**Electricity and Magnetism 2 and Statistical Thermodynamics (MP232)**
**Assignment 2**

*Please hand in your solutions no later than Tuesday, March 9, at the start of the 11am lecture. Late assignments will not be accepted. If you have questions about this assignment, please ask your tutor,*

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## Ex. 2.1 Flux, Induction and EMF

A circular loop of copper wire, of radius $a$ sits in a constant magnetic field $\mathbf{B} = (0, 0, B)$. We choose standard Cartesian coordinates with the origin at the center of the loop.

a. If the loop lies in the $(x, y)$-plane, what is the magnetic flux through the flat planar surface bounded by this loop?

The magnetic flux $\Phi$ through a surface $S$ is defined as the integral of $\mathbf{B} \cdot \hat{\mathbf{n}}$ over the surface, where $\hat{\mathbf{n}}$ is the unit normal vector to the surface,

$$\Phi = \int_S \mathbf{B} \cdot \hat{\mathbf{n}} \, dS.$$

In this case the normal vector to the surface is $\hat{\mathbf{n}} = (0, 0, 1)$ and so $\mathbf{B}\cdot\hat{\mathbf{n}} = B$, everywhere on the surface. This means the integral is just the product of the surface area of the loop and the constant $B$, so we have $\Phi = \pi a^2 B$.

b. The loop is rotated anticlockwise over an angle of $45°$ ($\frac{\pi}{4}$ radians) around the $x$-axis. After this rotation, what is the unit normal vector to the flat surface bounded by the loop? And what is the magnetic flux through this surface?

The new unit normal vector is $\frac{1}{\sqrt{2}}(0, -1, 1)$ (it probably helps to draw a sketch to see this). As a result, $\mathbf{B} \cdot \hat{\mathbf{n}} = \frac{1}{\sqrt{2}} B$, everywhere on the surface. This means the integral is the product of the surface area of the loop and the constant $\frac{B}{\sqrt{2}}$, giving $\Phi = \frac{\pi a^2 B}{\sqrt{2}}$.

The loop is brought back to its original position and then, at time $t = 0$, is set into motion rotating anticlockwise around the $x$-axis at constant angular velocity $\omega$ (in radians), so that at any time the normal vector to the flat planar surface bounded by the loop makes an angle $\omega t$ with the $z$-axis (and hence the magnetic field).

c. Find an expression for this normal vector as a function of time. Also find the corresponding expression for the magnetic flux through the loop.

The new unit normal vector at time $t$ is $(0, -\sin(\omega t), \cos(\omega t))$ (again, draw a sketch to see this). As a result, $\mathbf{B} \cdot \hat{\mathbf{n}} = \cos(\omega t)B$, everywhere on the surface. The flux is then the product of $\cos(\omega t)B$ with the surface area of the loop, giving $\Phi = \pi a^2 B \cos(\omega t)$.

d. Using Faraday's law, calculate the electromotive force in the loop as a function of time. There will be an alternating induction current in the loop as a result of this EMF. Use Lenz's law to find out the direction of this current when the motion is just starting.

Faraday's law says that the electromotive force through a loop at time $t$ is equal to the negative of the the time derivative of the magnetic flux through the loop:

$$EMF(loop) = -\frac{d\Phi}{dt}$$

Here, we know the flux $\Phi(t) = \pi a^2 B \cos(\omega t)$ so all we need to do to get the EMF is calculate the time derivative. This gives $EMF(loop, t) = \pi a^2 B\omega \sin(\omega t)$ (Note that the sign from the derivative of the cosine and the sign from the equation above cancel). Lenz's law says that the induction current caused by the change in the magnetic flux through the loop will be in such a direction as to counteract the change in flux. In this case, at the start of the motion, the upward flux is maximally positive, and it is decreasing due to the loop's motion. Hence the induction current in the loop will try to create more positive magnetic flux in the upward direction, to counteract the decrease. Using the right hand screw rule for finding the direction of the magnetic field caused by a current, we see that, in order to have a magnetic field pointing in the positive $z$-direction, the induction current must be flowing counterclockwise around the loop, when the loop is viewed from the positive $z$-direction (or clockwise, when the loop is viewed from the negative $z$-direction).

e. Assume that a resistor of resistance $0.1\Omega$ has been inserted in the loop and that the resistance of the loop itself can be neglected. If the strength of the magnetic field is $10mT$ and the radius of the loop is $10cm$, at what angular frequency $\omega$ must we rotate the loop to produce a maximal current of $50mA$?

