Electricity and Magnetism 2 and Statistical Thermodynamics (MP232)
Assignment 1

Please hand in your solutions no later than Monday, February 22. Late assignments will not be accepted. If you have questions about this assignment,
please ask your tutor,
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Ex. 1.1 Motion of a particle in constant electric and magnetic fields

A particle of charge q and mass m is moving through a constant magnetic field $\mathbf{B}$. We choose the z-axis along the magnetic field, so that we have $\mathbf{B} = (0, 0, B)$, with $B = |\mathbf{B}|$. The particle has velocity $\mathbf{v} = (v_x, v_y, v_z)$.

a. Assume that the only force acting on the particle is due to the magnetic field. Show that we must have

$$\begin{aligned} \frac{dv_x}{dt} &= \frac{q}{m} B v_y \\ \frac{dv_y}{dt} &= -\frac{q}{m} B v_x \\ \frac{dv_z}{dt} &= 0 \end{aligned}$$

The only force acting on the particle is due to the magnetic field, so it is the Lorentz force $\mathbf{F}_{Lor} = q(\mathbf{v} \times \mathbf{B})$. We then have $\mathbf{F}_{total} = m\frac{dv}{dt} = q(\mathbf{v} \times \mathbf{B})$ and consequently $\frac{dv}{dt} = \frac{q}{m}(\mathbf{v} \times \mathbf{B})$. We can now work out the components of the exterior product (using $\mathbf{B} = (0, 0, B)$) and then the three components of the equation for $\frac{dv}{dt}$ turn out to be precisely the three equations given above.

b. Find the general solution for the velocity $\mathbf{v}(t)$ from the equations in part **a**. From there, find the general solution for the position $\mathbf{r}(t)$ of the particle.
HINT: the general solution of the equation $\frac{d^2 f}{dt^2} = -\omega^2 f$ *is given by* $f(t) = A\sin(\omega t) + B\cos(\omega t)$*, with* A *and* B *arbitrary constants.*

First of all we have $\frac{dv_z}{dt} = 0$, so v_z is a constant. Also $\frac{dr_z}{dt} = v_z$ and hence, using that v_z does not depend on t, we have $r_z = v_z t + Z$, where Z is another constant (Z is the z-component of the particle's position at time $t = 0$). Now take the time derivative of the equation for $\frac{dv_x}{dt}$. This gives

$$\frac{d^2 v_x}{dt^2} = \frac{q}{m} B \frac{dv_y}{dt},$$

where we used that q, m and B are constant in time. We now substitute the equation for $\frac{dv_y}{dt}$ into the right hand side, giving

$$\frac{d^2 v_x}{dt^2} = -\left(\frac{qB}{m}\right)^2 v_x = -\omega_c^2 v_x.$$

Here we note the appearance of the *cyclotron frequency* $\omega_c = \frac{qB}{m}$. We can now apply the hint to find the general solution for v_x. We have $v_x(t) = A\sin(\omega_c t) + B\cos(\omega_c t)$, with A and B arbitrary constants. We can obviously apply the same trick to the equation for $\frac{dv_y}{dt}$ to get $v_y = A_y \sin(\omega_c t) + B_y \cos(\omega_c t)$ with new constants A_y and B_y. If we substitute the tentative solutions for v_x and v_y back into the original equations, we see that we get

$$\begin{aligned} A\omega_c \cos(\omega_c t) - B\omega_c \sin(\omega_c t) &= \omega_c A_y \sin(\omega_c t) + \omega_c B_y \cos(\omega_c t) \\ A_y\omega_c \cos(\omega_c t) - B_y\omega_c \sin(\omega_c t) &= -\omega_c Asin(\omega_c t) - \omega_c Bcos(\omega_c t) \end{aligned}$$

so that the equations are satisfied for all t precisely if we take $A_y = -B$ and $B_y = A$. Hence the general solution for $\mathbf{v}$ is given by

$$(v_x, v_y, v_z) = (A\sin(\omega_c t) + B\cos(\omega_c t), -B\sin(\omega_c t) + A\cos(\omega_c t), v_z)$$

Where v_z is constant. Integrating this with respect to t, we get the general solution for $\mathbf{r}$, which is

$$(r_x, r_y, r_z) = (-\frac{A}{\omega_c}\cos(\omega_c t) + \frac{B}{\omega_c}\sin(\omega_c t) + X, \frac{B}{\omega_c}\cos(\omega_c t) + \frac{A}{\omega_c}\sin(\omega_c t) + Y, v_z t + Z).$$

Here X,Y and Z are integration constants which are determined by the initial position of the particle.

