MP231 and MP232

Electromagnetic Theory

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CHAPTER I
From electric charges to potentials

§ 1. Preliminaries on books and units

IN this course we shall provide material which is intended to be self-contained but reference elsewhere is occasionally needed. It is also good practice to read around a subject as widely as one's time will allow.

There are a huge number of books on electromagnetic theory and so we only recommend three; the college library will provide one with many, many more. So our three titles are—the first one is the main text, the others are for subsidiary reading:

Grant and Phillips: Electromagnetism

Feynman: Feynman lectures in physics, volume 2.

Purcell: Berkeley lectures in physics, volume on electromagnetism

The units we shall use are MKSA units which are the standard units currently used in all the natural sciences and in engineering. For electromagnetism they are not ideal since (as we shall gradually see) their definitions contain arbitrary factors of π in situations where there is no circular, cylindrical or spherical symmetry. The most unfortunate consequence of this (as will become obvious at the time) factors of π then disappear from situations where there is some circular, cylindrical or spherical symmetry.

Finally a universal constant that occurs in electromagnetism is ϵ_0 known as the *permittivity of free space* [1]. In our MKSA units its value is given by

$$\begin{aligned} \epsilon_0 &= 8.85 \times 10^{-12} \text{ coulomb}^2/\text{newton-metre}^2 \\ \text{or entirely equivalently } \epsilon_0 &= 8.85 \times 10^{-12} \text{volt-metre/coulomb} \end{aligned} \tag{1.1}$$

The coulomb being the unit of charge as we shall see below. It is often useful, for numerical purposes, to know that, since c, the velocity of light is given by

$$c = 3 \times 10^8 \text{m/sec} \tag{1.2}$$

[1] The phrase *free space* often occurs in electromagnetic theory and it refers to *charge free space* i.e. a vacuum as opposed to a solid, liquid or gas.

then

$$\begin{aligned}\epsilon_0 c^2 &= \frac{10^7}{4\pi} \\ \Rightarrow \frac{1}{4\pi\epsilon_0} &\simeq 9 \times 10^9\end{aligned} \tag{1.3}$$

§ 2. Coulomb's law.

The fundamental fact lying at the base of all electromagnetism is that the forces between charges are of the inverse square type. The formal statement of this fact is known as *Coulomb's law.* Formally we have

Coulomb's law *If two charges of size q_1 and q_2 are located at $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively then the force $\mathbf{F}$ between them is given by*

$$\begin{aligned}\mathbf{F} &= \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|^2}(\widehat{\mathbf{r}_1 - \mathbf{r}_2}) \\ &= \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_1 - \mathbf{r}_2)\end{aligned} \tag{1.4}$$

When the force is computed with this formula the units of charge are called *Coulomb's.*

This law implies the familiar property that like charges repel and that unlike charges attract. With Coulomb's law under our belt we can immediately proceed to the notion of an electric field $\mathbf{E}$.

Definition (Electric field $\mathbf{E}$) *The electric field $\mathbf{E}$ at a point $\mathbf{r}$ exerted by any collection of charges is the force that would act on a unit charge placed at $\mathbf{r}$.*

Example *The electric field due to a single charge*

Combining this definition with Coulomb's law we can immediately compute the electric field $\mathbf{E}(\mathbf{r})$ at $\mathbf{r}$ exerted by a *single charge* q. For, if q is located at $\mathbf{r}_1$, then the force $\mathbf{F}$ on a unit charge at $\mathbf{r}$ is given by

$$\begin{aligned}\mathbf{F} &= \frac{q}{4\pi\epsilon_0}\frac{(\widehat{\mathbf{r} - \mathbf{r}_1})}{|\mathbf{r} - \mathbf{r}_1|^2} \\ \text{i.e. } \mathbf{E}(\mathbf{r}) &= \frac{q}{4\pi\epsilon_0}\frac{(\widehat{\mathbf{r} - \mathbf{r}_1})}{|\mathbf{r} - \mathbf{r}_1|^2}\end{aligned} \tag{1.5}$$

It now easily follows that if a charge Q is placed in an electric field $\mathbf{E}$ then it is acted on (at $\mathbf{r}$) with a force $\mathbf{F}(\mathbf{r})$ where

$$\mathbf{F}(\mathbf{r}) = Q\,\mathbf{E}(\mathbf{r}) \tag{1.6}$$

and if the precise point $\mathbf{r}$ meant is not important, we often abbreviate this to simply

$$\mathbf{F} = Q\,\mathbf{E} \tag{1.7}$$

Having obtained the electric field exerted by one charge we now want the field due to a collection of charges. For this purpose we need what is called the *principle of superposition.* This an *experimentally discovered fact* which says roughly that electric fields due to separate charges "add together". More formally we have

The principle of superposition. *If n charges $q_1, q_2, \ldots, q_n$ are located at $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ respectively, then their electric field $\mathbf{E}$ at r is additive, i.e. it is given by*

$$\mathbf{E}(\mathbf{r}) = \frac{q_1}{4\pi\epsilon_0}\frac{(\widehat{\mathbf{r}-\mathbf{r}_1})}{|\mathbf{r}-\mathbf{r}_1|^2} + \frac{q_2}{4\pi\epsilon_0}\frac{(\widehat{\mathbf{r}-\mathbf{r}_2})}{|\mathbf{r}-\mathbf{r}_2|^2} + \cdots + \frac{q_n}{4\pi\epsilon_0}\frac{(\widehat{\mathbf{r}-\mathbf{r}_n})}{|\mathbf{r}-\mathbf{r}_n|^2} \tag{1.8}$$

More briefly we can write

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_1 + \mathbf{E}_2 + \cdots + \mathbf{E}_n \\ &= \sum_{i=1}^{n} \mathbf{E}_i \\ \text{where } \ \mathbf{E}_i &= \frac{q_i}{4\pi\epsilon_0}\frac{(\widehat{\mathbf{r}-\mathbf{r}_i})}{|\mathbf{r}-\mathbf{r}_i|^2} \end{aligned} \tag{1.9}$$

The electric field, being a vector quantity, requires three quantities for its specification; actually this information triplet contains a lot of redundancy. We shall now see that only one scalar quantity is really needed to specify an electric field. this quantity is known as *the potential* and it is the next thing that we shall consider.

§ 3. The potential function V or Φ

There is a potential function V (also often denoted by Φ) associated with every electric field $\mathbf{E}$. For a given $\mathbf{E}$ it is defined by the equation

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= -\operatorname{grad} V(\mathbf{r}) \\ &\equiv -\nabla V(\mathbf{r}) \\ &= -\left(\frac{\partial V(\mathbf{r})}{\partial x}\mathbf{i} + \frac{\partial V(\mathbf{r})}{\partial y}\mathbf{j} + \frac{\partial V(\mathbf{r})}{\partial z}\mathbf{k}\right) \end{aligned} \tag{1.10}$$

where of course

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{1.11}$$

Example *The potential for a single charge*

If we place a single charge of size q_1, say, at the location $\mathbf{r}_1$then it is easy to check by direct differentiation that its potential at an arbitrary point $\mathbf{r}$ is given by

$$V(\mathbf{r}) = \frac{q_1}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}_1|} \tag{1.12}$$

i.e. one has that the electric field $\mathbf{E}$ of the charge, which we obtained above in 1.5 , is given by

$$\mathbf{E} = -\nabla\left(\frac{q_1}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}_1|}\right) \tag{1.13}$$

and indeed the differentiation gives us the result that

$$-\nabla\left(\frac{q_1}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}_1|}\right) = -\frac{q_1}{4\pi\epsilon_0}\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}_1|}\right) = \frac{q_1}{4\pi\epsilon_0}\frac{(\widehat{\mathbf{r}-\mathbf{r}_1})}{|\mathbf{r}-\mathbf{r}_1|^2} \tag{1.14}$$

in perfect agreement with 1.5.

The principle of superposition for potentials *It is easy to prove that the principle of superposition also applies to potentials. To prove this assume that we have, as in 1.8, n charges $q_1, q_2, \ldots, q_n$ at $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ respectively, then their potential $V(\mathbf{r})$ at $\mathbf{r}$ is given by*

$$\begin{aligned} V(\mathbf{r}) &= V_1(\mathbf{r}) + V_2(\mathbf{r}) + \cdots + V_n(\mathbf{r}) \\ &= \sum_{i=1}^{n} V_i(\mathbf{r}) \\ \textit{where}\ \ V_i(\mathbf{r}) &= \frac{q_i}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}_i|} \end{aligned} \tag{0.15}$$

This is easy to prove: we know know that the electric field produced by these charges is given by

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_1 + \mathbf{E}_2 + \cdots + \mathbf{E}_n \\ &= \sum_{i=1}^{n} \mathbf{E}_i \\ \text{where}\ \ \mathbf{E}_i &= \frac{q_i}{4\pi\epsilon_0}\frac{(\widehat{\mathbf{r}-\mathbf{r}_i})}{|\mathbf{r}-\mathbf{r}_i|^2} \end{aligned} \tag{1.16}$$

but we also know that

$$\mathbf{E}_i = -\nabla \left(\frac{q_i}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}_i|} \right) \tag{1.17}$$

Hence we can write

$$\begin{aligned} \mathbf{E} &= -\nabla \left(\frac{q_1}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}_1|} \right) - \nabla \left(\frac{q_2}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}_2|} \right) - \cdots - \nabla \left(\frac{q_n}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}_n|} \right) \\ &= -\nabla \left(\frac{q_1}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}_1|} + \frac{q_2}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}_2|} + \cdots + \frac{q_n}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}_n|} \right) \end{aligned} \tag{1.18}$$

In other words we have

$$\mathbf{E} = -\nabla V \tag{1.19}$$

where

$$\begin{aligned} V(\mathbf{r}) &= V_1(\mathbf{r}) + V_2(\mathbf{r}) + \cdots + V_n(\mathbf{r}) \\ \text{and } \; V_i(\mathbf{r}) &= \frac{q_i}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}_i|} \end{aligned} \tag{0.20}$$

which is indeed the principle of superposition.

Now we see that give a particular potential V it is easy to find the corresponding electric field $\mathbf{E}$: one just has to the differentiations appropriate for the expression $-\nabla V$. We would like to be able to go in the reverse direction: i.e. given the electric field $\mathbf{E}$ construct its associated potential V. This is indeed possible but is a little harder as, can easily be anticipated, it involves *integration* rather than *differentiation*. We are ready to digest the argument.

The argument rests on one technical piece of calculus. This is that if $f(x, y, z)$ is any differentiable function, then

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \tag{1.21}$$

which follows from Taylor's theorem. It also follows that

$$\int df = f \tag{1.22}$$

Suppose now that we are given an electric field $\mathbf{E}$. Let V be its potential function so that

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \tag{1.23}$$

But if we write

$$\mathbf{dr} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} \tag{1.24}$$

then we note that

$$\begin{aligned}\nabla V \cdot \mathbf{dr} &= \left(\frac{\partial V(\mathbf{r})}{\partial x} + \frac{\partial V(\mathbf{r})}{\partial y} + \frac{\partial V(\mathbf{r})}{\partial z}\right) \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}) \\ &= \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz \\ &= dV\end{aligned} \tag{1.25}$$

In other words we have

$$dV = -\mathbf{E} \cdot \mathbf{dr} \tag{1.26}$$

Finally we take a path Γ, say, beginning at an arbitrary but *fixed* point $\mathbf{r}_0$ and ending at $\mathbf{r}$ and we integrate along Γ. In this way we obtain

$$\begin{aligned}\int_\Gamma dV \equiv \int_{\mathbf{r}_0}^{\mathbf{r}} dV &= -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E} \cdot \mathbf{dr} \\ \Rightarrow V(\mathbf{r}) - V(\mathbf{r}_0) &= -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E} \cdot \mathbf{dr} \\ \Rightarrow V(\mathbf{r}) &= V(\mathbf{r}_0) - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E} \cdot \mathbf{dr}\end{aligned} \tag{1.27}$$

But we can discard the constant quantity $V(\mathbf{r}_0)$ on the right hand side of the last equation since we can always alter a potential V by a constant without changing its associated electric field: this is obvious if one simply notes that, if C is a constant, then

$$\nabla(V + C) = \nabla V, \quad \text{because } \nabla C = 0 \tag{1.28}$$

Thus we take our final expression for the potential V due to an electric field $\mathbf{E}$ to be simply

$$V(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E} \cdot \mathbf{dr} \tag{1.29}$$

Summarising the relations between $\mathbf{E}$ and V then gives us the pair of equations

$$\begin{aligned}\mathbf{E} &= -\nabla V \\ V &= -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E} \cdot \mathbf{dr}\end{aligned} \tag{1.30}$$

This brings the present chapter to a close.

CHAPTER II

Laplace's equation and the dipole potential

§ 1. Laplace's equation

OUR next topic will be to study an important equation due to Laplace and others which is obeyed by V. The potential V due to any (discrete) system of charges satisfies an equation known as *Laplace's equation.* This equation is

$$\nabla^2 V = 0 \tag{2.1}$$

or, spelled out in more detail,

$$\operatorname{div} \cdot \operatorname{grad} V = \nabla \cdot (\nabla V) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) V = 0 \tag{2.2}$$

The proof of Laplace's equation is not difficult; because of the superposition principle it requires just one simple calculation involving the potential due to a single charge. We now give the proof: take a general collection of n charges so that, as in 1.20, their potential at $\mathbf{r}$ is given by

$$\begin{aligned} V(\mathbf{r}) &= V_1(\mathbf{r}) + V_2(\mathbf{r}) + \cdots + V_n(\mathbf{r}) \\ \text{and} \quad V_i(\mathbf{r}) &= \frac{q_i}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}_i|} \end{aligned} \tag{0.3}$$

Hence

$$\begin{aligned} \nabla^2 V &= \nabla^2 \left(V_1(\mathbf{r}) + V_2(\mathbf{r}) + \cdots + V_n(\mathbf{r}) \right) \\ &= \sum_{i=1}^{n} \nabla^2 V_i(\mathbf{r}) \\ &= 0, \quad \text{since, as we shall now show,} \\ \nabla^2 V_i(\mathbf{r}) &= 0, \quad \text{for each } i \end{aligned} \tag{2.4}$$

All that remains is to show that

$$\nabla^2 V_i(\mathbf{r}) = 0 \tag{2.5}$$

and we do this by direct differentiation. We simply write

$$\begin{aligned}\mathbf{r} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\\ \mathbf{r}_i &= x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k}\end{aligned} \tag{2.6}$$

So that

$$\begin{aligned}|\mathbf{r} - \mathbf{r}_i| &= |\{(x - x_i)\mathbf{i} + (y - y_i)\mathbf{j} + (z - z_i)\mathbf{k}\}|\\ &= \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}\end{aligned} \tag{2.7}$$

This means that

$$\begin{aligned}\nabla^2 V_i(\mathbf{r}) &= \frac{q_i}{4\pi\epsilon_0}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) A\\ \text{where } A &= \left(\frac{1}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}}\right)\end{aligned} \tag{2.8}$$

and it is then a completely straightforward matter to verify that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}}\right) = 0 \tag{2.9}$$

So we have indeed proved Laplace's equation for an arbitrary *discrete* collection of charges.

It is worthwhile observing that Laplace's equation is really a consequence of the Coulomb's inverse square law of force. Hence the potential V for other situations where an inverse square law applies will also satisfy Laplace's equation. For example the *gravitational potential* produced by a system of masses also satisfies Laplace's equation.