we use Ohm's law to relate the current to the EMF, giving $I = \frac{EMF}{R}$ (In the electrostatic case we would have $EMF = V$ and fact this notation is often used also in the case of more general EMFs, even if there is no electrostatic potential). The maximal current occurs when the EMF is maximal and this happens when $\sin(\omega t) = \pm 1$, which gives $|EMF_{max}| = \pi a^2 B\omega$ and hence $I_{max} = \frac{\pi a^2 B}{R}\omega$. Now if we want to produce a maximal current of $50mA = 50 \times 10^{-3}A$ at $B = 10mT = 10 \times 10^{-3}T$ and $a = 10cm = 1.0 \times 10^{-2}m$ and $R = 0.1\Omega$, this means we must have

$$I_{max} = 50 \times 10^{-3}A = \frac{\pi a^2 B}{R}\omega = \frac{\pi\, 1.0 \times 10^{-4}m^2\, 10 \times 10^{-3}T}{0.1\Omega} = 1.0 \times 10^{-5}\omega\, As$$

and this yields $\omega = \frac{50 \times 10^2}{\pi}\ rad/s$ or in other words, the loop has to make approximately 253 full revolutions every second.

### Ex. 2.2 Maxwell's equations and continuity

a. State Maxwell's four equations for the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$. Give the names of all quantities that appear and give their units in terms of the SI units of mass ($kg$), time ($s$), length ($m$) and charge ($C$). For any constants that appear, give the value in SI units (look these up if necessary). For each of the four equations, give a single sentence description of some of the physical information it contains.

The equations are

$$\nabla \cdot E = \frac{\rho}{\epsilon_0} \qquad \nabla \times E = -\frac{\partial B}{\partial t}$$
$$\nabla \cdot B = 0 \qquad \nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

$\mathbf{E}$ is the electric field and its unit is $N/C = V/m = \frac{kg\, m}{s^2 C}$.
$\mathbf{B}$ is the magnetic field and its unit is $T = \frac{Ns}{Cm} = \frac{kg}{Cs}$.
$\mathbf{J}$ is the electric current density and its unit is $\frac{A}{m^2} = \frac{C}{m^2 s}$.
$\rho$ is the electric charge density and its unit is $\frac{C}{m^3}$.
The constant $\epsilon_0$ is the permittivity of free space, or of vacuum and has a value of approximately $8.854 \times 10^{-12} \frac{C^2}{N\, m^2} = 8.854 \times 10^{-12} \frac{C^2\, s^2}{kg\, m^3}$
The constant $\mu_0$ is the permeability of free space, also just called the magnetic constant and has a value of exactly $4\pi \times 10^{-7} \frac{N}{A^2} = 4\pi \times 10^{-7} \frac{kgm}{C^2}$
Note that the product $\epsilon_0 \mu_0$ has units $\frac{s^2}{m^2}$ and is exactly equal to $\frac{1}{c^2}$, where $c$ is the speed of light (approximately $3 \times 10^8\, m/s$).
For completeness, the differential operator $\nabla$ has unit $m^{-1}$ and $\frac{\partial}{\partial t}$ has unit $s^{-1}$.
Some short information on the equations. The equation for the divergence of $\mathbf{E}$ (often called Gauss' law) describes generation of electric fields due to the presence of electric charge. The equation for the divergence of $\mathbf{B}$ conveys the information that there is no such thing as magnetic charge (or at least it has not been observed yet). The equation for the curl of $\mathbf{E}$ (often called Faraday's law) describes the generation of induced Electric fields (and corresponding electromotive forces) due to changing magnetic fields. The equation for the curl of $\mathbf{B}$ describes how magnetic fields are generated due to electric currents, as well as due to changing electric fields. The term in this equation that involves the time derivative of the electric field is necessary for charge conservation and is called the *displacement current.*

b. State the equation of continuity (or charge conservation). Derive this equation from Maxwell's equations.

The continuity equation is

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0.$$

To derive this, take the divergence of both sides of the equation for the curl of $\mathbf{B}$. Since the divergence of the curl of a vector field vanishes, the left hand side disappears and we are left with

$$0 = \mu_0 \nabla \cdot \mathbf{J} + \mu_0 \epsilon_0 \nabla \cdot (\frac{\partial E}{\partial t}).$$

Moving the time derivative outside the divergence and substituting from the equation for the divergence of $E$, we obtain the continuity equation (after dividing out an overall factor $\mu_0$).

c. Give two examples of current densities $\mathbf{J}(\mathbf{x})$ which satisfy the continuity equation in the case that $\frac{\partial \rho}{\partial t} = 0$.