Describe the motion of the particle for the following sets of initial positions and velocities. Give the centre and radius of the orbit where applicable.

c. $\mathbf{v}(t=0) = (u, 0, 0)$ and $\mathbf{r}(t=0) = (0, 0, 0)$.

From the general solution for the velocity and position of the particle, we see that $\mathbf{v}(t=0) = (B, A, v_z)$ and $\mathbf{r}(t=0) = (\frac{-A}{\omega_c} + X, \frac{B}{\omega_c} + Y, Z)$. Setting these equal to $(u, 0, 0)$ and $(0, 0, 0)$, we get $B = u$, $A = 0$, $v_z = 0$, $X = 0$, $Y = -\frac{u}{\omega_c}$ and $Z = 0$, so the orbit is described by the formula

$$(r_x, r_y, r_z) = (\frac{u}{\omega_c}\sin(\omega_c t), \frac{u}{\omega_c}\cos(\omega_c t) - \frac{u}{\omega_c}, 0).$$

Thus the particle describes a circular orbit of radius $\frac{u}{\omega_c}$ in the (x, y)-plane, centered on the point $(0, -\frac{u}{\omega_c}, 0)$. Note that $u = |v|$, so the radius is given by the usual formula for the *cyclotron radius* $r = \frac{m|v|}{q|B|}$.

d. $\mathbf{v}(t=0) = (0, u, 0)$ and $\mathbf{r}(t=0) = (0, 0, 0)$.

In the same way as in part **c.** we find that $B = 0$, $A = u$, $v_z = 0$, $X = \frac{u}{\omega_c}$, $Y = 0$ and $Z = 0$, so the orbit is described by the formula

$$(r_x, r_y, r_z) = (-\frac{u}{\omega_c}\cos(\omega_c t) + \frac{u}{\omega_c}, \frac{u}{\omega_c}\sin(\omega_c t), 0),$$

so this time the orbit is again a circle of radius $\frac{u}{\omega_c}$ in the (x, y)-plane, but now centered on the point $(\frac{u}{\omega_c}, 0, 0)$.

e. $\mathbf{v}(t=0) = (0,0,u)$ and $\mathbf{r}(t=0) = (x,y,0)$.

In the same way as in part **c.** we find that $B = 0$, $A = 0$, $v_z = u$, $X = x$, $Y = y$ and $Z = 0$, so the orbit is described by the formula

$$(r_x, r_y, r_z) = (x, y, ut),$$

In other words, the particle moves at constant velocity u along a line in the positive z direction which intersects the $(z = 0)$-plane at $(x, y, 0)$.

f. $\mathbf{v}(t=0) = (u,0,w)$ and $\mathbf{r}(t=0) = (0,0,0)$. What is the angle $\theta(t)$ that the orbit makes with the magnetic field in this case?

In the same way as in part **c.** we find that $B = u$, $A = 0$, $v_z = w$, $X = 0$, $Y = -\frac{u}{\omega_c}$ and $Z = 0$, so the orbit is described by the formula

$$(r_x, r_y, r_z) = (\frac{u}{\omega_c}\sin(\omega_c t), \frac{u}{\omega_c}\cos(\omega_c t) - \frac{u}{\omega_c}, wt).$$

So the orbit is a helix radius $\frac{u}{\omega_c}$, centered on the vertical line through $(0, -\frac{u}{\omega_c}, 0)$. The angle $\theta(t)$ between the orbit and the magnetic field can be calculated by taking the inner product between $\mathbf{v}$ and $\mathbf{B}$. We have $\mathbf{v}\cdot\mathbf{B} = |\mathbf{v}||\mathbf{B}|\cos(\theta)$, so $\theta = \arccos(\frac{\mathbf{v}\cdot\mathbf{B}}{|\mathbf{v}||\mathbf{B}|})$. in this case $\mathbf{v} = (u\cos(\omega_c t), -u\sin(\omega_c t), w)$ and $\mathbf{B} = (0,0,B)$ so $|\mathbf{v}| = \sqrt{u^2+w^2}$, $|\mathbf{B}| = B$ and $\mathbf{v}\cdot\mathbf{B} = wB$, so that $\theta = \arccos(\frac{w}{\sqrt{u^2+w^2}})$. Note that θ is constant in time and does not depend on the magnetic field.