§ 2. Laplace's equation, stability and atomic models

The requirement that a potential V satisfy Laplace's equation is quite a strong one: in particular it means that such a V cannot have a minimum (or indeed a maximum). We shall now show that has an immediate consequence for a simple atomic model for the element hydrogen.

First we shall show that V cannot have a minimum. Simple calculus suffices to accomplish this. First recall that if $f(x)$ is a function of the single variable x then, if x_0 is a *minimum* of $f(x)$, one has

$$\left.\frac{df(x)}{dx}\right|_{x=x_0} = 0, \quad \text{and} \quad \left.\frac{d^2 f(x)}{dx^2}\right|_{x=x_0} > 0 \tag{2.10}$$

Now consider a function of several variables such as the potential $V = V(x,y,z)$. The potential has a minimum at the point $(x_0, y_0, z_0) = \mathbf{p}$, say, if [1],

$$\begin{aligned} & \left.\frac{\partial V}{\partial x}\right|_{\mathbf{r}=\mathbf{p}} = 0, \ \left.\frac{\partial V}{\partial y}\right|_{\mathbf{r}=\mathbf{p}} = 0, \ \left.\frac{\partial V}{\partial z}\right|_{\mathbf{r}=\mathbf{p}} = 0 \\ \text{and} \quad & \left.\frac{\partial^2 V}{\partial x^2}\right|_{\mathbf{r}=\mathbf{p}} > 0, \ \left.\frac{\partial^2 V}{\partial y^2}\right|_{\mathbf{r}=\mathbf{p}} > 0, \ \left.\frac{\partial^2 V}{\partial z^2}\right|_{\mathbf{r}=\mathbf{p}} > 0, \end{aligned} \tag{2.11}$$

Hence when $\mathbf{r} = \mathbf{p}$ is a minimum, then, at $\mathbf{p}$, V satisfies

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) V > 0 \tag{2.12}$$

because of 2.11 and this simply contradicts Laplace's equation. Thus the potential V can never have a minimum (or indeed a maximum since maxima just correspond to negative rather than positive second derivatives.

The best that can happen to the potential that is consistent with its obeying Laplace's equation is that it can have a point $\mathbf{p}$ with the following properties.

$$\begin{aligned} & \left.\frac{\partial V}{\partial x}\right|_{\mathbf{r}=\mathbf{p}} = 0, \ \left.\frac{\partial V}{\partial y}\right|_{\mathbf{r}=\mathbf{p}} = 0, \ \left.\frac{\partial V}{\partial x}\right|_{\mathbf{r}=\mathbf{p}} = 0 \\ \text{and} \quad & \left.\frac{\partial^2 V}{\partial x^2}\right|_{\mathbf{r}=\mathbf{p}} > 0, \ \left.\frac{\partial^2 V}{\partial y^2}\right|_{\mathbf{r}=\mathbf{p}} > 0, \ \left.\frac{\partial^2 V}{\partial z^2}\right|_{\mathbf{r}=\mathbf{p}} < 0, \end{aligned} \tag{2.13}$$

or one of the other 5 cases where the signs of the second derivatives are not all the same.

Points $\mathbf{p}$ that satisfy 2.13 are called *saddle points.* They clearly correspond to *minima* when viewed from a direction whose second derivative is *positive* and *maxima* when viewed from a direction whose second derivative is *negative.*

The makes the origin of the term *saddle point* transparent: if $Sx, y)$ is a function which represents the surface of a horse's saddle then such a surface is obviously rising to a maximum when traversed along a curve parallel to the rib of the horse, while it drops to a minimum when traversed along a curve following the spine of the horse. The "lowest point" point on the surface of the saddle is then a saddle point.

The relevance for atomic models of this result is that the energy [2] $\mathcal{E}$of a charge q in the electric field produced by a potential V is given by

$$\mathcal{E} = qV \tag{2.14}$$

[1] Actually the conditions for a minimum are not quite these but we shall not go into that here and shall just assume that, for our particular V, these conditions can be used.

[2] We shall prove this shortly in chapter 3

Hence a minimum of V, if allowed, would imply a minimum of the energy $\mathcal{E}$. So Laplace's equation outlaws a minimum energy for a system consisting of a charge q in a potential V. This means that such a system can *never* be in a state of *stable equilibrium* because this requires a *minimum* of the energy.

To see this just think of the example of a ball, or a mass m, at rest at the bottom of a hill or valley. If the gravitational potential is also V then its energy is mV and this is a minimum at the bottom. The ball is also stable against small perturbations because a small kick, or perturbation, of the ball will cause it to move up the hill a little but then it will roll back. Conversely a ball at the saddle point on an actual saddle is at a point of *unstable equilibrium*: if it is displaced along the spine of the horse it will roll back under gravity, but if it is displaced parallel to a rib, or indeed in any direction other than along the spine, it will roll away and never come back.

Example *Laplace's equation and the Hydrogen atom*

Finally consider the hydrogen atom. This consists of two charges: the electron of *negative* charge e and the proton of *positive* charge $-e$. Let the electrostatic potential produced by the proton be V the energy $\mathcal{E}$ of the atom is given by

$$\mathcal{E} = eV \tag{2.15}$$

and we see that

$$\begin{aligned} \nabla^2 \mathcal{E} &= \nabla^2 (eV) \\ &= e\nabla^2 V \\ &= 0 \end{aligned} \tag{2.16}$$

hence $\mathcal{E}$ can never have a minimum; at best it can have a saddle point. The consequence that this has for the hydrogen atom is that there is no state of the two charges where they can be at rest and in stable equilibrium. This is why the electron is in rapid motion around the proton, there is no equilibrium state at rest. In general, then, we expect all atomic models to consist of charges *in motion*: stable equilibrium is not achievable with potentials that satisfy Laplace's equation.

This is taken for granted nowadays, but this was not so in the late nineteenth and early twentieth century when atomic models were not so well understood. Remember that the electron itself was only discovered in 1896. Finally the key ingredient in establishing the detail of atomic models was *quantum mechanics*; this was needed to show that an electron could not radiate away all its energy as it orbited the proton, this, however, is another story.

§ 3. The dipole potential and its atomic significance

A pair of magnetic or electric charges is often referred to as an electric or magnetic dipole, or simply a dipole when the electrical or magnetic context is clear. We want, now, to consider electrical dipoles and such a dipole has a potential V given by the superposition principle as a sum of two terms. This potential we shall call the *dipole potential* and it is therefore given by V where

$$V(\mathbf{r}) = \frac{q_1}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}_1|} + \frac{q_2}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}_2|} \tag{2.17}$$

Now we turn to atomic physics and show how any atom gives rise to a dipole. Taking any atom we have a clump of positive charge q, say, at the nucleus which we shall approximate by a single point charge q at the centre of the nucleus. But the electrons much further out form a cloud of negative charge whose total charge is $-q$; we shall approximate the electron cloud by a single charge $-q$ at the centre of the cloud. The key *experimental fact* is that the centre of the electron cloud is not at the centre of the nucleus but is only *near* to the centre of the nucleus.

This means that a simple picture of the charge distribution if the atom is that it consists of a pair of equal and opposite charges $\pm q$ separated by a small distance d (d being the distance between the centre of the nucleus and the centre of the electron cloud). In this way we have reduced[3] the electric charge structure of the atom to an electric dipole with $q_1 = q$, $q_2 = -q$ and $|\mathbf{r}_1 - \mathbf{r}_2| = d$.

Now we want to study the dipole potential for this atomic picture and so we set

$$q_1 = q, \quad q_2 = -q \quad \text{and } \mathbf{r}_2 = \mathbf{r}_1 + \mathbf{d} \tag{2.18}$$

where $\mathbf{d}$ is a small vector joining the centres of positive and negative charge. With these details in place we find that

$$\begin{aligned} V(\mathbf{r}) &= \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}_1|} - \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}_1-\mathbf{d}|} \\ &= \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{|\mathbf{r}-\mathbf{r}_1|} - \frac{1}{|\mathbf{r}-\mathbf{r}_1-\mathbf{d}|}\right\} \end{aligned} \tag{2.19}$$

Finally we want to exploit the fact that $|d|$ is *small* by which we mean small compared to $|\mathbf{r}|$; i.e. we are considering the value of $V(\mathbf{r})$ measured at a point $\mathbf{r}$ *outside* the atom. The only mathematics we shall need is a judicious use of the binomial theorem and we now set to work.

[3] This whole picture is obviously a simplification: it ignores quantum mechanics and the fact that the electrons are in rapid motion relative to each other and to the nucleus. It is even necessary to be aware that quantum mechanics shows that the nucleons are not static but oscillate somewhat. Nevertheless this picture does give useful theoretical and experimental results.

We have

$$\begin{aligned}
\frac{1}{|\mathbf{r}-\mathbf{r}_1-\mathbf{d}|} &= |(\mathbf{r}-\mathbf{r}_1)-\mathbf{d}|^{-1} \\
&= \left\{(\mathbf{r}-\mathbf{r}_1)^2 - 2\mathbf{d}\cdot(\mathbf{r}-\mathbf{r}_1)+\mathbf{d}^2\right\}^{-1/2} \\
&\simeq \left\{(\mathbf{r}-\mathbf{r}_1)^2 - 2\mathbf{d}\cdot(\mathbf{r}-\mathbf{r}_1)\right\}^{-1/2}, \quad \text{since } \mathbf{d}^2 \text{ is small} \\
&= |\mathbf{r}-\mathbf{r}_1|^{-1}\left\{1-\frac{2\mathbf{d}\cdot(\mathbf{r}-\mathbf{r}_1)}{(\mathbf{r}-\mathbf{r}_1)^2}\right\}^{-1/2} \\
&= |\mathbf{r}-\mathbf{r}_1|^{-1}\left\{1+\left(\frac{1}{2}\right)\frac{2\mathbf{d}\cdot(\mathbf{r}-\mathbf{r}_1)}{(\mathbf{r}+\mathbf{r}_1)^2}+\cdots\right\}, \text{ (binomial th}^m\text{)} \\
&= \frac{1}{|\mathbf{r}-\mathbf{r}_1|}+\frac{\mathbf{d}\cdot(\mathbf{r}-\mathbf{r}_1)}{(\mathbf{r}-\mathbf{r}_1)^3}+\cdots \\
&\simeq \frac{1}{|\mathbf{r}-\mathbf{r}_1|}+\frac{\mathbf{d}\cdot(\mathbf{r}-\mathbf{r}_1)}{(\mathbf{r}-\mathbf{r}_1)^3}
\end{aligned} \tag{2.20}$$

where we have neglected all terms containing powers of **d**

Having done our approximation work we now substitute 2.20 in the expression 2.19 above for the atomic potential to obtain an approximated atomic potential which we now formally call the *dipole potential* and denote it by V_{dipole}. Hence V_{dipole} is given by

$$\begin{aligned}
V_{dipole}(\mathbf{r}) &= \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{|\mathbf{r}-\mathbf{r}_1|}-\frac{1}{|\mathbf{r}-\mathbf{r}_1|}-\frac{\mathbf{d}\cdot(\mathbf{r}-\mathbf{r}_1)}{(\mathbf{r}-\mathbf{r}_1)^3}\right\} \\
\Rightarrow V_{dipole}(\mathbf{r}) &= -\frac{q}{4\pi\epsilon_0}\frac{\mathbf{d}\cdot(\mathbf{r}-\mathbf{r}_1)}{(\mathbf{r}-\mathbf{r}_1)^3}
\end{aligned} \tag{2.21}$$

It is customary to use the expression for the unit vector $(\mathbf{r}\hat{-}\mathbf{r}_1)$ in writing out V_{dipole}. So we note that

$$(\mathbf{r}\hat{-}\mathbf{r}_1) = \frac{\mathbf{r}-\mathbf{r}_1)}{|\mathbf{r}-\mathbf{r}_1|} \tag{2.22}$$

and that this immediately gives us our final expression for V_{dipole} which is

$$V_{dipole}(\mathbf{r}) = -\frac{q}{4\pi\epsilon_0}\frac{\mathbf{d}\cdot(\mathbf{r}\hat{-}\mathbf{r}_1)}{(\mathbf{r}-\mathbf{r}_1)^2}$$

The key point of significance for atomic physics is to note that for large distances from the atom, i.e. $|\mathbf{r}-\mathbf{r}_1|$ large we have

$$V_{dipole}(\mathbf{r}) \propto \frac{1}{|\mathbf{r}-\mathbf{r}_1|^2} \tag{2.23}$$

rather than $|\mathbf{r} - \mathbf{r}_1|^{-1}$; this is a much faster decrease with distance than might have been guessed and has obvious implications for the size of inter-atomic forces.

Example *The dipole potential and the Van der Waal's force*

In 1873 the Dutch physicist J. Van der Waals completed his thesis *Over de continuïteit van de gas- en vloeistoftoestand*: On the continuity of the gaseous and the liquid state; this contained the famous work [4] which described what are nowadays referred to as *Van der Waals forces*.

The Van der Waals attractive forces between molecules are now known to be due to interactions between the dipoles of pairs of molecules. These forces depend on the dipole potentials of each molecule and are found to be large only over a short range. This short range is because when one calculates the dependence of the dipole–dipole interaction on the separation r between the molecules one finds [5] that the force is proportional to

$$\frac{1}{r^7} \tag{2.24}$$

and this quantity decreases rapidly as r increases: for example if the distance between the molecules is doubled, the force is 2^7 times smaller, i.e. 128 times smaller. Hence these intermolecular forces will only be appreciable when the molecules are close enough together—this agrees well with experiment: the general picture of van der Waal is correct.

We see, then, that the dipole potential is of real practical significance.

[4] Van der Waals received the Nobel prize in 1910 for this work.

[5] Some quantum mechanics needs to be added the the electromagnetic theory to get the power 7 seen here.

CHAPTER III

Energy and work for systems of charges

§ 1. The energy of a system of charges

WE can obtain a precise expression for the energy of a system of charges by idealising slightly and *defining* the energy of such a system as being the work done in appropriately assembling the system.

The words *idealising* and *appropriately* are key words here as we now explain: We take an arbitrary system of n charges $q_1, q_2, \ldots, q_n$ located at $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ respectively. To *assemble* this system we first *disassemble it* by requiring that all the charges are moved off to infinity so that the distance between any pair of charges also tends to infinity. This latter point is easy to achieve if one thinks of sending each charge to infinity along a straight ray connecting the origin to its location. Then the appropriate assembly procedure is to assemble the system by moving in charges one at a time to their final locations and computing the *total* work W done in this entire process. This work W is them defined to be the energy $\mp\mathcal{E}$, say, of the system of charges; notice that we do not commit ourselves as to the sign in front of $\mathcal{E}$. We shall, in the end, *define* $\mathcal{E}$ by choosing the *minus* sign i.e. by writing

$$\mathcal{E} = -W \tag{3.1}$$

This choice of sign is natural from the physical point of view and we shall show below that it can be understood easily if one takes a simple system of two charges.

