Electricity and Magnetism 2 and Statistical Thermodynamics (MP232)
Assignment 3

Please hand in your solutions no later than Tuesday, March 30, at the start of the 11am lecture. Late assignments will not be accepted. If you have questions about this assignment, please ask your tutor,
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or your lecturer,
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Ex. 3.1: Wave equations and such

a. From Maxwell's equations, derive that the magnetic field satisfies the following equation

$$\nabla^2 \mathbf{B} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = -\mu_0 \nabla \times \mathbf{J}$$

The Maxwell equations are

$$\begin{array}{ll} \nabla \cdot E = \frac{\rho}{\epsilon_0} & \nabla \times E = -\frac{\partial B}{\partial t} \\ \nabla \cdot B = 0 & \nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t} \end{array}$$

and we note that $\mu_0 \epsilon_0 = \frac{1}{c^2}$.
To derive the equation given in the question, we take the curl of the equation for the curl of $\mathbf{B}$. This yields

$$\nabla \times \nabla \times B = \mu_0 \nabla \times J + \frac{1}{c^2}\frac{\partial(\nabla \times E)}{\partial t}$$

where we exchanged the space and time derivatives. On the right hand side we can now substitute from the Maxwell equation for $\nabla \times \mathbf{E}$, to obtain an equation in terms of $\mathbf{B}$ only. This gives $\frac{\partial(\nabla \times E)}{\partial t} = -\frac{\partial^2 \mathbf{B}}{\partial t^2}$.
The left hand side can also be simplified. We have

$$\nabla \times \nabla \times \mathbf{B} = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = -\nabla^2 \mathbf{B}.$$

The first equality is a general vector identity for the curl of the curl of a vector field and the second equality is true because $\nabla \cdot \mathbf{B} = 0$. Putting our results together, we get

$$\nabla^2 \mathbf{B} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = -\mu_0 \nabla \times \mathbf{J}$$

b. Inside a homogeneously charged insulator that rotates around the z-axis at angular velocity ω, the current density has the form $\mathbf{J} = \rho\,\omega(y, -x, 0)$, where ρ is the constant charge density in the object. Assuming this form of $\mathbf{J}$ and assuming that $\mathbf{B}$ does not depend on x and y, show that the equation from part **a.** reduces to

$$\frac{\partial^2 \mathbf{B}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = 2\mu_0 \rho\,\omega \hat{\mathbf{z}}$$

Because $\mathbf{B}$ does not depend on x and y, $\nabla^2 \mathbf{B} = \frac{\partial^2 \mathbf{B}}{\partial z^2}$.
Also, $\nabla \times \mathbf{J} = \rho\omega \nabla \times (y, -x, 0) = -2\rho\omega\hat{\mathbf{z}}$.

We see that that, for B_x and B_y, we have simply the wave equations

$$\frac{\partial^2 B_x}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 B_x}{\partial t^2} = 0 \qquad \frac{\partial^2 B_y}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 B_y}{\partial t^2} = 0,$$

that is, the same equations as for the magnetic field in vacuum. To solve the wave equation for B_x (or B_y), we make a change of variables.

c. Define $u = z + ct$ and $v = z - ct$. Show that, for any function f

$$\frac{\partial^2 f}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 4\frac{\partial^2 f}{\partial u \partial v}$$

We have $\partial_z f = (\partial_u f)\frac{\partial u}{\partial z} + (\partial_v f)\frac{\partial v}{\partial z} = \partial_u f + \partial_v f$. This also gives

$$\begin{aligned}\partial_z^2 f &= \partial_z(\partial_u f + \partial_v f)\\ &= \partial_z(\partial_u f) + \partial_z(\partial_v f)\\ &= \partial_u\partial_u f + \partial_v\partial_u f + \partial_u\partial_v f + \partial_v\partial_v f = \partial_u^2 f + \partial_v^2 f + 2\partial_u\partial_v f.\end{aligned}$$

Similarly $\partial_t f = (\partial_u f)\frac{\partial u}{\partial t} + (\partial_v f)\frac{\partial v}{\partial t} = c\partial_u f - c\partial_v f$. This gives

$$\begin{aligned}\partial_t^2 f &= \partial_t(c\partial_u f - c\partial_v f)\\ &= c\partial_t(\partial_u f) - c\partial_t(\partial_v f)\\ &= c^2\partial_u\partial_u f - c^2\partial_v\partial_u f - c^2\partial_u\partial_v f + c^2\partial_v\partial_v f = c^2(\partial_u^2 f + \partial_v^2 f) - 2c^2\partial_u\partial_v f.\end{aligned}$$

Using these results, we see that indeed

$$\frac{\partial^2 f}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 4\frac{\partial^2 f}{\partial u \partial v}$$

d. Show that $B_x(z,t) = f_+(z+ct) + f_-(z-ct)$, for some functions f_+ and f_-.