We now add an electric field $\mathbf{E}$ in addition to the magnetic field and orthogonal to it, say in the y-direction: $\mathbf{E} = (0, E, 0)$ with $E = |\mathbf{E}|$.

g. Derive the equations of motion for the new situation and compare with part **a.** If the particle is initially stationary at the origin ($\mathbf{r} = (0,0,0)$), describe qualitatively how it will move away from there. How does the motion depend on the charge q?

The equations of motion are now given by $F_{total} = m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$. Working this out for the given electric field and the same magnetic field as before we get

$$\begin{aligned}
\frac{dv_x}{dt} &= \frac{q}{m}Bv_y \\
\frac{dv_y}{dt} &= \frac{q}{m}E - \frac{q}{m}Bv_x \\
\frac{dv_z}{dt} &= 0
\end{aligned}$$

If the particle is initially at rest then, initially, the only force that acts is the force due to the $\mathbf{E}$ field. This will accelerate the particle in the positive y-direction. As soon as the particle develops a nonzero velocity, the force due to the magnetic field starts to play a role and the direction of the particle's motion is changed. Depending on the sign of the particle's charge it will be bent away from its motion in the positive y-direction towards the positive or negative x-direction. For a standard right handed coordinate system, particles with positive charge will bend toward the positive x-direction and particles with negative charge towards the negative x-direction. Much more can of course be said and in fact the equations of motion can be solved exactly (solutions are discussed for example in the book by Griffiths).

We now remove the electric field in the y direction and instead add an electric field $\mathbf{E}$ parallel to the magnetic field, in the z-direction: $\mathbf{E} = (0, 0, E)$.

h. Derive the equations of motion for the new situation and give the general solution for the velocity $\mathbf{v}(t)$. Give an expression for the angle $\theta(t)$ that the orbit makes with the magnetic field in this case.

Working out the equations of motion as before we get

$$\begin{aligned}\frac{dv_x}{dt} &= \frac{q}{m}Bv_y \\ \frac{dv_y}{dt} &= -\frac{q}{m}Bv_x \\ \frac{dv_z}{dt} &= \frac{q}{m}E\end{aligned}$$

The equations for v_x and v_y are the same as in part **b.** and so the general solution for those velocity components as well as those position components is the same as before. For the z-direction, we find by integration that $v_z = \frac{q}{m}Et + V_z$ where V_z is a constant, and by further integration that $r_z = \frac{q}{2m}Et^2 + V_z t + Z$, where Z is another constant. We see that the x and y components of the particle's position still move in a circle, but the particle is now accelerating in the z-direction. Note that this does not influence the motion in the other directions, because the Lorentz force due to the $\mathbf{B}$-field depends only on the coordinates of the velocity perpendicular to the $\mathbf{B}$-field. We have, as before, that $\theta = \arccos(\frac{\mathbf{v}\cdot\mathbf{B}}{|\mathbf{v}||\mathbf{B}|})$. In this case

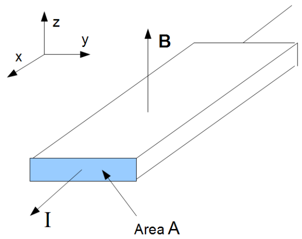
$$\mathbf{v} = (A\sin(\omega_c t) + B\cos(\omega_c t), -B\sin(\omega_c t) + A\cos(\omega_c t), \frac{q}{m}Et + V_z)$$

and as before, $\mathbf{B} = (0, 0, B)$ so $|\mathbf{v}| = \sqrt{A^2 + B^2 + \left(\frac{q}{m}Et + V_z\right)^2}$, $|\mathbf{B}| = B$ and $\mathbf{v}\cdot\mathbf{B} = \left(\frac{q}{m}Et + V_z\right)B$, so that

$$\theta = \arccos\left(\frac{\frac{q}{m}Et + V_z}{\sqrt{A^2 + B^2 + \left(\frac{q}{m}Et + V_z\right)^2}}\right).$$

Now θ is no longer constant in time. In fact, it decreases with time and approaches 0 as the velocity turns more and more in the vertical direction. However, θ still does not depend on the magnetic field. (Do not be confused by the B that appears in the formula for θ - it is a constant unrelated to the magnetic field - unfortunate notation!)