The continutity equation reduces to $\nabla \cdot \mathbf{J} = 0$ in this case. The simplest example is a constant current density (constant in space, but not necessarily in time), which obviously has zero divergence. However a very large class of examples exist. In particular, we can start from any vector field $A$ and take $J = \nabla \times A$. Since the divergence of a curl vanishes, this satisfies $\nabla \cdot \mathbf{J} = 0$.

In the next two parts, we will consider Maxwell's equations in vacuum (with $\rho = 0$ and $\mathbf{J} = \mathbf{0}$) and their relation to the equations with nonzero charge and current density.

d. Show that the vacuum equations are *linear* in the fields $\mathbf{E}$ and $\mathbf{B}$, that is,

– Show that if $(\mathbf{E}, \mathbf{B})$ is a solution to the equations, then, for any constant $\lambda$, $(\lambda\mathbf{E}, \lambda\mathbf{B})$ is also a solution

– Show that if $(\mathbf{E}, \mathbf{B})$ and $(\mathbf{E}', \mathbf{B}')$ are two solutions to the equations, then their *superposition* $(\mathbf{E} + \mathbf{E}', \mathbf{B} + \mathbf{B}')$ is also a solution.

The crucial ingredient here is to notice that multiplication by a constant factor and addition both commute with taking derivatives, so that for example for any constant $\lambda$, we have $\nabla \cdot \lambda E = \lambda \nabla \cdot E = 0$, since we were given that $E$ solves the vacuum Maxwell equations and hence $\nabla \cdot E = 0$. Similarly $\nabla \cdot (E + E') = \nabla \cdot E + \nabla \cdot E' = 0 + 0 = 0$, etc.

e. Indicate briefly why the Maxwell equations with currents are *not* linear in the sense of part **d.** Now assume we have a solution $(\mathbf{E}, \mathbf{B})$ to the full Maxwell equations with given charge density $\rho$ and current density $\mathbf{J}$ and also a solution $(\mathbf{E_{vac}}, \mathbf{B_{vac}})$ to Maxwell's equation in vacuum. Show that $(\mathbf{E} + \mathbf{E_{vac}}, \mathbf{B} + \mathbf{B_{vac}})$ is a solution to Maxwell's equations for the same charge and current densities as $(\mathbf{E}, \mathbf{B})$.

The Maxwell equations with currents are not linear in the sense described above, since for example if we have two electric fields $E$ and $E'$ with $\nabla \cdot E = \nabla \cdot E' = \frac{\rho}{\epsilon_0}$, then $\nabla \cdot (E + E') = \nabla \cdot E + \nabla \cdot E' = 2\frac{\rho}{\epsilon_0}$. Of course the full equations are linear if we add not only the fields but also the charge and current densities.
For the addition of vacuum solutions to solutions with nonzero charge and current densities, we proceed as before, so for example $\nabla \cdot (E + E_{vac}) = \nabla \cdot E + \nabla \cdot E_{vac} = \frac{\rho}{\epsilon_0} + 0 = \frac{\rho}{\epsilon_0}$, showing that $E + E_{vac}$ solves the first of the Maxwell equations.

f. (BONUS) The Maxwell equations for vacuum have an extraordinary amount of symmetry between the magnetic and electric fields. This is reflected in the set of solutions. Show that, if $(\mathbf{E}, \mathbf{B})$ is a solution to the equations, then for any angle $\theta$, the fields $(\mathbf{E}', \mathbf{B}')$ given by the equation below are also a solution.

$$\mathbf{E}' = \sin(\theta)\mathbf{E} + c\cos(\theta)\mathbf{B} \qquad \mathbf{B}' = \frac{1}{c}\cos(\theta)\mathbf{E} + \sin(\theta)\mathbf{B}$$

Here, $c$ is the speed of light (note that $\epsilon_0\mu_0 = \frac{1}{c^2}$).
What happens for the special case $\theta = \frac{\pi}{2}$?

This is the simplest manifestation of a class of mathematically very interesting symmetries/dualities, called *electric-magnetic dualities*, which occur in so called *gauge theories* (field theories which describe for example the standard model of elementary particle physics). Since this is a bonus question, there is no worked solution (but of course you can always ask for help)