Now we supply the details. Let all the charges be at infinity as just described. Now bring in the first charge q_1 from infinity to its final destination $\mathbf{r}_1$. As q_1 moves in to $\mathbf{r}_1$ it sees a Universe with zero electric field and hence no *electrical* work has to be done to get q_1 into place. So far, so good. Now we start q_2 off on its journey; q_2 immediately sees the electric field $\mathbf{E}_1$ created by q_1 and so has to work against this as it moves. More precisely, if q_2 moves an infinitesimal amount along the direction $\mathbf{dl}$ it does electrical work

$$\mathbf{E}_1 \cdot dl \tag{3.2}$$

This in turn means that the *total* work done in moving q_2 along some path Γ_2 from infinity to its final position $\mathbf{r}_2$ is given by the integral

$$q_2 \int_{\Gamma_2} \mathbf{E_1} \cdot \mathbf{dl} \tag{3.3}$$

Now we bring in q_3 and it has to work against the *combined* electric fields of q_1 and q_2 which, by superposition gives the field

$$\mathbf{E_1} + \mathbf{E_2} \tag{3.4}$$

Hence to bring in q_3 along a path Γ_3 the work that has to be done is

$$q_3 \int_{\Gamma_2} (\mathbf{E_1} + \mathbf{E_2}) \cdot \mathbf{dl} \tag{3.5}$$

It should now be clear that the work done to bring in the charge q_i along Γ_i is given by

$$q_i \int_{\Gamma_i} (\mathbf{E_1} + \mathbf{E_2} + \cdots + \mathbf{E_{i-1}}) \cdot \mathbf{dl} \tag{3.6}$$

Therefore the grand total for the work W done in assembling all n charges is given by summing all the entries in the list

$$\begin{aligned}
&q_2 \int_{\Gamma_2} \mathbf{E_1} \cdot \mathbf{dl} \\
&q_3 \int_{\Gamma_2} (\mathbf{E_1} + \mathbf{E_2}) \cdot \mathbf{dl} \\
&\vdots \\
&q_i \int_{\Gamma_i} (\mathbf{E_1} + \mathbf{E_2} + \cdots + \mathbf{E_{i-1}}) \cdot \mathbf{dl} \\
&\vdots \\
&q_n \int_{\Gamma_n} (\mathbf{E_1} + \mathbf{E_2} + \cdots + \mathbf{E_{n-1}}) \cdot \mathbf{dl}
\end{aligned} \tag{3.7}$$

It transpires that we can do all the integrals in the above sum. They are all of the form

$$q_i \int_{\Gamma_i} \mathbf{E_j} \cdot \mathbf{dl} \tag{3.8}$$

for suitable i and j. But we know already that

$$\begin{aligned}
\mathbf{E_j}(\mathbf{r}) &= \frac{q_j}{4\pi\epsilon_0}\frac{(\widehat{\mathbf{r}-\mathbf{r}_j})}{|\mathbf{r}-\mathbf{r}_j|^2} \\
&= -\nabla V_j(\mathbf{r}) \qquad (3.9)\\
\text{where} \qquad V_j(\mathbf{r}) &= \frac{q_j}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}_j|}
\end{aligned}$$

This means that

$$\begin{aligned}
q_i\int_{\Gamma_i}\mathbf{E_j}\cdot\mathbf{dl} &= -q_i\int_{\Gamma_i}\nabla V_j(\mathbf{r})\cdot\mathbf{dl} \\
&= -q_i\int_{\Gamma_i} dV_j(\mathbf{r}) \qquad (3.10)\\
&= -q_iV_j(\mathbf{r_i}) \\
&= -\frac{q_iq_j}{4\pi\epsilon_0}\frac{1}{|\mathbf{r_i}-\mathbf{r}_j|}
\end{aligned}$$

So all the terms in the sum above are of this kind and it easy to check that there are exactly $n(n-1)/2$ of them corresponding to all possible pairs of charges (q_i, q_j). Recalling that we *defined* $\mathcal{E}$ above by writing

$$\mathcal{E} = -W \qquad (3.11)$$

This gives our final expression for the energy $\mathcal{E}$ contained in a system of n charges; it is

$$\mathcal{E} = \sum_{\text{all pairs}}\frac{q_iq_j}{4\pi\epsilon_0}\frac{1}{|\mathbf{r_i}-\mathbf{r}_j|} \qquad (3.12)$$

We shall now explain why we chose above to define $\mathcal{E}$ by writing

$$\mathcal{E} = -W \qquad (3.13)$$

rather than $\mathcal{E} = +W$. Consider a system of just two charges each with the same charge e. If such a system is constructed and left to its own devices the two charges will repel one another and move off to infinity. In that case their energy will then be zero using *either* of the definitions $\mathcal{E} = \mp W$. Now if we define $\mathcal{E}$ by writing $\mathcal{E} = -W$, then, when the charges move to infinity their energy will have *decreased* to zero from the *positive* value

$$+\frac{e^2}{4\pi\epsilon_0}\frac{1}{|\mathbf{r_1}-\mathbf{r}_2|} \qquad (3.14)$$

This is the correct behaviour of a general physical system: it seeks to attain a stable state by lowering its energy, i.e. it tries to attain a *minimum*. Had we used the other sign in the definition of $\mathcal{E}$ the energy would have *increased* to zero which would be wrong.

One could also consider a system of two equal and opposite charge $\mp e$. These would have negative energy $\mathcal{E}$ where

$$\mathcal{E} = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{|\mathbf{r_1} - \mathbf{r}_2|} \tag{3.15}$$

These charges will try to move closer together and thus lower their energy making it even more negative, this is also correct; the fact that two such charges do not collapse into one another and give themselves an energy of $-\infty$ is something which is only properly understood with quantum mechanics.

§ 2. Path independence of the work done in electric fields

We now address a question which has not yet been mentioned but which must be faced sooner or later. This is the question of how the work done in moving a charge q from A to B through an electric field $\mathbf{E}$ depends on the path Γ_{AB} followed to get from A to B. The somewhat striking answer is that the work done is *independent* of the particular path Γ_{AB} chosen. This is indeed fortunate because we have to choose one or more paths when computing $\mathcal{E}$ for any physical system so, if $\mathcal{E}$ were to depend on the paths chosen, we would still have matters to discuss.

The proof of this matter just requires us to use Stokes' theorem which we recall says that

$$\int_C \mathbf{V}\cdot\mathbf{dl} = \int_S \nabla\times\mathbf{V}\cdot\mathbf{dS} \tag{3.16}$$

where $\mathbf{V}$ is any vector and S is any surface and C is its boundary.

We take $\mathbf{V} = \mathbf{E}$, the electric field, giving us

$$\int_C \mathbf{E}\cdot\mathbf{dl} = \int_S \nabla\times\mathbf{E}\cdot\mathbf{dS} \tag{3.17}$$

But, if V is the potential associated with $\mathbf{E}$ then

$$\begin{aligned}\mathbf{E} &= -\nabla V\\ \Rightarrow \nabla\times\mathbf{E} &= -\nabla\times\nabla V\\ &= 0\end{aligned} \tag{3.18}$$

so that $\mathbf{E}$ satisfies

$$\int_C \mathbf{E}\cdot\mathbf{dl} = 0 \tag{3.19}$$

where C is any *closed* curve. Now let us state what we have to prove it is this. If Γ_{AB} and $\tilde{\Gamma}_{AB}$ are any two paths joining the points A and B then the work done in moving a charge q from A to B through the electric field $\mathbf{E}$ is independent of which path we choose. In other words we want to prove that

$$q\int_{\Gamma_{AB}} \mathbf{E}\cdot\mathbf{dl} = q\int_{\tilde{\Gamma}_{AB}} \mathbf{E}\cdot\mathbf{dl} \tag{3.20}$$

or simply

$$\int_{\Gamma_{AB}} \mathbf{E}\cdot\mathbf{dl} = \int_{\tilde{\Gamma}_{AB}} \mathbf{E}\cdot\mathbf{dl} \tag{3.21}$$

But the *paths* Γ_{AB} and $\tilde{\Gamma}_{AB}$ can be joined together to form the *closed curve* C. C is constructed by going from A to B along Γ_{AB} and then back from B to A along $\tilde{\Gamma}_{AB}$. With this done we can write

$$\int_{C} \mathbf{E}\cdot\mathbf{dl} = \int_{\Gamma_{AB}} \mathbf{E}\cdot\mathbf{dl} - \int_{\tilde{\Gamma}_{AB}} \mathbf{E}\cdot\mathbf{dl} \tag{3.22}$$

where he minus sign in front of second term on the RHS of 3.22 is because the path $\tilde{\Gamma}_{AB}$ has to be traversed from B to A rather than from A to B. But we have just seen that the LHS of 3.22 is zero i.e. we have the result

$$\int_{\Gamma_{AB}} \mathbf{E}\cdot\mathbf{dl} = \int_{\tilde{\Gamma}_{AB}} \mathbf{E}\cdot\mathbf{dl} \tag{3.23}$$

which is just as we wished.

CHAPTER IV

Calculating electric fields: Gauss's theorem

§ 1. Gauss' dielectric flux theorem

WE are now ready to consider a remarkable result whose existence is directly traceable to the inverse square force law between charges—were this force law to be an inverse *cube* law, or indeed were this force to decrease with distance at any rate *other* than an inverse square, then Gauss' dielectric flux theorem would not hold but would be replaced by something more complicated. The theorem states the following

Theorem (Gauss' dielectric flux theorem) *If a closed surface S contains a total amount of electric charge Q then the flux $\int_S \mathbf{E} \cdot \mathbf{dS}$ of the electric field $\mathbf{E}$ out of S is given by*

$$\int_S \mathbf{E} \cdot \mathbf{dS} = \frac{Q}{\epsilon_0} \tag{4.1}$$

Proof: We shall give a proof which is valid for a *discrete* collection of charges. Hence we shall now assume that Q is collection of a finite number n of charges q_1, q_2, …, q_n.

Now let us take just one of these charges q_i, say, which is located at the point O. Now consider an arbitrary infinitesimal patch $\mathbf{dS}$ on the surface S which is a distance r from O. The flux of q_i through this patch is

$$\mathbf{E} \cdot \mathbf{dS} \tag{4.2}$$

But $\mathbf{E}$ on $\mathbf{dS}$ is given by

$$\mathbf{E} = \frac{q_i}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2} \tag{4.3}$$

so that

$$\mathbf{E}\cdot\mathbf{dS} = \frac{q_i}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}\cdot\mathbf{dS}}{r^2} \tag{4.4}$$

However this flux is simply related to a certain solid angle as follows: the solid angle subtended by $\mathbf{dS}$ at O is $d\Omega$ where

$$d\Omega = \frac{dA}{r^2} \tag{4.5}$$

and dA denotes the area of a spherical cap normal to $\mathbf{E}$, cf. diagram. This means that we have

$$dA = |\mathbf{dS}|\cos\theta \tag{4.6}$$

where θ is the angle between $\mathbf{E}$ and $\mathbf{dS}$. Since we also have

$$\hat{\mathbf{r}}\cdot\mathbf{dS} = |\mathbf{dS}|\cos\theta \tag{4.7}$$

then we have the equation

$$\mathbf{E}\cdot\mathbf{dS} = \frac{q_i}{4\pi\epsilon_0}d\Omega \tag{4.8}$$

We now immediately integrate to obtain

$$\int_S \mathbf{E}\cdot\mathbf{dS} = \frac{q_i}{4\pi\epsilon_0}\int_S d\Omega \tag{4.9}$$

But it is an elementary fact that

$$\int_S d\Omega = 4\pi \tag{4.10}$$

Hence we have proved that

$$\int_S \mathbf{E}\cdot\mathbf{dS} = \frac{q_i}{\epsilon_0} \tag{4.11}$$

Now all we have to do is to sum both sides of 4.11 over i so as to include all charges; all this does is to replace the q_i on the RHS by the total charge Q giving us the desired result

$$\int_S \mathbf{E}\cdot\mathbf{dS} = \frac{Q}{\epsilon_0} \tag{4.12}$$

and the proof is complete.

§ 2. Maxwell's first equation and Poisson's equation

We are now ready to derive Maxwell's first equation which is simply

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{4.13}$$

One starts with Gauss's flux theorem

$$\int_S \mathbf{E} \cdot \mathbf{dS} = \frac{Q}{\epsilon_0} \tag{4.14}$$

then we suppose that the charge Q comes totally from charge inside, or on the surface of, a closed volume V whose boundary is the surface S above. This means that, if ρ is the charge density per unit volume, then

$$Q = \int_V \rho \, dV \tag{4.15}$$

so that we have immediately that

$$\int_S \mathbf{E} \cdot \mathbf{dS} = \frac{1}{\epsilon_0} \int_V \rho \, dV \tag{4.16}$$

Gauss's divergence theorem applied to the LHS then yields the equation

$$\int_V \nabla \cdot \mathbf{E} \, dV = \int_V \frac{\rho}{\epsilon_0} \, dV \tag{4.17}$$

But since the volume V is arbitrary then the integrands on both sides of 4.17 must be equal; hence we have

Maxwell's first equation

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \text{Maxwell's first equation} \tag{4.18}$$

as desired.

If we write this equation in terms of the potential V rather than the electric field $\mathbf{E}$ then the equations that we get is called *Poisson's equation*: recalling that

$$\mathbf{E} = -\nabla V \tag{4.19}$$

we substitute for $\mathbf{E}$ in 4.17 and obtain an equation for V which is

$$\begin{aligned} \nabla \cdot (-\nabla V) &= \frac{\rho}{\epsilon_0} \\ \Rightarrow \nabla^2 V &= -\frac{\rho}{\epsilon_0} \end{aligned} \tag{4.20}$$

Poisson's equation is the last of the two equations above, that is the following equation for V

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}, \quad \text{(Poisson's equation)} \tag{4.21}$$

§ 3. Gauss's theorem at work

Gauss's theorem is a very useful tool for calculating the electric field in a variety of situations. We shall now consider some examples of this.

Example *The electric field due to a sphere of charge*

Our task now is to compute the electric field created by a sphere of charge, where the total charge on the sphere is Q. We suppose that there exists a *solid* sphere of charge, of radius a and centre O; we then wish to calculate the electric field $\mathbf{E}$ at an arbitrary point P where P is a distance r from O. We also assume that $r > a$ so that P is a point *outside* the sphere. Later we shall show how to compute $\mathbf{E}$ when P is *inside* the sphere.

The technique used is just a judicious use of Gauss's theorem

$$\int_S \mathbf{E}\cdot\mathbf{dS} = \frac{Q}{\epsilon_0} \tag{4.22}$$

The key matter is to choose the right surface S over which to integrate $\mathbf{E}$. We take S to be a sphere of radius r and *centre* O so that it is concentric to the sphere of charge.

Now we can deduce that spherical symmetry demands that $\mathbf{E}$ be *radial* on the surface of S, i.e. $\mathbf{E}$ is parallel to $\mathbf{dS}$ on S since $\mathbf{dS}$, by definition, is always radial. Hence we have

$$\int_S \mathbf{E}\cdot\mathbf{dS} = \int_S |\mathbf{E}||\mathbf{dS}| \tag{4.23}$$

But all points on S are equidistant from O so $|\mathbf{E}|$ must be constant on S, therefore we can write

$$\begin{aligned}\int_S |\mathbf{E}||\mathbf{dS}| &= |\mathbf{E}|\int_S |\mathbf{dS}| \\ &= |\mathbf{E}|4\pi r^2\end{aligned} \tag{4.24}$$

where we have used the obvious fact that $\int_S |\mathbf{dS}|$ is just the total surface area of S. We have now deduced that

$$|\mathbf{E}|4\pi r^2 = \frac{Q}{\epsilon_0} \tag{4.25}$$

and since we already know that $\mathbf{E}$ is radial we have the complete expression for $\mathbf{E}$ which is

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{|\mathbf{r}|^2} \tag{4.26}$$

It is noteworthy that this expression expresses the eminently reasonable fact that a sphere of charge behaves as if all the charge is concentrated at its centre.