Using the result of the previous part, the wave equation for B_x reduces to $\frac{\partial^2 B_x}{\partial u \partial v} = 0$. Integrating with respect to u gives a constant c, which, however, may still be an arbitrary function of v, so we find $\frac{\partial B_x}{\partial v} = c(v)$. Now integrating with respect to v gives a function $f_+(v)$ of v whose derivative is $c(v)$, plus a constant, which can be an arbitrary function $f_-(u)$ of u. Hence we find indeed that $B_x(z,t) = f_+(v) + f_-(u) = f_+(z+ct) + f_-(z-ct)$, for some functions f_+ and f_-.

e. Use the Maxwell equation for $\nabla \cdot \mathbf{B}$ to simplify the equation B_z and then solve for B_z. Using the solution, explain why the situation described in this problem cannot continue indefinitely.

We have $\nabla \cdot \mathbf{B} = \partial_x B_x + \partial_y B_y + \partial_z B_z 0$ and since $\mathbf{B}$ does not depend on x and y this means $\partial_z B_z = 0$. Subsituting this in the equation from part **b.** we find that $\frac{1}{c^2}\frac{\partial^2 B_z}{\partial t^2} = 2\mu_0 \rho\,\omega$, or $\frac{\partial^2 B_z}{\partial t^2} = 2\mu_0\rho\, c^2\omega$. Integrating twice with respect to t, we find that $B_z = \mu_0\rho\, c^2\omega t^2 + c_1 t + c_2$, where c_1 and c_2 are constants of integration. This would mean that the B_z would grow quadratically with t, for all times. Obviously this cannot continue forever, since first of all, an infinite amount of energy would be stored in the B-field and secondly, there would be a linearly (and indefinitely) growing electromotive force on any horizontal loop. It seems likely that when one tries to maintain the situation, either the torque available for rotation will eventually be insufficient, or there will be an electric discharge, changing the charge density on the insulator.

Ex. 3.2: Plane waves

An electromagnetic wave has an electric field given by

$$\mathbf{E} = E_0 \cos(kx - \omega t)\, \hat{\mathbf{z}}$$

a. Write the frequency ν, the wavelength λ, and the velocity c of this wave in terms of ω and k.

   We have $\lambda = \frac{2\pi}{k}$, $\nu = \frac{\omega}{2\pi}$ and $c = \frac{\omega}{k}$.

b. Calculate the frequency of a wave with $\lambda = 510nm$ (green light). Also calculate the wavelength of a wave with frequency $88.2MHz$ (RTÉ radio 1 FM).

   We have $\lambda\nu = c \approx 3.0 \times 10^8 m/s$,
   so $\lambda = c/\nu \approx (3.0 \times 10^8 (m/s)/\nu)$ and $\nu = c/\lambda \approx (3.0 \times 10^8 (m/s)/\lambda)Hz$.
   For green light, this gives $\nu = (3.0 \times 10^8 (m/s)/(510 \times 10^{-9} m) \approx 5.9 \times 10^{14} Hz$
   For RTÉ radio 1, we get $\lambda \approx (3.0 \times 10^8 (m/s)/(88.2 \times 10^6 Hz)) \approx 3.4m$

c. Calculate the **B**-field of this wave by direct use of the Maxwell equations

   We can us the equation for $\nabla \times E$ to get the time derivative of the B-field. We have

$$\frac{\partial B}{\partial t} = -\nabla \times E$$

   We have, in this case, $\nabla \times E = -(\partial_x E_z)\, \hat{\mathbf{y}}$, so we get

$$\frac{\partial B}{\partial t} = +(\partial_x E_z)\, \hat{\mathbf{y}} = -kE_0 \sin(kx - \omega t)\, \hat{\mathbf{y}}$$

   and it follows that, up to a constant, which we would not usually consider part of the wave's field,

$$\mathbf{B} = -\frac{k}{\omega} E_0 \cos(kx - \omega t)\, \hat{\mathbf{y}} = -\frac{E_0}{c} \cos(kx - \omega t)\, \hat{\mathbf{y}}$$

d. Calculate the energy density u and Poynting vector **S** of this wave.

   We have

$$\begin{aligned} u &= \frac{\epsilon_0}{2}\left(E^2 + c^2 B^2\right) \\ &= \frac{\epsilon_0}{2}\left(E_0^2 \cos^2(kx - \omega t) + c^2 \frac{E_0^2}{c^2} \cos^2(kx - \omega t)\right) \\ &= \epsilon_0 E_0^2 \cos^2(kx - \omega t) \end{aligned}$$

   and

$$\begin{aligned} \mathbf{S} &= \epsilon_0 c^2 \left(\mathbf{E} \times \mathbf{B}\right) \\ &= -\epsilon_0 c^2 \frac{E_0^2}{c} \cos^2(kx - \omega t)\, \hat{\mathbf{z}} \times \hat{\mathbf{y}} \\ &= +\epsilon_0 c E_0^2 \cos^2(kx - \omega t)\, \hat{\mathbf{x}} \end{aligned}$$