (but approach asymptotically) the cone $y = \pm\frac{b}{a}x$

Answer:

The energy of a satellite with mass m, orbiting the earth at a distance r and travelling with a speed v is

$$E = \frac{1}{2}mv^2 - G\frac{mM_\oplus}{r}\,. \tag{4.71}$$

The angular momentum is

$$\ell = |\vec{L}| = m|\vec{v} \times \vec{r}| = mvr\sin\theta_{vr}\,. \tag{4.72}$$

At perigee or apogee the velocity (and momentum) is perpendicular to the radial direction, so $\ell = mvr$ at these points of the orbit.

Since the mass of the satellite is much smaller than the mass of the earth, the reduced mass μ can be replaced by m:

$$\mu = \frac{mM_\oplus}{m + M_\oplus} \approx \frac{mM_\oplus}{M_\oplus} = m\,. \tag{4.73}$$

The constant k in the inverse-square force is in this case $k = GmM_\oplus$. With this knowledge we can work out the parameters α (or major semiaxis a) and eccentricity e for the orbit. There are several ways doing this. The simplest is probably using the relation between the energy and semimajor axis, which holds for an elliptical orbit:

$$a = -\frac{k}{2E} = -\frac{GmM_\oplus}{mv^2 - 2GmM_\oplus/r} = \frac{GM_\oplus r}{2GM_\oplus - rv^2}\,. \tag{4.74}$$

In our problem, the speed of the satellite at apogee is $v = 7400\text{m/s}$. The distance from the centre of the earth is $r = 630$ km $+ R_\oplus = 7.000 \cdot 10^6$ km. This gives

$$a = \frac{6.674 \cdot 10^{-11}\text{m}^3/\text{kg}\,\text{s}^2 \cdot 5.9736 \cdot 10^{24}\text{kg} \cdot 7.000 \cdot 10^6\text{m}}{2 \cdot 6.674 \cdot 10^{-11}\text{m}^3/\text{kg}\,\text{s}^2 \cdot 5.9736 \cdot 10^{24}\text{kg} - 7.000 \cdot 10^6\text{m} \cdot (7400\text{m/s})^2}$$
$$= 6740\text{km}\,. \tag{4.75}$$

The sum of the apogee distance r_+ and the perigee distance r_- is twice the major semiaxis,

$$r_+ + r_- = 2a \quad \Longleftrightarrow \quad r_- = 2a - r_+ = 2 \cdot 6740\text{km} - 7000\text{km} = 6481\text{km}\,. \tag{4.76}$$

Therefore, the height of the satellite above earth at perigee is $r_- - R_\oplus$ =111km.

The speed may be found from the angular momentum,

$$\ell = mv_+r_+ = mv_-r_- \quad \Longrightarrow \quad v_+ = \frac{r_-}{r_+}v_- = \frac{7000}{6481}7400\text{m/s} = 7994\text{m/s}\,. \tag{4.77}$$

4.7 Kepler's third law

Let us look again at the elliptic orbit. We can use the relations we have found above, together with Kepler's second law, to derive an expression for the period, the tine T it takes to complete one full orbit. From Kepler's second law we have

$$\frac{dA}{dt} = \frac{\ell}{2\mu} \quad \Longrightarrow \quad A = \int_0^T \frac{dA}{dt}dt = \frac{\ell}{2\mu}T\,. \tag{4.78}$$

The area of an ellipse is $A = \pi ab$, so we will need to find a suitable expression for the minor semiaxis b. We know from above that $b^2 = a^2(1 - e^2)$. Using the expression for the eccentricity e, we get

$$b = a\sqrt{1 - e^2} = a\sqrt{\frac{-2E\ell^2}{\mu k^2}} = a\sqrt{\frac{\ell^2}{\mu ka}} = \frac{\ell}{\sqrt{\mu k}}\,a^{\frac{1}{2}}\,. \tag{4.79}$$