Example *The electric field inside a hollow charged closed conductor*

Now let us take a *hollow* conductor and place an arbitrary charge distribution on its surface. Provided we consider points *outside* the conductor then this makes no difference to calculations of the electric field which use Gauss's theorem. However, since the conductor is hollow we can now go inside and, for points *inside*, the situation is radically different. In fact the electric field is always *zero* for all points inside a hollow closed charged conductor.

We shall not give a general proof [1] of the above facts but shall prove them for the case when the conductor is spherical of radius a.

First we choose a point P outside the conductor. In this case there is nothing new to say the field $\mathbf{E}$ is exactly the same as before and given by the expression

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{|\mathbf{r}|^2} \tag{4.27}$$

where Q is the total charge on the conductor.

Next we suppose that the point P at which we want the electric field is inside the sphere. To this end let the distance from the centre of the sphere to P be r where $r < a$. Then we choose S to be the sphere of radius r centre O and apply Gauss's theorem from which we obtain the result

$$|\mathbf{E}|4\pi r^2 = \frac{Q}{\epsilon_0} \tag{4.28}$$

where $\mathbf{E}$ is the field at P and Q is the charge inside S. But Q has to be zero since we are inside a hollow conductor hence we immediately deduce that

$$\begin{aligned} |\mathbf{E}| &= 0 \\ \Rightarrow \mathbf{E} &= 0 \end{aligned} \tag{4.29}$$

as claimed.

[1] The general proof follows fairly easily from the fact that a solution of Poisson's equation for the electric field of an arbitrary charge distribution is uniquely specified by the values of the potential on some closed surface (in this case the closed conducting surface).

Example *The electric field due to an infinite cylinder of charge*

In this example we shall compute the electric field $\mathbf{E}$ a distance r from the axis of an infinitely long charged cylinder of radius a; we shall assume that the cylinder carries a charge of λ per unit length.

This is another application of Gauss's theorem and all we have to do is to make a sensible choice for the surface S that appears in the statement of the theorem.

Cylindrical symmetry makes is reasonable that we should choose S to be a cylinder coaxial to the first, but of radius r and length L, and placed so that the point P lies on its surface. We remind the reader that Gauss's theorem requires S to be *closed* so that this cylinder consists of a curved piece of area $2\pi rL$ plus two circular discs each of area πr^2.

Applying the theorem we have

$$\int_S \mathbf{E}\cdot\mathbf{dS} = \frac{Q}{\epsilon_0} \tag{4.30}$$

but Q must be the charge inside a length L of the charged cylinder so that

$$Q = \lambda L \tag{4.31}$$

Also, if we break the integral up into two natural pieces, we get

$$\int_S \mathbf{E}\cdot\mathbf{dS} = \int_{\text{curved part}} \mathbf{E}\cdot\mathbf{dS} + \int_{\text{two discs}} \mathbf{E}\cdot\mathbf{dS} \tag{4.32}$$

Now cylindrical symmetry means that $\mathbf{E}$ must point in the *radial* direction; hence on the two discs $\mathbf{E}$ is *perpendicular* to $\mathbf{dS}$, while on the curved part $\mathbf{E}$ is *parallel* to $\mathbf{dS}$. These two observations mean that

$$\begin{aligned}\int_{\text{two discs}} \mathbf{E}\cdot\mathbf{dS} &= 0\\ \int_{\text{curved part}} \mathbf{E}\cdot\mathbf{dS} &= \int_{\text{curved part}} |\mathbf{E}||\mathbf{dS}|\\ &= |\mathbf{E}|\int_{\text{curved part}} |\mathbf{dS}|\\ &= |\mathbf{E}|2\pi rL\end{aligned} \tag{4.33}$$

where, in the second integral, we have used the fact that all points on the curved side are equidistant from the axis of the cylinder so that $|\mathbf{E}|$ must be constant throughout the integral. We now have computed both the LHS and RHS of the expressions entering Gauss's theorem and, using these computations, we find that

$$|\mathbf{E}|2\pi r L = \frac{\lambda L}{\epsilon_0} \tag{4.34}$$

We deduce at once that

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{|\mathbf{r}|} \tag{4.35}$$

and it is useful to remember that the cylindrical geometry has rendered $|\mathbf{E}|$ proportional to $1/r$ rather than $1/r^2$.

Example *The electric field due to an infinite plane of charge*

Perhaps the easiest example, though it is important, is the present one where we compute the electric field a distance d from an infinite charged plane.

Let the plane have charge σ per unit area. We shall calculate $\mathbf{E}$ at a point P where P is a vertical distance d from the plane.

Select a circular disc of area A whose centre meets a perpendicular from the point P. Then, on this disk, erect a cylinder, with base of area A, which extends a height d *above* the plane and also a height d *below* it. This closed cylinder is chosen to be the surface S for Gauss's theorem. We note that left–right symmetry of the infinite plane forbids the field $\mathbf{E}$ from pointing in any direction other than *perpendicular* to the plane. This means that the integral over the curved part of S drops out since $\mathbf{E}$ is perpendicular to $\mathbf{dS}$ there. More precisely we find that

$$\begin{aligned}\int_S \mathbf{E}\cdot\mathbf{dS} &= \int_{\text{two discs}} \mathbf{E}\cdot\mathbf{dS} \\ &= \int_{\text{two discs}} |\mathbf{E}||\mathbf{dS}| \\ &= |\mathbf{E}|\int_{\text{two discs}} |\mathbf{dS}| \\ &= |\mathbf{E}|2A\end{aligned} \tag{4.36}$$

Hence we have

$$|\mathbf{E}|2A = \frac{Q}{\epsilon_0} \tag{4.37}$$

where Q is the charge on the disc of area A. So, if we let σ be the density of charge per unit area on the plates, then we deduce at once that

$$\begin{aligned}|\mathbf{E}|2A &= \frac{\sigma A}{\epsilon_0} \\ \Rightarrow |\mathbf{E}| &= \frac{\sigma}{2\epsilon_0} \\ \Rightarrow \mathbf{E} &= \frac{\sigma}{2\epsilon_0}\mathbf{n}\end{aligned} \tag{4.38}$$

where $\mathbf{n}$ is a unit vector perpendicular to the plate.

We draw attention to the fact that $\mathbf{E}$ has been found to be *independent* of the distance d that the point P is from the plate. This artificial result is only because we have taken an infinite rather than a finite plate; nevertheless our result is still numerically reasonable for finite plates with P subject to the following restrictions: P is opposite the plate, not near any of its edges but a horizontal distance h away from the nearest edge with

$$d/h << 1 \tag{4.39}$$

CHAPTER V

The method of images

§ 1. Introduction

IN this chapter we explain a very interesting technique for calculating electric fields and their associated potentials. It is known as the *method of images* and its essence is that it is a transform method, that is to say that it transforms one difficult formulation of a problem into a much easier but totally equivalent one.

We need just one mathematical result in order to understand the transform which comprises the method of images. This concerns the uniqueness of solutions to boundary value problems. In this case the boundary value problem is Laplace's equation for the potential V. The theorem we need, which we quote without proof, is

Theorem (Uniqueness for solutions to Laplace's equation) *If V satisfies Laplace's equation*

$$\nabla^2 V = 0 \tag{5.1}$$

and obeys the boundary condition $V = f$ on some closed surface S, then the solution V is unique.

The basic idea in the method of images is to take two *apparently different* electrical situations with potentials V_1 and V_2 satisfying Laplace's equation, i.e.

$$\nabla^2 V_1 = \nabla^2 V_2 = 0 \tag{5.2}$$

Now suppose that

$$V_1 = V_2, \quad \text{on some closed surface } S \tag{5.3}$$

It then follows that V_1 and V_2 are uniquely determined by the theorem above; hence since they are equal on the surface S they must be equal everywhere so that we will have

$$V_1 = V_2, \qquad \text{everywhere} \tag{5.4}$$

In other words the two electrical situations are, after all, *identical.* It is now time to see what happens when we take an example of the method of images in action and this we now do.

We begin with the important definition of a conducting surface which we shall make use of below.

Definition (A conducting surface) *The electric field* $\mathbf{E}$ *at a conducting surface is always perpendicular to that surface.*

One deduces immediately from this that the *potential* V on the surface of the conductor is always *a constant* since, if it were not, then ∇V would be non zero in directions *tangential* to the surface of the conductor and hence the electric field $\mathbf{E}$ (remember $\mathbf{E} = -\nabla V$) would have a component *tangential* to the surface contrary to the definition of a conductor.

Note, too, that electrical *lines of force* are those lines whose tangent, at a point P, give the electric field $\mathbf{E}$ at that same point P.

§ 2. The method of images illustrated

Let us now consider some examples in order to get to grips with all the details.

Example *A charge in front of an earthed conducting plane*

Take the following simple electrical system consisting of a single charge q placed a distance d from an infinite earthed conducting plane. We just want to know what the potential $V(\mathbf{r})$ is where $\mathbf{r}$ is a general point in front of the plane.

It turns out that the conducting plane and the charge q are *completely equivalent* to a pair of equal and opposite charges $\mp q$ placed a distance $2d$ apart.

In other words the charge $-q$ is able to exactly mimic the presence of the conducting plane. If one thinks of the plane as a *mirror* with the charge q a distance d away from its surface then the charge $-q$ is in the same position as the image of q would be in this mirror. This where the terminology *method of images* has its origin.

Finally we need to understand why the pair of charges $\mp q$ form the correct image system. To this end consider the charges $\mp q$ and, in particular, consider the infinite plane that bisects, at right angles, the line joining q to $-q$—clearly this plane is in precisely the position previously occupied by the conducting plane. Now note that the potential is zero at all points on this plane which we shall now call S. This is because, on S, the potential is given by

$$\begin{aligned} V(\mathbf{r}) &= \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r_q}|} - \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r_{-q}}|} \\ &= 0, \qquad \text{since} \qquad |\mathbf{r}-\mathbf{r_q}| = |\mathbf{r}-\mathbf{r_{-q}}|, \quad \text{on } S \end{aligned} \tag{5.5}$$

But this means that the image system and the original system have the same potential on the surface[1] S and so the potentials of the two systems coincide everywhere according to the theory explained in theorem 5.1.

Example *A charge in the presence of an earthed spherical conductor*

Our second image situation consists of a single charge q placed in front of an earthed conducting sphere.

More precisely we have an earthed spherical conductor of radius a, and centre O, outside of which we place a charge q, with q a distance d from O so that $d > a$. It again turns out that the conductor can be replaced by a single charge Q if Q is placed appropriately.

Suppose then that we discard the conducting sphere, and that Q is placed a distance x from the point O and on the line joining q to O. This charge Q will be able to mimic the presence of the earthed sphere but we must find its size and location—i.e. we must find the values of Q and x. To do this we just need two equations which we can obtain by choosing two points on the sphere $r = a$ and demanding that the potential V produced by Q and q vanishes. We choose these points as follows.

Let a straight line be drawn from q through the centre O of the sphere and continued until it intersects the sphere again. This line now intersects the sphere in two places which we denote by A and B where we take A to be the nearer one to q. If V_A and V_B are the potentials at these two points then we have

$$V_A = 0, \qquad V_B = 0 \tag{5.6}$$

But, applying the principle of superposition, these two equations become

$$\begin{aligned} \frac{q}{4\pi\epsilon_0}\frac{1}{(d-a)} + \frac{Q}{4\pi\epsilon_0}\frac{1}{(a-x)} &= 0 \\ \frac{q}{4\pi\epsilon_0}\frac{1}{(d+a)} + \frac{Q}{4\pi\epsilon_0}\frac{1}{(a+x)} &= 0 \end{aligned} \tag{5.7}$$

We can easily solve these two equations and now do so.

From the first we obtain

$$Q = -q\frac{(a-x)}{(d-a)} \tag{5.8}$$

[1] In 5.1 the surface S is described as being closed and this is not the case here. However this does not matter because the plane is an infinite surface and it does divide the whole of three dimensional space $\mathbf{R}^3$ into two distinct regions as does a closed surface, this latter being the key point.

which we substitute in the second to give

$$
\begin{aligned}
\frac{q}{(d+a)} &= \frac{q(a-x)}{(d-a)(a+x)} \\
\Rightarrow (d-a)(a+x) &= (d+a)(a-x) \\
\Rightarrow 2dx &= 2a^2 \\
\Rightarrow x &= \frac{a^2}{d}
\end{aligned} \tag{5.9}
$$

So x is now found and we deduce immediately that

$$
\begin{aligned}
Q &= -q\frac{(a-a^2/d)}{(d-a)} = -q\frac{(ad-a^2)}{d(d-a)} \\
&= -q\frac{a}{d}
\end{aligned} \tag{5.10}
$$

So our image system is now completely determined and the values of Q and x are

$$
\begin{aligned}
Q &= -q\frac{a}{d} \\
x &= \frac{a^2}{d}
\end{aligned} \tag{5.11}
$$

We would like to examine one more example from the method of images. It is a refinement of the previous problem which results when we do not insist that the spherical conductor be earthed. So let us now consider

Example *A charge in the presence of a spherical conductor at constant potential*

The only difference between this example and the previous one is that the surface of the sphere is now at a constant potential V, say where V need no longer be zero.

It is not difficult to see that the image system which will mimic such a sphere consists of precisely two charges. In fact all one has to do is to add to the image system of the previous problem a charge Q' placed at the centre of the sphere. If we do this, and consider the potential at any point the sphere, the principle of superposition gives this potential as

$$
\begin{aligned}
&V(\mathbf{r})_{|_{sphere}} = V_{Q'}(\mathbf{r}) + V_Q(\mathbf{r}) + V_q(\mathbf{r}) \\
&\text{where } \mathbf{r} \text{ is a point on the sphere}
\end{aligned} \tag{5.12}
$$

But, by the previous example, when $\mathbf{r}$ is on the sphere we have

$$
V_Q(\mathbf{r}) + V_q(\mathbf{r}) = 0 \tag{5.13}
$$

Hence 5.12 reduces to

$$V(\mathbf{r})|_{sphere} = V_{Q'}(\mathbf{r}) \tag{5.14}$$

But since the sphere is and equipotential with potential V we must have

$$V_{Q'}(\mathbf{r}) = V \tag{5.15}$$

In addition, since Q' is a point charge at the centre of a sphere of radius a, we have

$$V_{Q'}(\mathbf{r}) = \frac{Q'}{4\pi\epsilon_0}\frac{1}{a} \tag{5.16}$$

So we determine the size of Q' at once by requiring that

$$\begin{aligned} \frac{Q'}{4\pi\epsilon_0}\frac{1}{a} &= V \\ \Rightarrow Q' &= 4\pi\epsilon_0 a V \end{aligned} \tag{5.17}$$

The problem is now completely solved

CHAPTER VI
Capacitors

§ 1. Capacitance and capacitors

IN this chapter we study capacitors which are, at a first glance, simply devices for storing charge. However, when alternating voltages are applied to them, they play an absolutely vital role as components of amplifiers and of almost all electronic circuitry. Unfortunately we shall not have space to describe these latter applications.

The basic idea of a capacitor is that it is comprised of two conductors separated by an insulating material (this is often just air). Then, when a potential difference V is applied across the two conductors a certain quantity of charge Q is deposited on each plate or terminal; clearly if an amount -Q is deposited on one plate then $+Q$ will be deposited on the other. The central property of the capacitor is that Q and V then obey a simple proportionality or linear law, namely one has

$$Q = CV \tag{6.1}$$

where C is a constant known as the *capacitance* of the capacitor. This constant C will vary from capacitor to capacitor but it can be computed in many situations and we shall now examine some of these.

§ 2. The parallel plate capacitor

We now turn to one of the simplest capacitors, it is made by placing two flat conductors of area A parallel to one another so that they are a distance d apart.

