Electricity and Magnetism 2 and Statistical Thermodynamics (MP232)
Assignment 2

Please hand in your solutions no later than Tuesday, March 9, at the start of the 11am lecture. Late assignments will not be accepted. If you have questions about this assignment, please ask your tutor,
Glen Burrella, (`glen-dot-burella-at-nuim-dot-ie`), Office 1.1, Mathematical Physics,
or your lecturer,
Joost Slingerland, (`joost-at-thphys-dot-nuim-dot-ie`), Office 1.7D, Mathematical Physics.

Ex. 2.1 Flux, Induction and EMF

A circular loop of copper wire, of radius a sits in a constant magnetic field $\mathbf{B} = (0, 0, B)$. We choose standard Cartesian coordinates with the origin at the center of the loop.

a. If the loop lies in the (x, y)-plane, what is the magnetic flux through the flat planar surface bounded by this loop?

b. The loop is rotated anticlockwise over an angle of $45°$ ($\frac{\pi}{4}$ radians) around the x-axis. After this rotation, what is the unit normal vector to the flat surface bounded by the loop? And what is the magnetic flux through this surface?

The loop is brought back to its original position and then, at time $t = 0$, is set into motion rotating anticlockwise around the x-axis at constant angular velocity ω (in radians), so that at any time the normal vector to the flat planar surface bounded by the loop makes an angle ωt with the z-axis (and hence the magnetic field).

c. Find an expression for this normal vector as a function of time. Also find the corresponding expression for the magnetic flux through the loop.

d. Using Faraday's law, calculate the electromotive force in the loop as a function of time. There will be an alternating induction current in the loop as a result of this EMF. Use Lenz's law to find out the direction of this current when the motion is just starting.

e. Assume that a resistor of resistance 0.1Ω has been inserted in the loop and that the resistance of the loop itself can be neglected. If the strength of the magnetic field is $10mT$ and the radius of the loop is $10cm$, at what angular frequency ω must we rotate the loop to produce a maximal current of $50mA$?

Ex. 2.2 Maxwell's equations and continuity

a. State Maxwell's four equations for the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$. Give the names of all quantities that appear and give their units in terms of the SI units of mass (kg), time (s), length (m) and charge (C). For any constants that appear, give the value in SI units (look these up if necessary). For each of the four equations, give a single sentence description of some of the physical information it contains.

b. State the equation of continuity (or charge conservation). Derive this equation from Maxwell's equations.

c. Give two examples of current densities $\mathbf{J}(\mathbf{x})$ which satisfy the continuity equation in the case that $\frac{\partial \rho}{\partial t} = 0$.

In the next two parts, we will consider Maxwell's equations in vacuum (with $\rho = 0$ and $\mathbf{J} = \mathbf{0}$) and their relation to the equations with nonzero charge and current density.

d. Show that the vacuum equations are *linear* in the fields $\mathbf{E}$ and $\mathbf{B}$, that is,

- Show that if $(\mathbf{E}, \mathbf{B})$ is a solution to the equations, then, for any constant λ, $(\lambda\mathbf{E}, \lambda\mathbf{B})$ is also a solution
- Show that if $(\mathbf{E}, \mathbf{B})$ and $(\mathbf{E}', \mathbf{B}')$ are two solutions to the equations, then their *superposition* $(\mathbf{E} + \mathbf{E}', \mathbf{B} + \mathbf{B}')$ is also a solution.

e. Indicate briefly why the Maxwell equations with currents are *not* linear in the sense of part **d.** Now assume we have a solution $(\mathbf{E}, \mathbf{B})$ to the full Maxwell equations with given charge density ρ and current density $\mathbf{J}$ and also a solution $(\mathbf{E}_{\mathbf{vac}}, \mathbf{B}_{\mathbf{vac}})$ to Maxwell's equation in vacuum. Show that $(\mathbf{E} + \mathbf{E}_{\mathbf{vac}}, \mathbf{B} + \mathbf{B}_{\mathbf{vac}})$ is a solution to Maxwell's equations for the same charge and current densities as $(\mathbf{E}, \mathbf{B})$.

f. (BONUS) The Maxwell equations for vacuum have an extraordinary amount of symmetry between the magnetic and electric fields. This is reflected in the set of solutions. Show that, if $(\mathbf{E}, \mathbf{B})$ is a solution to the equations, then for any angle θ, the fields $(\mathbf{E}', \mathbf{B}')$ given by the equation below are also a solution.

$$\mathbf{E}' = \sin(\theta)\mathbf{E} + c\cos(\theta)\mathbf{B} \qquad \mathbf{B}' = \frac{1}{c}\cos(\theta)\mathbf{E} + \sin(\theta)\mathbf{B}$$

Here, c is the speed of light (note that $\epsilon_0\mu_0 = \frac{1}{c^2}$).
What happens for the special case $\theta = \frac{\pi}{2}$?