Putting this together, we have

$$T = \frac{2\mu}{\ell}A = \frac{2\pi\mu}{\ell}ab = \frac{2\pi\mu}{\ell}\frac{\ell}{\sqrt{\mu k}}a^{3/2} = 2\pi\sqrt{\frac{\mu}{k}}a^{3/2}\,, \tag{4.80}$$

or

$$T^2 = \frac{4\pi^2\mu}{k}a^3\,. \tag{4.81}$$

For planets orbiting the sun, the ratio $\mu/k = 1/G(m + M_\odot) \approx 1/(GM_\odot)$ to a very high approximation, so the proportionality constant will be the same for all planets! This is Kepler's third law: *The square of the orbital period varies like the cube of the major axis.*

Example 4.4

A comet is observed travelling at a speed of 64.0 km/s at is closest approach to the sun, 64.5 million km from the sun. Will this comet ever be seen again, and if so, when?

What would the answer be if the closest distance to the sun was 65.0 million km?

Answer: The solution of this problem follows the same lines as that of Example 3, with the mass of the sun, $M_\odot$, replacing the mass of the earth, $M_\oplus$. The comet will be seen again if the orbit is closed, which happens if the total energy $E < 0$. Alternatively, it is possible to calculate the eccentricity e and determine whether $e < 1$ (closed orbit) or $e > 1$ (open orbit).

We may first note that Newton's constant G and the mass of the sun $M_\odot$ always occur in the combination $GM_\odot$ so we may calculate this product once and for all,

$$\begin{aligned} GM_\odot &= 6.674 \cdot 10^{-11}\mathrm{m}^3/\mathrm{kg\,s}^2 \cdot 1.9881 \cdot 10^{30}\mathrm{kg} = 1.3275 \cdot 10^{20}\mathrm{m}^3/\mathrm{s}^2 \\ &= 1.3275 \cdot 10^{11}\mathrm{km}^3/\mathrm{s}^2 \,. \end{aligned} \tag{4.82}$$

If the speed of the comet at perihelion is $v = 64.0$ km/s and the distance is $r = 64.5 \cdot 10^6$ km, we find that

$$\frac{E}{m} = \frac{v^2}{2} - \frac{GM_\odot}{r} = \frac{64.0^2}{2}\mathrm{km}^2/\mathrm{s}^2 - \frac{1.3275 \cdot 10^{11}\mathrm{km}^3/\mathrm{s}^2}{64.5 \cdot 10^6\mathrm{km}} = -10.14\mathrm{km}^2/\mathrm{s}^2 \,. \tag{4.83}$$

So $E < 0$ and the orbit is closed (an ellipse). The comet will therefore be seen again. To find out when, we use Kepler's third law,

$$T = \frac{2\pi}{\sqrt{GM_\odot}} a^{3/2} \,. \tag{4.84}$$

We must first find the semimajor axis a, which is given by

$$a = -\frac{GM_\odot}{2E/m} = \frac{1.3275 \cdot 10^{11}\mathrm{km}^3/\mathrm{s}^2}{2 \cdot 10.14\mathrm{km}^2/\mathrm{s}^2} = 6.5152 \cdot 10^9\mathrm{km} \,. \tag{4.85}$$

Inserting this into (4.84) gives

$$T = \frac{2\pi}{\sqrt{1.3275 \cdot 10^{11}\mathrm{km}^3/\mathrm{s}^2}} (6.5152 \cdot 10^9\mathrm{km})^{3/2} = 9.069 \cdot 10^9\mathrm{s} = 104963\mathrm{d} = 287\mathrm{yr} \,. \tag{4.86}$$

The comet will be seen again in 287 years.

Making the perihelion distance just a bit larger, 65.0 million km, we find that

$$\frac{E}{m} = \left(\frac{64^2}{2} - \frac{1.3275 \cdot 10^{11}}{65 \cdot 10^6}\right)\mathrm{km}^2/\mathrm{s}^2 = 5.69\mathrm{km}^2/\mathrm{s}^2 \,. \tag{4.87}$$

This comet now moves in a hyperbolic orbit, and will never be seen again.