The first step in the computation of the capacitance C for any capacitor is always to obtain an expression for the electric field $\mathbf{E}$ between the plates. We shall now *approximate* the field $\mathbf{E}$ and take it to be the same as the field due to two infinite plates which are oppositely charged. In that case, as we have seen in 4.38 above a single infinite plane with charge density σ produces a field $\mathbf{E}$ given by

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0}\mathbf{n} \tag{6.2}$$

where $\mathbf{n}$ is a unit vector perpendicular to the plate. Since we have two plates with charge densities σ and $-\sigma$, whose normal vectors will also be of *opposite signs*, then superposition gives the field $\mathbf{E}$ between the plates as a sum of two terms: we have

$$\begin{aligned}\mathbf{E} &= \frac{\sigma}{2\epsilon_0}\mathbf{n} + \left(\frac{-\sigma}{2\epsilon_0}\right)(-\mathbf{n}) \\ &= \frac{\sigma}{\epsilon_0}\mathbf{n}\end{aligned} \tag{6.3}$$

The potential difference V across the plates is given by integration of $\mathbf{E}$ so that, if the x axis is taken to be normal to the plates, we have

$$\begin{aligned}V &= \int_0^d |\mathbf{E}| dx = \frac{\sigma}{\epsilon_0}\int_0^d dx \\ &= \frac{\sigma d}{\epsilon_0}\end{aligned} \tag{6.4}$$

However, if Q is the total charge on one of the plates and we recall that A is its area, then

$$Q = \sigma A \tag{6.5}$$

Hence we can write V as

$$\begin{aligned}V &= \frac{Qd}{A\epsilon_0} \\ \Rightarrow Q &= CV \\ \text{with} \quad C &= \frac{\epsilon_0 A}{d}\end{aligned} \tag{6.6}$$

Summarising we have deduced that the capacitance C of a parallel plate capacitor, with plates of area A which are a distance d apart, is given by

$$C = \frac{\epsilon_0 A}{d} \tag{6.7}$$

We are now ready to move on to another example.

§ 3. The coaxial cylinder capacitor

This time our capacitor consists of two coaxial cylindrical conductors of length L but with radii R_1 and R_2 respectively, where we assume that $R_1 < R_2$.

As in the previous example we want an expression for the field $\mathbf{E}$ between the plates, i.e. in the space between the cylinders. Now let the plates be charged with charges $\mp Q$. We approximate to the situation where both cylinders have infinite length instead of length L. In this case we know that the *outer* cylinder being at constant potential contributes zero to the field $\mathbf{E}$ inside. However the field due to the *inner* cylinder at a point P inside is given by the expression 4.35, in other words we have

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{|\mathbf{r}|} \tag{6.8}$$

where r is the distance of the point P from the common axis, $\hat{\mathbf{r}}$ is a unit vector in the radial direction, λ is the charge per unit length and, of course, $R_1 < r < R_2$.

Proceeding to the calculation of the usual potential difference V across the plates gives us the relation

$$\begin{aligned} V &= \int_{R_1}^{R_2} \mathbf{E}\cdot\mathbf{dr} \\ &= \frac{\lambda}{2\pi\epsilon_0}\int_{R_1}^{R_2}\frac{dr}{r} \\ &= \frac{\lambda}{2\pi\epsilon_0}\left[\ln(r)\right]_{R_1}^{R_2} \\ &= \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{R_2}{R_1}\right) \end{aligned} \tag{6.9}$$

Now, reverting to the finite plates of length L, we have

$$Q = \lambda L \tag{6.10}$$

and this means that

$$\begin{aligned} V &= \frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{R_2}{R_1}\right) \\ \Rightarrow Q &= CV \\ \text{with} \qquad C &= \frac{2\pi\epsilon_0 L}{\ln\left(\frac{R_2}{R_1}\right)} \end{aligned} \tag{6.11}$$

So we now have the capacitance C of the coaxial cylinders; this information is often quoted by saying the capacitance per unit length of the coaxial cylinders is

$$\frac{2\pi\epsilon_0}{\ln\left(\frac{R_2}{R_1}\right)} \tag{6.12}$$

We move on once more.

§ 4. The concentric sphere capacitor

In this capacitor we replace the coaxial cylinders by two concentric conducting spheres with radii R_1 and R_2 and with $R_1 < R_2$.

The field $\mathbf{E}$ between the spheres is entirely that due to the inner sphere which we take to have a charge Q. Hence if P is an interior point a distance r from the centre of the spheres, by spherical symmetry, we have

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2} \tag{6.13}$$

The potential difference V between the plates is given by

$$\begin{aligned} V &= \int_{R_1}^{R_2} \mathbf{E}\cdot d\mathbf{l} \\ &= \int_{R_1}^{R_2} \frac{Q}{4\pi\epsilon_0}\frac{dr}{r^2} \\ &= -\frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r}\right]_{R_1}^{R_2} \\ &= +\frac{Q}{4\pi\epsilon_0}\frac{(R_2 - R_1)}{R_1 R_2} \end{aligned} \tag{6.14}$$

But this means that

$$Q = CV \tag{6.15}$$

with

$$C = \frac{4\pi\epsilon_0 R_1 R_2}{(R_2 - R_1)} \tag{6.16}$$

which is therefore the capacitance of the spherical plate capacitor.

The last matter to discuss in this chapter is the energy that is stored when a capacitor charged up or, equally, the energy that is released when a capacitor is allowed to discharge.

§ 5. The energy stored by a capacitor

The act of charging up a capacitor to some charge Q requires the electrical system doing the charging to do some work. This means that a capacitor stores this amount of work as internal energy which can be released later on discharge. We shall now calculate this *stored energy*.

Let us begin by having a *completely uncharged* capacitor with capacitance C. Suppose that we now add a small charge dQ to the capacitor thereby creating a potential difference of V across its plates. This will require us to expend energy, let us say an amount dU where dU will be given by [1]

$$dU = V dQ \tag{6.19}$$

But since

$$Q = CV \tag{6.20}$$

then

$$dQ = C dV \tag{6.21}$$

and so dU can be written as

$$dU = CV dV \tag{6.22}$$

Finally we integrate with respect to V from 0 to the final voltage V (this is of course exactly equivalent to integrating with respect to Q from 0 to the final charge Q). This gives us

$$U = \frac{1}{2}CV^2 \tag{6.23}$$

which is the usually quoted expression for the energy stored in a capacitor of capacitance C across whose plates there is a potential difference V.

[1] We can see why this is the correct expression for dU by referring to 3.12 where we calculate the energy $\mathcal{E}$ of a system of charges: Think of assembling a sphere of charge by building it up from successive concentric spherical layers of thickness dr. If $Q(r)$ is the charge of the sphere when it has attained a radius r and we now add a further charge dQ then 3.12 tells us that the work done dU in adding dQ is just

$$\frac{dQQ(r)}{4\pi\epsilon_0 r} \tag{6.17}$$

But Gauss's flux theorem tells us that the potential at r created by the sphere is $Q(r)/4\pi\epsilon_0 r$ hence we have

$$dU = V dQ \tag{6.18}$$

as claimed.

CHAPTER VII

Charges in motion: electric currents

§ 1. Electric currents and resistance

WE are now ready to depart from the realm of electrostatics and to consider *moving charges*. Among other things this will allow us to discuss electric currents for the first time and we shall do this now.

When an electric current moves in a conductor such as a copper wire, for example, there is one conduction electron per copper atom and this large number of electrons makes it useful to define a vector $\mathbf{J}$ known as the *current density* cf. Fig. 1. below. So we now have the following definition.

Definition (Current density $\mathbf{J}$) *The current density* $\mathbf{J}$ *is a vector whose direction coincides with that of the velocity vector* $\mathbf{v}$ *of the conduction electrons in the conductor. Its magnitude is given by the amount of charge crossing a unit area within the conductor per unit time.*

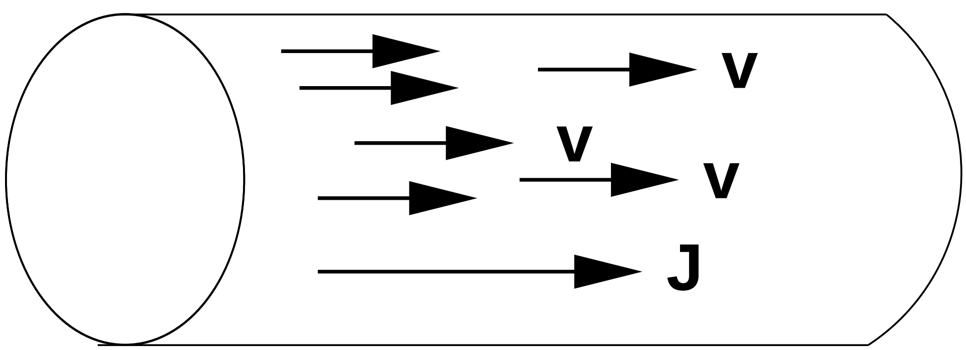

Fig. 1: Inside a conductor: electrons and the current density

We can easily find an expression for $\mathbf{J}$ and we now proceed to do just that: Let there be N electrons per unit volume inside the conductor then, in unit time, the electrons that cross a unit area travel a distance $|\mathbf{v}|$ (since they have velocity

$\mathbf{v}$). Thus they trace out a cylinder of length $|\mathbf{v}|$ and base of unit area. The volume of this cylinder is therefore just

$$|\mathbf{v}| \times \mathbf{1} = |\mathbf{v}| \tag{7.1}$$

Hence the number of (conduction) electrons in this cylinder is precisely

$$N \times |\mathbf{v}| = N|\mathbf{v}| \tag{7.2}$$

and this means that the total *charge* in this cylinder is got by multiplying by e, where e is the electric charge; so this charge is

$$eN|\mathbf{v}| \tag{7.3}$$

But this number is, by the definition of $\mathbf{J}$ above, equal to the magnitude of $\mathbf{J}$ so we have deduced that

$$|\mathbf{J}| = eN|\mathbf{v}| \tag{7.4}$$

Finally the direction of $\mathbf{J}$ coincides with that of $\mathbf{v}$ so that the completed expression for the current density $\mathbf{J}$ is

$$\mathbf{J} = eN\mathbf{v} \tag{7.5}$$

The electron velocity vector $\mathbf{v}$ is usually referred to as the *drift velocity* of the electrons; this is because in most situations it has a rather small magnitude, we shall demonstrate this shortly in an example below.

Now the *electric current* I through any surface S is defined as follows:

Definition (Electric current I) *The electric current I through any surface S is defined to be the charge Q passing through S per unit time, i.e.*

$$I = \frac{dQ}{dt} \tag{7.6}$$

Also I is measured in amps.

Now if we take S to be the total cross section of a conductor, such as a copper wire, we see that I and $\mathbf{J}$ are related by integration over S giving us the equation

$$I = \int_S \mathbf{J} \cdot \mathbf{dS} \tag{7.7}$$

This brings us to the point where we can examine some of the details of the passage of current through a piece of conducting wire.

Example *The drift of electrons through a uniform straight copper wire*

Suppose that we pass a current of I amps through a uniform copper wire of cross sectional area A. Applying what we have just learned we write

$$\begin{aligned} I &= \int_S \mathbf{J} \cdot \mathbf{dS} \\ &= \int_S Ne\mathbf{v} \cdot \mathbf{dS} \end{aligned} \tag{7.8}$$

But in a straight wire $\mathbf{v}$ will be parallel to $\mathbf{dS}$ giving

$$\mathbf{v} \cdot \mathbf{dS} = |\mathbf{v}||\mathbf{dS}| \tag{7.9}$$

and we obtain the result that

$$\begin{aligned} I &= \int_S Ne|\mathbf{v}||\mathbf{dS}| \\ &= Ne|\mathbf{v}| \int_S |dS|, \quad \text{since } N,\ e \text{ and } |\mathbf{v}| \text{ are all constants} \\ \Rightarrow I &= Ne|\mathbf{v}|A, \quad \text{where } A \text{ is the cross sectional area of the wire} \end{aligned} \tag{7.10}$$

Now we can put in some typical numbers and see how small the drift velocity $|\mathbf{v}|$ actually is. Let

$$I = 1.5 \quad \text{amps, } A = 1 \text{ square mm} = 10^{-6} m^2 \tag{7.11}$$

Further we know that, for copper,

$$N = 8 \times 10^{28}, \quad \text{electrons per } m^3 \tag{7.12}$$

and the charge e on an electron is given by

$$e = 1.6 \times 10^{-19} coul. \tag{7.13}$$

Hence since we can deduce from 7.10 that

$$|\mathbf{v}| = \frac{I}{NeA} \tag{7.14}$$

we find that

$$\begin{aligned} |\mathbf{v}| &= \frac{1.5}{8 \times 10^{28} \times 1.6 \times 10^{-19} \times 10^{-6}}, \quad m/sec \\ \Rightarrow |\mathbf{v}| &\simeq 10^{-4}, \quad m/sec \end{aligned} \tag{7.15}$$

which is indeed small justifying the name drift velocity for $|\mathbf{v}|$.

Continuing in our examination of the inner workings of a copper wire we now turn to the celebrated Ohm's law.

§ 2. Ohm's law

If a conductor has a potential V applied to it, causing a current I to flow, then this potential difference creates an internal electric field $\mathbf{E}$ where $\mathbf{E} = -\nabla V$. For most conductors this internal field $\mathbf{E}$ and the current density $\mathbf{J}$ are parallel. In other words, inside the conductor, one has[1]

$$\mathbf{J} = \sigma \mathbf{E}, \quad \sigma \text{ a constant} \tag{7.16}$$

This constant σ is called the *conductivity* of the conductor and its units of measurement are $ohm^{-1}\, m^{-1}$; for copper one has

$$\sigma = 5.9 \times 10^7\, ohm^{-1}\, m^{-1} \tag{7.17}$$

The *inverse* of σ is also used; it is denoted by ρ and is called the *resistivity* so that we can write

$$\rho = \frac{1}{\sigma} \tag{7.18}$$

Example *A wire of length L and cross section A*

Consider a wire of length L and cross sectional area A to which a potential difference V is applied. The potential V produces an internal electric field $\mathbf{E}$ and the two are related by

$$\begin{aligned} V &= \int_0^L \mathbf{E} \cdot \mathbf{dl} = |\mathbf{E}| \int_{\mathbf{0}}^{\mathbf{L}} |\mathbf{dl}|, \quad \text{since } \mathbf{E} \text{ is parallel to } \mathbf{dl} \\ \Rightarrow V &= |\mathbf{E}| L \end{aligned} \tag{7.19}$$

Also I is related to $\mathbf{J}$ by

$$\begin{aligned} I &= \int_S \mathbf{J} \cdot \mathbf{dS} \\ &= \sigma \int_S \mathbf{E} \cdot \mathbf{dS}, \quad \text{using 7.16} \\ \Rightarrow I &= \sigma |\mathbf{E}| A \end{aligned} \tag{7.20}$$

[1] This equation $\mathbf{J} = \sigma \mathbf{E}$ is sometimes referred to as Ohm's law as well as the more familiar equation $V = RI$; we shall see below that the former implies the latter so that there is some justification in such a nomenclature.