Ex. 1.1 The Hall effect



Charged particles of charge q move through a conducting slab of material, called a *Hall bar*, giving rise to a current I in the x-direction.

a. Assume that

   - the Hall bar has a constant cross sectional area A (orthogonal to the x-direction)
   - there is a constant density ρ of moving particles per unit volume
   - the particles move with (average) velocity v in the x direction.

   Now argue that we must have $I = qA\rho v$

   The current through a cross section of the bar is the amount of charge Q that passes that cross section per unit time. For a small time interval Δt we can write $I = \frac{Q}{\Delta t}$. If the charged particles move at a speed v then all charges in a block of material of length $v\Delta t$ pass through a given cross section during a time Δt. These are in fact all particles in a block of material of volume $Av\Delta t$. Since the particle density is ρ, the number of charges in this block is $\rho Av\Delta t$ and since each particle carries charge q, this represents a total charge $Q = q\rho Av\Delta t$. Finally $I = \frac{Q}{\Delta t} = qA\rho v$.

We now apply a magnetic field in the direction ortogonal to the current, say $\mathbf{B} = (0, 0, B)$. This gives rise to a Lorentz force on the particles which carry the current and causes their trajectories to bend in the y-direction. As a result, the density of charge carriers is raised one side of the Hall bar and lowered on the other side.

b. You are given that the current I flows in the positive x-direction. Now suppose that q is positive. On which side of the Hall bar will ρ be raised/lowered? And if q is negative?

   Using the right hand rule for the Lorentz force, we see that positive charges are pushed in the negative y-direction (left in the figure), so for positively charged particles, ρ will be raised on the left and lowered on the right. For negatively charged particles, we have to take into account that if the *current* is in the positive x-direction, this means that the charge particles are moving in the negative x-direction. Using the right hand rule with this input (and of course reversing the direction indicated by this rule due to the negative charge in the force formula $q\mathbf{v} \times \mathbf{B}$), we see that negatively charged particles are also pushed to the left, so the change in the *particle density* is the same in both cases. However, note that the effect on the *charge density* is different!

If the magnetic field is kept in place, an equilibrium charge density profile is reached; the "excess" of charge carriers on one side and the "shortage" of charge carriers on the other side cause an electric field $\mathbf{E}_{Hall} = (0, E_{Hall}, 0)$ whose action on the particles compensates for the force due to the magnetic field.

c. Show that, in the equilibrium situation, $E_{Hall} = \frac{IB}{qA\rho}$

In equilibrium, the total force on the particles is zero, so we must have $q(\mathbf{E}+\mathbf{v}\times\mathbf{B}) = 0$ In the current situation, this means $E_{Hall} = vB$ (check the sign for example by the right hand rule). Using the result of part **a.** we have $v = \frac{I}{qA\rho}$ and so we get $E_{Hall} = \frac{IB}{qA\rho}$.

d. Many simple devices for measuring the magnetic field are based on the Hall effect[1]. Explain how you could measure the magnetic field somewhere in space using a Hall bar with known shape and material properties. The voltage over the Hall bar in the directions orthogonal to the current is proportional to the component of the magnetic field in the direction orthogonal to the direction of the current and the measured voltage. We can get all components of $\mathbf{B}$ for example by rotating the Hall bar or using multiple Hall bars.

e. In some semiconductors, the current is carried by positively charged "holes" of charge $q = e$ (rather than electrons of charge $q = -e$). Can you tell this from the observed Hall field $\mathbf{E}_{Hall}$?

Yes, the direction of the Hall field (at given current I) is different for differently charged particles. This is of closely related to the answer in part **b.**

f. We have made a number of extremely crude assumptions and simplifications in this exercise, mostly implicitly. Give at least two examples of complications that should be taken into account in a more thorough treatment of the Hall effect.

Some examples: The charge carriers are usually microscopic particles that should be treated quantum mechanically. They interact with stationary charged and neutral parts of the medium they move in (metal ions etc.). The particles do not all move at the same speed, but rather there will be some distribution of speeds. Many materials will have all kinds of inhomogeneities and impurities which influence the behaviour of the moving particles, etc. etc. etc.

[1]Usually one does not measure the electric field but rather the voltage V_{Hall} or the resistance R_{Hall} over the width of the bar, but this does not matter for the principle