4.8 Kepler's equations

It is possible to integrate the angle equation,

$$\frac{d\theta}{dt} = \frac{\ell}{\mu r^2} = \frac{\ell(1+e\cos\theta)^2}{\mu(\ell^2/\mu k)^2} \quad \Longleftrightarrow \quad \frac{d\theta}{(1+e\cos\theta)^2} = \frac{\mu k^2}{\ell^2}dt\,. \tag{4.88}$$

However, the integral that we obtain from this is very ugly and must be expressed in terms of special functions (elliptic integrals) that cannot be straightforwardly inverted to obtain θ as a function of the time t.

Instead, we can go back to the energy equation, and the integral (4.29) we obtained from this,

$$\begin{aligned} t - t_0 &= \sqrt{\frac{\mu}{2}} \int\limits_{r_{\min}}^{r} \frac{dr}{\sqrt{E + \frac{k}{r} - \frac{\ell^2}{2\mu r^2}}} \\ &= \sqrt{\frac{\mu}{-2E}} \int\limits_{r_{\min}}^{r} \frac{rdr}{\sqrt{-r^2 - \frac{k}{E}r - \frac{k}{4E^2} + \frac{k^2}{4E^2}(1 + \frac{2\ell^2 E}{\mu k^2})}} \\ &= \sqrt{\frac{\mu}{-2E}} \int\limits_{r_{\min}}^{r} \frac{rdr}{\sqrt{a^2e^2 - (r-a)^2}}\,. \end{aligned} \tag{4.89}$$

We can now make the substitution

$$r - a = -ae\cos\psi \tag{4.90}$$

$$\Longrightarrow \quad \sqrt{a^2e^2 - (r-a)^2} = ae\sin\psi\,, \tag{4.91}$$

$$rdr = (r - a + a)dr = (-ae\cos\psi + a)(ae\sin\psi d\psi) \tag{4.92}$$

Putting this into (4.89) we get

$$\begin{aligned} t &= \sqrt{\frac{\mu}{2|E|}} \int (-ae\cos\psi + a)d\psi = \sqrt{\frac{\mu}{k}}\sqrt{\frac{k}{2|E}}a(\psi - e\sin\psi) + C \\ &= \sqrt{\frac{\mu}{k}}a^{3/2}(\psi - e\sin\psi) + C\,. \end{aligned} \tag{4.93}$$

To determine the integration constant C, we take $t = 0$ at perihelion. At this point we have

$$r_0 - a = a(1-e) - a = -ae = -ae\cos\psi_0 \Longrightarrow \quad \psi_0 = 0\,,\ C = 0\,. \tag{4.94}$$

This gives us

$$\boxed{\begin{aligned} r &= a(1 - e\cos\psi) \\ t &= \sqrt{\frac{\mu}{k}}a^{3/2}(\psi - e\sin\psi) \end{aligned}} \qquad \textbf{Kepler's equations} \tag{4.95}$$

The parameter ψ is called the *eccentric anomaly*. This name dates back to medieval, ptolemean astronomy where all the heavenly bodies were assumed to move in perfect circles. To rescue this assumption the planets were assumed to sit on circles which were themselves orbiting the earth (or the sun in the Copernicus picture). This epicycle motion motion was called the 'anomaly'. The angle θ is called the *true anomaly*. We have already seen that $\theta = \psi = 0$ at the perihelion, and we can also see that $\theta = \psi = \pi$ at aphelion. If $e = 0$ (circular motion) we have $\psi = \theta$ always.

Solving Kepler's equations is not straightforward. In fact, Kepler himself said:

> *I am sufficiently satisfied that it cannot be solved a priori, on account of the different nature of the arc and the sine. But if I am mistaken, and any one shall point out the way to me, he will be in my eyes the great Apollonius.*