But since $V = |\mathbf{E}|L$ then we can write

$$\begin{aligned} I &= \frac{\sigma V A}{L} \\ \Rightarrow V &= \left(\frac{L}{\sigma A}\right) I \end{aligned} \tag{7.21}$$

hence we have deduced the familiar version of Ohm's law

$$\begin{aligned} V &= RI \\ \text{with} \quad R &= \frac{L}{\sigma A} \end{aligned} \tag{7.22}$$

We recognise R as the *resistance* of the material; its units of measurement are Ohms which are denoted by Ω. It useful to note that

$$R \propto \frac{1}{A} \tag{7.23}$$

but, by contrast,

$$R \propto L \tag{7.24}$$

It is useful to be aware of the conductivities of some of the more common substances in the world and we provide a table below.

Material	**Conductivity**	
Copper	5.9×10^7	
Gold	4.1×10^7	
Germanium	2.2	(semiconductor)
NaCl solution	23.0	
Glass	10^{-10}—10^{-14}	(insulator)
Quartz	1.3×10^{-18}	(piezo-electric effect)
Wood	10^{-8}—10^{-11}	

Conductivity table for substances $(293^0 K)$**.**

§ 3. The power dissipated in a wire

It is of great importance to be able to calculate the power dissipated by the passage of a current through a wire,

This is not difficult to do and one proceeds as follows. Let a voltage difference V be applied to wire of resistance R producing a current I. If U is the internal energy of the total charge Q in the wire then we know that

$$U = QV \tag{7.25}$$

The rate of change of U with time is the energy consumed per unit time by the passage of the current—i.e. it is the power dissipated. I we denote the power dissipated by P then we have

$$\begin{aligned} P &= \frac{d}{dt} QV \\ &= \frac{dQ}{dt} V \\ &= IV, \qquad \text{since } I = \frac{dQ}{dt} \\ \Rightarrow P &= VI \end{aligned} \tag{7.26}$$

But since Ohm's law says that $V = RI$ we can use Ohm's law to obtain three completely equivalent expressions for P and these are

$$\begin{aligned} P &= VI \\ P &= \frac{V^2}{R} \\ P &= I^2 R \end{aligned} \tag{7.27}$$

CHAPTER VIII
Magnetic fields

§ 1. Magnetic fields and Maxwell's second equation

THE forces between magnetic poles obey an inverse square law just as is the case for electric charges. This means that the magnetic field $\mathbf{B}$ obeys a flux law similar to that for electric fields $\mathbf{E}$. Recall that for electric fields the inverse square law leads directly to Gauss's dielectric flux theorem which states that, for a closed surface S containing a total charge Q, the flux of $\mathbf{E}$ through S satisfies

$$\int_S \mathbf{E} \cdot \mathbf{dS} = \frac{Q}{\epsilon_0} \tag{8.1}$$

Hence for magnetic fields the inverse square law says that, for a closed surface S containing a total *magnetic* charge Q_M, the flux of $\mathbf{B}$ through S satisfies

$$\int_S \mathbf{B} \cdot \mathbf{dS} = \frac{Q_M}{C} \tag{8.2}$$

where C is some constant which is the magnetic equivalent of ϵ_0. However *experimentally* it is found that *all magnetic charges occur in equal and opposite pairs.* This is often stated as the fact that no *magnetic monopoles* have ever been discovered. The conclusion that one draws from this is that the total amount of magnetic charge inside a surface is always exactly *zero* i.e. we *always* have

$$Q_M = 0 \tag{8.3}$$

But this means that

$$\int_S \mathbf{B} \cdot \mathbf{dS} = 0 \tag{8.4}$$

and so applying Gauss's divergence theorem to V, the volume contained inside S, we have

$$\begin{aligned} \int_S \mathbf{B}\cdot\mathbf{dS} &= \int_V \nabla\cdot\mathbf{B}\,dV \\ \Rightarrow \int_V \nabla\cdot\mathbf{B}\,dV &= 0 \\ \Rightarrow \nabla\cdot\mathbf{B} &= 0, \quad \text{since } V \text{ is arbitrary} \end{aligned} \tag{8.5}$$

This last result is very important as it is the second of Maxwell's four equations. Emphasising this we write

$$\nabla\cdot\mathbf{B} = 0, \quad \text{Maxwell's second equation} \tag{8.6}$$

Just to summarise, the two (of the four) Maxwell equations that we have derived so far are

Maxwell's first two equations

$$\begin{aligned} \nabla\cdot\mathbf{E} &= \frac{\rho}{\epsilon_0}, \quad \text{Maxwell's first equation} \\ \nabla\cdot\mathbf{B} &= 0, \quad \text{Maxwell's second equation} \end{aligned} \tag{8.7}$$

We shall come to the last two in due course.

§ 2. The magnetic field produced by electric currents and Ampère's Theorem

A key result—due to Ampère—which helps one to measure and calculate the magnetic field $\mathbf{B}$ that is created when a current I passes through a wire is called Ampère's law (or Ampére's theorem) and its formal statement is;

Ampère's law or theorem *Let a current I pass through a wire thereby producing a magnetic field $\mathbf{B}$. Then if C is a closed curve we have the fact that*

$$\int_C \mathbf{B}\cdot\mathbf{dl} = \begin{cases} \mu_0 N I, & \text{if } C \text{ links } N \text{ times with the wire} \\ 0, & \text{otherwise} \end{cases} \tag{8.8}$$

Ampère's law is as useful for calculating magnetic fields as Gauss's dielectric flux theorem is for calculating electric fields. The most fruitful way to appreciate the importance of this result is to use it to obtain the magnetic field in a specific example. Let us now do this. Here is our first example.

Example *The magnetic field due to a long straight current carrying wire*

Let P be a point which is a perpendicular distance r from an infinite wire through which is passing current I; we want to calculate the magnetic field $\mathbf{B}$ at P. Since we wish to use Ampère's theorem we must first select a curve C around which to integrate the magnetic field $\mathbf{B}$; we choose C to be circle of radius $r = |\mathbf{r}|$ with its centre on the wire cf. Fig. 2.

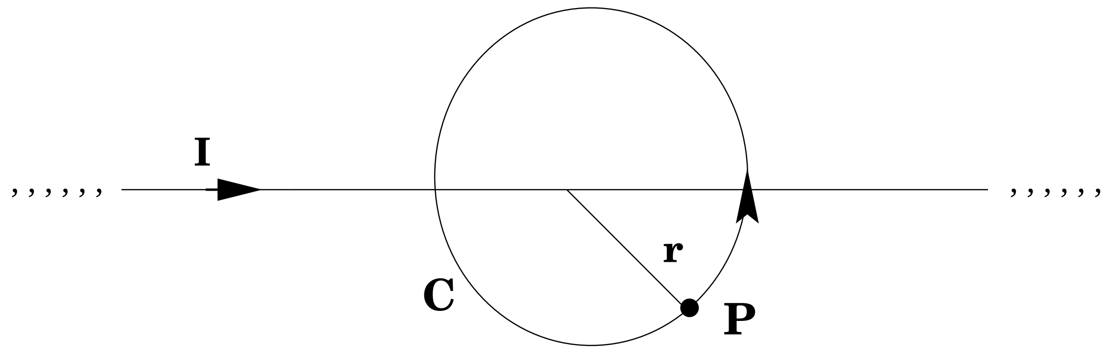

Fig. 2: The magnetic field produced by an infinite straight wire

Since this circle links the wire precisely once we have

$$\int_C \mathbf{B} \cdot \mathbf{dl} = \mu_0 I \tag{8.9}$$

Now we *assume* [1] that experiment has shown us that the magnetic lines of force are circles centred on the wire. This means that the $\mathbf{B}$ vectors are *tangential* to the curve C; but so are the $\mathbf{dl}$ vectors by their definition—i.e. $\mathbf{B}$ and $\mathbf{dl}$ are parallel. Hence

$$\mathbf{B} \cdot \mathbf{dl} = ||\mathbf{B}||\mathbf{dl}| \tag{8.10}$$

Now we can evaluate the integral for, using this parallelism, we have

$$\begin{aligned} \int_C \mathbf{B} \cdot \mathbf{dl} &= \int_C ||\mathbf{B}||\mathbf{dl}| \\ &= |\mathbf{B}| \int_{\mathbf{C}} |\mathbf{dl}|, \quad \text{since } |\mathbf{B}| \text{ is constant on } C \\ &= |\mathbf{B}| 2\pi r \end{aligned} \tag{8.11}$$

where we explain that $\mathbf{B}$ is constant on C since all points on C are the same perpendicular distance r from the wire; also we used the fact that $\int_C |dl|$ is just the total length of C—i.e. the circumference $2\pi r$ of the circle. Finally Ampère's law tells that the integral is equal to $\mu_0 I$ so we can say that

$$\begin{aligned} |\mathbf{B}| 2\pi r &= \mu_0 I \\ \Rightarrow |\mathbf{B}| &= \frac{\mu_0 I}{2\pi r} \end{aligned} \tag{8.12}$$

[1] We shall redo this calculation without this assumption in the next section using the more powerful (but not always needed) Biot-Savart law.

Thus, since we know that $\mathbf{B}$ is tangential to C, we let $\mathbf{e}$ denote a unit vector tangent to the curve C and the complete expression for the magnetic field $\mathbf{B}$ at P is now

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\mathbf{e} \tag{8.13}$$

We move on to another example, this one involves a *solenoid.*

Example *The magnetic field inside an infinite solenoid*

This time we pass a current I through an infinitely long solenoid—cf. Fig. 3 for a picture of a short piece of the solenoid viewed from a skew angle.

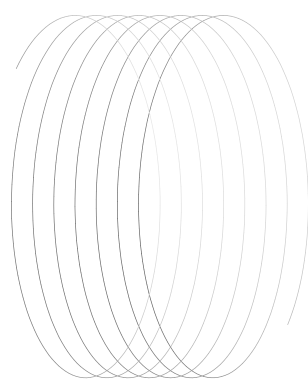

Fig. 3: A loosely wound solenoid

If we move round exactly perpendicular to the axis of the solenoid and stretch it out somewhat it will then look as shown in Fig. 4 below.

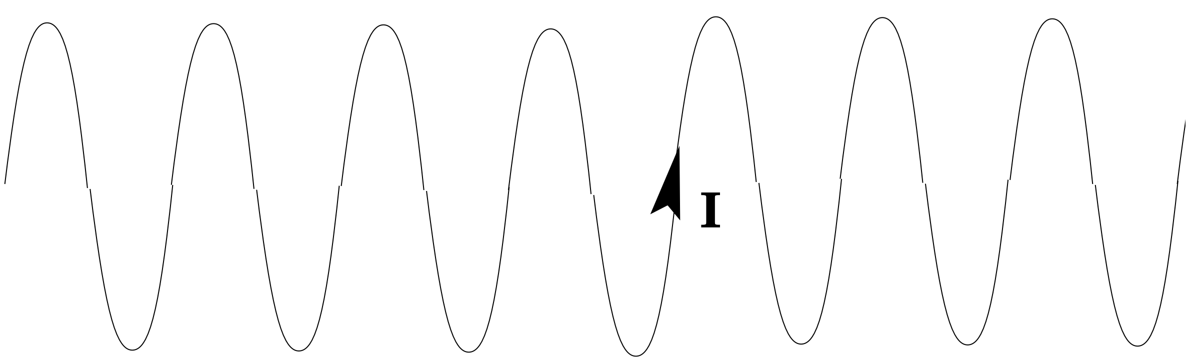

Fig. 4: A loosely wound solenoid carrying a current I

The current I produces a magnetic field $\mathbf{B}$. We want an expression for the field $\mathbf{B}$ at the point P where P is a point on the axis of the solenoid. We are going to use Ampère's theorem and so must choose a curve C and then integrate $\mathbf{B}$ around C. We choose C to be the rectangle $EFGH$, cf. Fig. 5.

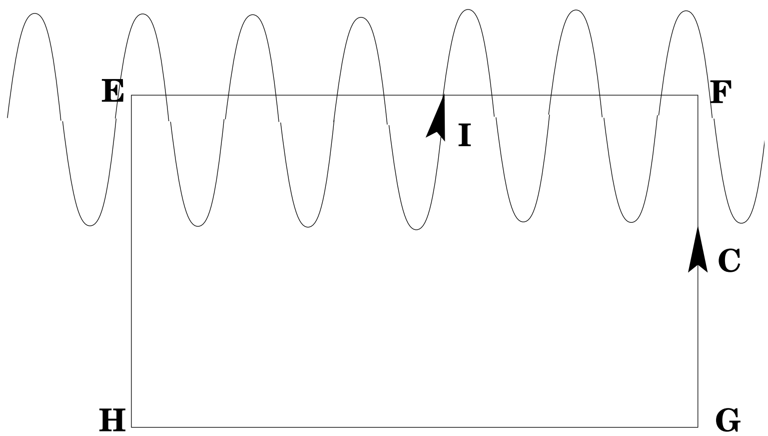

Fig. 5: The solenoid and the rectangular path C

Next we must specify *how tightly* the solenoid is wound and so we define the integer N by saying that N is the *number of turns per unit length* of the solenoid. In addition we wish to specify the *width* of the rectangle, i.e the length of the line EF; we shall denote the length of EF by L.

We note, now. that all this means that the line EF, having length L, passes through exactly

$$NL \tag{8.14}$$

turns of the solenoid. This in turn means that Ampère's theorem applied to the curve C gives the result that

$$\int_C \mathbf{B} \cdot \mathbf{dl} = \mu_0 NLI \tag{8.15}$$

since the rectangle is linked with all the NL turns passed through by the line EF.

The final short task that we have is to evaluate the integral $\int_C \mathbf{B} \cdot \mathbf{dl}$. The key to doing this is comprised of two observations and these are

(i) The integral $\int_C \mathbf{B} \cdot \mathbf{dl}$ is *independent* of the *length* FG of C; hence we may make this length as large as we like.

(ii) Axial symmetry of an infinitely long object, such as this solenoid, means that the magnetic field $\mathbf{B}$ must point along the axis of the solenoid. This then means that $\mathbf{B}$ is perpendicular to the $\mathbf{dl}$ vectors on the two vertical sides of C: namely the sides GF and EH. To use these observations we first decompose the integral into four pieces—one for each side of the rectangle—giving

$$\int_C \mathbf{B} \cdot \mathbf{dl} = \int_{HG} \mathbf{B} \cdot \mathbf{dl} + \int_{GF} \mathbf{B} \cdot \mathbf{dl} + \int_{FE} \mathbf{B} \cdot \mathbf{dl} + \int_{EH} \mathbf{B} \cdot \mathbf{dl} \tag{8.16}$$

Now, because of the perpendicularity mentioned in (ii) above, the integrals along the sides GF and EH vanish, so we have

$$\int_{GF} \mathbf{B} \cdot \mathbf{dl} = \int_{EH} \mathbf{B} \cdot \mathbf{dl} = 0 \tag{8.17}$$

Also, exploiting point (i) above, we make the length GF tend to infinity this make $\mathbf{B}$ tend to zero along HG so the integral for this side vanishes giving

$$\int_{HG} \mathbf{B} \cdot \mathbf{dl} = 0 \tag{8.18}$$

All that remains of the integral around C is the portion FE; but this portion is along the axis where $\mathbf{B}$ is parallel to the $\mathbf{dl}$ vectors. Hence we can immediately compute that

$$\int_{FE} \mathbf{B} \cdot \mathbf{dl} = \int_{FE} |\mathbf{B}||\mathbf{dl}| = |\mathbf{B}| \int_{FE} |\mathbf{dl}| = |\mathbf{B}|L \tag{8.19}$$

So the upshot of doing these four integrals, when combined with 8.15, is that

$$\begin{aligned} \int_{C} \mathbf{B} \cdot \mathbf{dl} &= |\mathbf{B}|L = \mu_0 NLI \\ \Rightarrow |\mathbf{B}| &= \mu_0 NI \end{aligned} \tag{8.20}$$

This gives us the magnitude of the field $\mathbf{B}$ inside the solenoid, but we already know that $\mathbf{B}$ point along the axis of the solenoid; hence, if $\mathbf{e}$ denotes a unit vector along the axis of the solenoid, our final result for $\mathbf{B}$ is that

$$\mathbf{B} = \mu_0 NI\mathbf{e} \tag{8.21}$$

§ 3. The Biot–Savart law

The Ampère law can sometimes not yield an easy route to the calculation of the magnetic field produced by a current carrying wire. When this is so there is a more powerful result that one can have recourse to; this result is called the Biot–Savart law and it is the following statement.

Biot-Savart law *Let a current I pass through a wire thereby producing a magnetic field* $\mathbf{B}$. *Then if* $\mathbf{dl}$ *is an element of length along the wire, located at* $\mathbf{r}'$, *it produces a field* $\mathbf{dB}$ *at the point* $\mathbf{r}$ *where*

$$\mathbf{dB} = \frac{\mu_0 I}{4\pi} \frac{\mathbf{dl} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \tag{8.22}$$

The best way to understand this law is to move straight on to an example. We shall work through two examples as illustrations of the Biot–Savart law; as our first example we shall redo the calculation of the field $\mathbf{B}$ due to a straight wire that we computed using Ampère's law.

Example *The magnetic field due to a long straight current carrying wire redone using the Biot–Savart law*

We shall take the wire to coincide with the x'-axis, cf. Fig. 6 (note that we have to call this axis the x'-axis rather than the x-axis because we have already used the variable x in the expression for the vector $\mathbf{r}$ which we recall is given by $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$), so that

$$\mathbf{dl} = dx'\mathbf{i} \tag{8.23}$$

In addition we shall choose (without any loss of generality) the point $\mathbf{r}$ to lie in the $x - z$ plane. With these notational conventions we have

$$\mathbf{r}' = x'\mathbf{i}, \quad \mathbf{r} = x\mathbf{i} + z\mathbf{k} \tag{8.24}$$

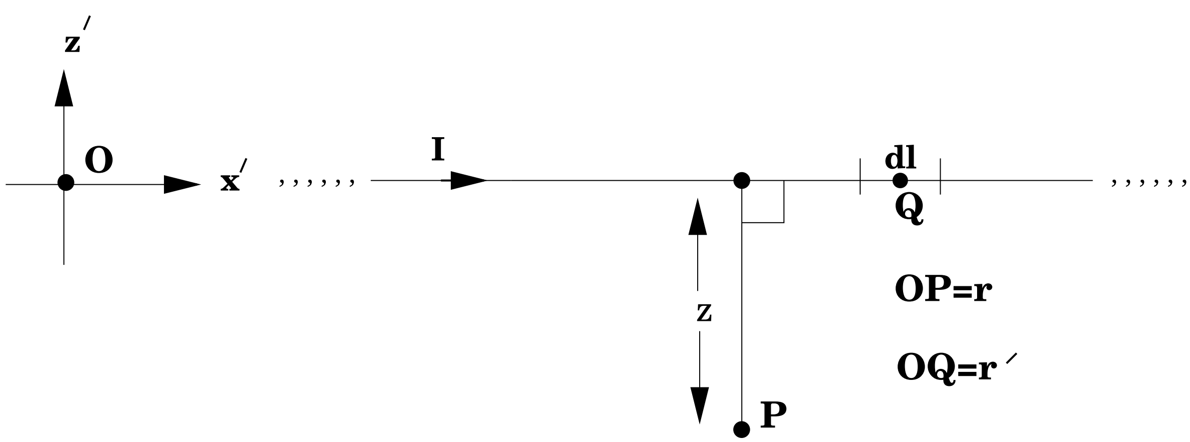

Fig. 6: The straight wire for the Biot-Savart calculation

It is now straightforward to calculate that

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{(x - x')^2 + z^2} \tag{8.25}$$

and that

$$\begin{aligned} \mathbf{dl} \times (\mathbf{r} - \mathbf{r}') &= dx'\mathbf{i} \times \{(x - x')\mathbf{i} + z\mathbf{k}\} \\ &= z dx'\mathbf{i} \times \mathbf{k} = -z dx'\mathbf{j} \end{aligned} \tag{8.26}$$

So now we have

$$\mathbf{dB}(\mathbf{r}) = -\frac{\mu_0 I}{4\pi} \frac{z dx'}{\{(x - x')^2 + z^2\}^{3/2}} \mathbf{j} \tag{8.27}$$

The last step in the calculation is to integrate **dB**: we have

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0 I z}{4\pi}\mathbf{j}\int_{-\infty}^{\infty}\frac{dx'}{\{(x-x')^2+z^2\}^{3/2}} \tag{8.28}$$

Evaluating the integral is routine enough: we make the substitution

$$\begin{aligned} x - x' &= z\tan(\theta) \\ \Rightarrow dx' &= -z\sec^2(\theta)d\theta \end{aligned} \tag{8.29}$$

and then we find that

$$\begin{aligned} \int_{-\infty}^{\infty}\frac{dx'}{\{(x-x')^2+z^2\}^{3/2}} &= -\int_{-\pi/2}^{\pi/2} d\theta\,\frac{z\sec^2(\theta)}{z^3\sec^3(\theta)} \\ &= -\frac{1}{z^2}\int_{-\pi/2}^{\pi/2}\frac{d\theta}{\sec(\theta)} = -\frac{1}{z^2}\int_{-\pi/2}^{\pi/2} d\theta\,\cos(\theta) \\ &= -\frac{2}{z^2} \end{aligned} \tag{8.30}$$

The resulting expression for **B** is given by

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{2\pi z}\mathbf{j} \tag{8.31}$$

and this is in complete agreement with the expression 8.13 obtained using Ampère's law except that in 8.13 the variable z was denoted by r and the unit vector **e** is here identified as **j**.

Example *The magnetic field at the centre of a circular loop carrying a current I*

Let a current I pass through a closed wire which is bent into a circular shape, the circle having a radius R. We want to calculate the magnetic field **B** produced at the centre of the circle.

The basic Biot–Savart expression for the field due to the element of wire **dl** is

$$\mathbf{dB} = \frac{\mu_0 I}{4\pi}\frac{\mathbf{dl}\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{8.32}$$

Now we choose the wire to lie in the $x-z$ plane so that the *centre* of the circle coincides with the *origin* of the coordinate system. This has the great simplification that the point **r** at which we want to calculate **B** is the zero vector—i.e. we have

$$\mathbf{r} = \mathbf{0} \tag{8.33}$$

This means that **dB** is now of the form

$$\mathbf{dB} = -\frac{\mu_0 I}{4\pi}\frac{\mathbf{dl} \times \mathbf{r}'}{|\mathbf{r}'|^3} \tag{8.34}$$

Now the vector $\mathbf{r}'$ lies on the circle of radius R so it is given by the formula

$$\mathbf{r}' = R\cos(\theta)\mathbf{i} + R\sin(\theta)\mathbf{k} \tag{8.35}$$

from which we see immediately that

$$|\mathbf{r}'| = R \tag{8.36}$$

Hence

$$\mathbf{dB} = -\frac{\mu_0 I}{4\pi}\frac{\mathbf{dl} \times (R\cos(\theta)\mathbf{i} + R\sin(\theta)\mathbf{k})}{R^3} \tag{8.37}$$

Now the derivative

$$\frac{\mathbf{dr}'}{d\theta} \tag{8.38}$$

has to be tangential to the circle at the point $\mathbf{r}'$ so the vector **dl**, which is also tangent at the same point, but has length $Rd\theta$, is given by

$$\begin{aligned} \mathbf{dl} &= \frac{\mathbf{dr}'}{d\theta}d\theta \\ &= (-R\sin(\theta)\mathbf{i} + R\cos(\theta)\mathbf{k})d\theta \\ &= R(-\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{k})d\theta \end{aligned} \tag{8.39}$$

So, putting all this together, we have

$$\mathbf{dB} = -\frac{\mu_0 I}{4\pi}\frac{R\,(-\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{k})\,d\theta \times (R\cos(\theta)\mathbf{i} + R\sin(\theta)\mathbf{k})}{R^3} \tag{8.40}$$

Tidying up, and doing the cross products, we find that

$$\begin{aligned} \mathbf{dB} &= -\frac{\mu_0 I}{4\pi R}\left(\sin^2(\theta)\mathbf{j} + \cos^2(\theta)\mathbf{j}\right)d\theta \\ &= -\frac{\mu_0 I}{4\pi R}\left(\sin^2(\theta) + \cos^2(\theta)\right)\mathbf{j}d\theta \\ &= -\frac{\mu_0 I}{4\pi R}\mathbf{j}d\theta \end{aligned} \tag{8.41}$$

It is now a trivial matter to integrate and obtain

$$\begin{aligned}
\mathbf{B} &= \int \mathbf{dB} \\
&= -\int_0^{2\pi} \frac{\mu_0 I}{4\pi R}\mathbf{j}d\theta \\
&= 2\pi \\
\Rightarrow \mathbf{B} &= -\frac{\mu_0 I}{2R}\mathbf{j}
\end{aligned} \tag{8.42}$$

and so our calculation is complete; note, before leaving this example, that $|\mathbf{B}| \propto 1/R$.

An equation for $\nabla \times \mathbf{B}$

We are now in a position to derive a useful equation for $\nabla \times \mathbf{B}$ which will turn out later to be a stepping stone to Maxwell's fourth equation. The equation we are after is

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{8.43}$$

To derive it we start with a curve C encircling a current carrying wire just once; this means that Ampère's law says that

$$\int_C \mathbf{B} \cdot \mathbf{dl} = \mu_0 I \tag{8.44}$$

But, if S is the surface interior to C, and $\mathbf{J}$ is the current density, then I is given by

$$I = \int_S \mathbf{J} \cdot \mathbf{dS} \tag{8.45}$$

Inserting this into Ampère's law gives

$$\int_C \mathbf{B} \cdot \mathbf{dl} = \mu_0 \int_S \mathbf{J} \cdot \mathbf{dS} \tag{8.46}$$

where I is the current carried by the wire. But applying Stokes' theorem to the $\mathbf{B}$ integral we then obtain

$$\int_S \nabla \times \mathbf{B} \cdot \mathbf{dS} = \mu_0 \int_S \mathbf{J} \cdot \mathbf{dS} \tag{8.47}$$

Finally, since the surface S is arbitrary—except that it must be cut by the wire—we find that the integrands are equal, i.e. we have the desired result

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{8.48}$$

CHAPTER IX
Maxwell's third and fourth equations

§ 1. Electromagnetic induction and Maxwell's third equation

WE now begin a discussion which will end with the derivation of Maxwell's third equation. As a prerequisite we want to describe a vital experimental fact which tells one how a moving charge is acted on by a magnetic field. This action is usually called the *Lorentz force law*. More formally we have

Lorentz force law *A charge q which moves with velocity* $\mathbf{v}$ *in a magnetic field* $\mathbf{B}$ *experiences a force* $\mathbf{F}$*, known as the Lorentz force, where*

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \tag{9.1}$$

We turn now to electromagnetic induction. The great experimentalist Faraday discovered that if one takes a closed loop of wire through which *no current is passing* then, if one *moves the loop* in a magnetic field $\mathbf{B}$, a current I can be induced.

Let the loop of wire enclose a surface S, then as the loop moves the magnetic flux $\int_S \mathbf{B} \cdot \mathbf{dS}$ changes with time. Faraday's very ingenious and careful experiments established that the voltage[1] $\mathcal{E}$, producing this current is related to the rate of change of the magnetic flux Φ by the equation

$$\begin{aligned} \mathcal{E} &= -\frac{\partial}{\partial t}\int_S \mathbf{B} \cdot \mathbf{dS} \\ \text{i.e.} \qquad \mathcal{E} &= -\frac{\partial \Phi}{\partial t} \end{aligned} \tag{9.2}$$

[1] $\mathcal{E}$ is also often called an *emf*, this stands for the term *electromotive force*

We shall now use the Lorentz force to rederive Faraday's result and to derive Maxwell's third equation which happens to be

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{9.3}$$

Let us take a rectangular loop $ABCD$ placed in a constant magnetic field $\mathbf{B}$ which is *perpendicular* to the plane of the rectangle so that is parallel to the $\mathbf{dS}$ vector. In addition we want the side AB to be moveable and be able to slide along at a constant speed v cf. Fig. 7.

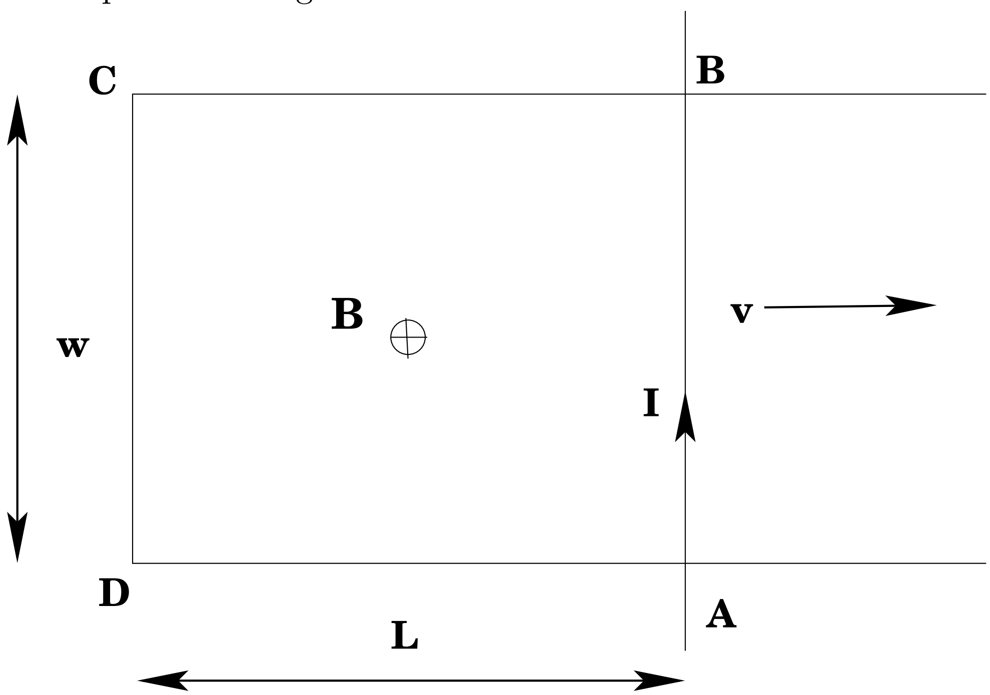

Fig. 7: The rectangular loop with the moving side

Now, since $\mathbf{B}$ is constant and perpendicular to the rectangle, the flux Φ through this loop is just $|\mathbf{B}|$ times the area of the loop, so that we have

$$\Phi = \int_S \mathbf{B} \cdot \mathbf{dS} = |\mathbf{B}| wL \tag{9.4}$$

the area of the loop being wL. Further if we differentiate ϕ with respect to t, and notice that

$$\frac{dL}{dt} = |\mathbf{v}| \tag{9.5}$$

since the side AB is moving with velocity $\mathbf{v}$, then we see at once that

$$\frac{\partial \Phi}{\partial t} = |\mathbf{B}| w |\mathbf{v}| \tag{9.6}$$

Next note that the electrons in the moving side AB are moving (with velocity $\mathbf{v}$) in a magnetic field and so are subject to the Lorentz force law. They receive, therefore, a force

$$\mathbf{F} = \mathbf{e}\mathbf{v} \times \mathbf{B} \tag{9.7}$$

But this force is the same as if they were subject to an electric field $\mathbf{E}$ given by the expression

$$\mathbf{E} = \mathbf{v} \times \mathbf{B} \tag{9.8}$$

and such an electric field would be produced by applying a potential difference $\mathcal{E}$ across the ends of AB where[2]

$$\mathcal{E} = \int_{AB} \mathbf{v} \times \mathbf{B} \cdot \mathbf{dl} \tag{9.9}$$

However $\mathbf{v}$, $\mathbf{B}$ and $\mathbf{dl}$ are all constant and mutually perpendicular so we have immediately the result that[3]

$$\begin{aligned} \int_{AB} \mathbf{v} \times \mathbf{B} \cdot \mathbf{dl} &= -|\mathbf{v}||\mathbf{B}|w \\ \text{i.e.} \qquad \mathcal{E} &= -|\mathbf{v}||\mathbf{B}|w \end{aligned} \tag{9.10}$$

Now if we compare equations 9.6 and 9.10 for Φ and $\mathcal{E}$ respectively we find that we do indeed have

$$\mathcal{E} = -\frac{\partial \Phi}{\partial t} \tag{9.11}$$

in agreement with Faraday's experimental law.

Since the other three sides of the rectangle do not move, we can change the expression for $\mathcal{E}$ to be an integral all the way around the rectangle and not just along the side AB; we shall do this and, denoting the rectangle $ABCD$ by just C, we have

$$\mathcal{E} = \int_C \mathbf{E} \cdot \mathbf{dl} \tag{9.12}$$

Maxwell's equation follows from being more formal and writing

$$\Phi = \int_S \mathbf{B} \cdot \mathbf{dS}, \qquad \text{and} \qquad \mathcal{E} = \int_C \mathbf{E} \cdot \mathbf{dl} \tag{9.13}$$

[2] Note that the convention is to define $\mathcal{E} = \int_{AB} \mathbf{E} \cdot \mathbf{dl}$ rather than $\mathcal{E} = -\int_{AB} \mathbf{E} \cdot \mathbf{dl}$ so that $\mathcal{E}$ is actually the opposite sign to what we normally call a voltage.

[3] To get the signs on the RHS work out you have choose $\mathbf{B}$ 'perpendicular and upwards' to $ABCD$ and recall that $\mathbf{dl}$ points from A to B; this means that the direction of $\mathbf{E} = \mathbf{v} \times \mathbf{B}$ is *opposite* to that of $\mathbf{dl}$. Other choices can be made but the final result will be unaltered.

With this notation 9.11 becomes

$$\begin{aligned}
\frac{\partial}{\partial t}\int_S \mathbf{B}\cdot\mathbf{dS} &= -\int_C \mathbf{E}\cdot\mathbf{dl}\\
\Rightarrow \int_S \frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{dS} &= -\int_C \mathbf{E}\cdot\mathbf{dl}\\
\Rightarrow \int_S \frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{dS} &= -\int_S \nabla\times\mathbf{E}\cdot\mathbf{dS}, \quad \text{by Stokes' theorem}\\
\Rightarrow \nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}, \quad \text{since } S \text{ is arbitrary}
\end{aligned} \tag{9.14}$$

and so we have the *third* of Maxwell's four equations. Stating it again for emphasis, it is the equation

Maxwell's third equation

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \tag{9.15}$$

We turn at once to the derivation of the fourth and last Maxwell equation.

§ 2. Maxwell's fourth equation—the story of the displacement current

Maxwell's fourth equation involves the quantity $\nabla\times\mathbf{B}$; now we have already obtained an equation for $\nabla\times\mathbf{B}$ namely

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} \tag{9.16}$$

however we shall now see that this equation is incomplete. It is the corrected, or completed, form of this equation that we are after.

Let us see what is wrong with

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} \tag{9.17}$$

First take the divergence of both sides yielding

$$\nabla\cdot\nabla\times\mathbf{B} = \mu_0\nabla\cdot\mathbf{J} \tag{9.18}$$

But $\nabla\cdot\nabla\times\mathbf{B} = 0$ since $\nabla\cdot\nabla\times\mathbf{A} = 0$ for *any* $\mathbf{A}$. Hence we have deduced that

$$\nabla\cdot\mathbf{J} = 0 \tag{9.19}$$

Unfortunately this deduction is a *disaster*, and is definitely mistaken, since it contradicts the extremely well established experimental fact that charge is conserved. We now have to put this right.

Suppose, then, that the density of charge inside an arbitrary volume V is ρ and that the charges are inside V are not static but in motion giving rise to a current density $\mathbf{J}$. Now some charges may flow out across the surface S of V and escape thus reducing the total charge Q of V. The corresponding *decrease* in Q per unit time is therefore

$$-\frac{dQ}{dt} \tag{9.20}$$

This outward flow is a current I across the surface S and we know that

$$I = \int_S \mathbf{J} \cdot \mathbf{dS} \tag{9.21}$$

But I is also a measured in units of charge per unit time and, *since charge is conserved*, this current must be equal to the decrease in charge of V. In other words we must have

$$-\frac{dQ}{dt} = \int_S \mathbf{J} \cdot \mathbf{dS} \tag{9.22}$$

But we can express Q in terms of the charge density ρ by writing

$$Q = \int_V \rho dV \tag{9.23}$$

so we have

$$\begin{aligned} -\frac{\partial}{\partial t}\int_V \rho\, dV &= \int_S \mathbf{J} \cdot \mathbf{dS} \\ \Rightarrow -\int_V \frac{\partial \rho}{\partial t}\, dV &= \int_S \mathbf{J} \cdot \mathbf{dS} \end{aligned} \tag{9.24}$$

But Gauss's divergence theorem applied to $\mathbf{J}$ says that

$$\int_S \mathbf{J} \cdot \mathbf{dS} = \int_V \nabla \cdot \mathbf{J}\, dV \tag{9.25}$$

and so we find that

$$\begin{aligned} -\int_V \frac{\partial \rho}{\partial t}\, dV &= \int_V \nabla \cdot \mathbf{J}\, dV \\ \Rightarrow \int_V \left\{ \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} \right\} dV &= 0 \\ \Rightarrow \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} &= 0, \quad \text{since } V \text{ is arbitrary} \end{aligned} \tag{9.26}$$

and this equation expresses the conservation of charge. So we see that, rather than having $\nabla \cdot \mathbf{J} = 0$ we have

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \tag{9.27}$$

Our strategy now is to add a term to our equation for $\nabla \times \mathbf{B}$ and to try and derive the form of this term from what we know already. To this end we denote this added term by $\mathbf{J_D}$ and call it (following the common practice) the *displacement current*; the equation for $\nabla \times \mathbf{B}$ then becomes

$$\nabla \times \mathbf{B} = \mu_0 \, \mathbf{J} + \mathbf{J_D} \tag{9.28}$$

Proceeding as before we take the divergence of both sides giving

$$\begin{aligned} \nabla \cdot \nabla \times \mathbf{B} &= \mu_0 \nabla \cdot \mathbf{J} + \nabla \cdot \mathbf{J_D} \\ \Rightarrow 0 &= \mu_0 \, \nabla \cdot \mathbf{J} + \nabla \cdot \mathbf{J_D} \\ \Rightarrow 0 &= -\mu_0 \, \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J_D}. \quad \text{using 9.27} \end{aligned} \tag{9.29}$$

Here we take time off to point out that experimentally [4] it is found that the constant μ_0 is related to the constant ϵ_0; in fact one knows that

$$\mu_0 = \frac{1}{\epsilon_0 c^2} \tag{9.30}$$

where c is the velocity of light in vacuo. In any case we now have an equation for the displacement current $\mathbf{J_D}$, it is simply that

$$\nabla \cdot \mathbf{J_D} = \frac{1}{\epsilon_0 c^2} \frac{\partial \rho}{\partial t} \tag{9.31}$$

Now we integrate both sides of this equation over the volume V giving

$$\int_V \nabla \cdot \mathbf{J_D} \, dV = \frac{1}{\epsilon_0 c^2} \int_V \frac{\partial \rho}{\partial t} \, dV \tag{9.32}$$

[4] For the record this was work done in 1856 by Weber and Kohlrausch—cf. Weber, W., Kohlrausch, R., *Über die Elektrizitätsmenge, welche bei galvanischen Strömen durch den Querschnitt der Kette fließt*, Poggendorffs Annalen, **99**, 10–25, (1856)—this result of course strongly suggests that light has something to do with electromagnetic phenomena; however the velocity of light was not known all that well in 1856 though it had become much more accurately determined by the year 1864: the year when Maxwell did his famous work, cf. below.

But we know from Gauss's divergence theorem that

$$\int_V \nabla \cdot \mathbf{J_D}\, dV = \int_S \mathbf{J_D} \cdot \mathbf{dS} \tag{9.33}$$

and from Maxwell's first equation that

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \\ \Rightarrow \rho &= \epsilon_0 \nabla \cdot \mathbf{E} \end{aligned} \tag{9.34}$$

Using both these facts we get

$$\begin{aligned} &\int_S \mathbf{J_D} \cdot \mathbf{dS} = \frac{1}{\epsilon_0 c^2} \int_V \epsilon_0 \frac{\partial(\nabla \cdot \mathbf{E})}{\partial t}\, dV \\ \Rightarrow &\int_S \mathbf{J_D} \cdot \mathbf{dS} = \frac{1}{c^2} \frac{\partial}{\partial t} \int_V \nabla \cdot \mathbf{E}\, dV \\ \Rightarrow &\int_S \mathbf{J_D} \cdot \mathbf{dS} = \frac{1}{c^2} \frac{\partial}{\partial t} \int_S \mathbf{E} \cdot \mathbf{dS}, \quad \text{(Gauss's divergence theorem)} \\ \Rightarrow &\int_S \mathbf{J_D} \cdot \mathbf{dS} = \int_S \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{dS} \end{aligned} \tag{9.35}$$

Then, as usual, since S is arbitrary we conclude that

$$J_D = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \tag{9.36}$$

and we have successfully found the displacement current $\mathbf{J_D}$. This means that the correct equation for $\nabla \times \mathbf{B}$, which is *Maxwell's fourth equation* is

Maxwell's fourth equation

$$\nabla \times \mathbf{B} = \frac{1}{\epsilon_0 c^2} \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \tag{9.37}$$

So we now have all four of Maxwell's equations.

In the next section we shall make some comments on each equation and examine the displacement current term a little more closely.

§ 3. Maxwell's four equations

The four Maxwell equations should now be viewed together and some thought given to how they were derived. To this end the four equations are listed below with each accompanied by a short relevant comment relating to their origins.

Maxwell's celebrated four equations

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0}, \qquad \text{(Coulomb's law)} \\
\nabla \cdot \mathbf{B} &= 0, \qquad \text{(absence of magnetic monopoles)} \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \qquad \text{(electromagnetic induction)} \\
\nabla \times \mathbf{B} &= \frac{1}{\epsilon_0 c^2}\mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}, \qquad \text{(displacement current)}
\end{aligned} \tag{9.38}$$

Example *The displacement current term in action*

The displacement current's rôle in electromagnetic phenomena can be quite subtle; because this is so we now present an example of the working of this term

$$\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{9.39}$$

We take a spherically symmetric situation. Let us have a radioactive β-source in the centre of a containing sphere. This source simply serves as a source of electrons which, by the spherical symmetry of the situation, are emitted outwardly along the radial directions giving a radial current density $\mathbf{J}$ cf. Fig. 8.

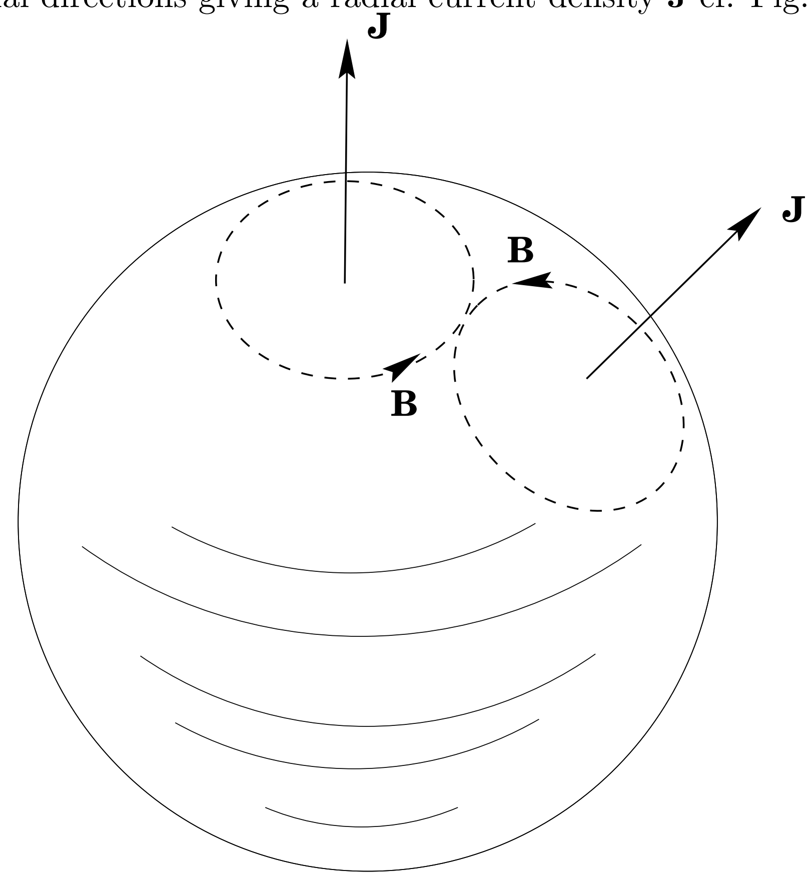

Fig. 8: The β-source with its radial current density $\mathbf{J}$.

In Fig. 8 the dotted lines show the magnetic lines of force produced by each vector $\mathbf{J}$. Note carefully that where the a pair of such lines touch the magnetic field the will be cancellation between the two causing the magnetic field $\mathbf{B}$ to be zero there. Since such a point of intersection could be anywhere on the sphere this suggests that $\mathbf{B}$ should be zero everywhere on the sphere. This is actually the case but we shall not prove it we shall just show that

$$\nabla \times \mathbf{B} = \mathbf{0} \tag{9.40}$$

which is, of course, implied by $\mathbf{B} = \mathbf{0}$.

The point is that $\nabla \times \mathbf{B}$ is the LHS of Maxwell's equation and so if it vanishes there must be a *cancellation* between the two terms on the RHS. In other words we will have

$$\begin{aligned} \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} &= -\frac{1}{\epsilon_0 c^2}\mathbf{J} \\ \Rightarrow \frac{\partial \mathbf{E}}{\partial t} &= -\frac{\mathbf{J}}{\epsilon_0} \end{aligned} \tag{9.41}$$

so that the displacement current term cannot be zero since $\mathbf{J}$ is, by construction, non-zero. We proceed to the calculation.

Let the sphere have radius r so and let it contain a total charge $Q(t)$ at time t. The *outward* electrical current I from the radioactive decay is given by

$$I = -\frac{\partial Q(t)}{\partial t} \tag{9.42}$$

But

$$I = \int_S \mathbf{J} \cdot \mathbf{dS} = 4\pi r^2 |\mathbf{J}| \tag{9.43}$$

so we have

$$-\frac{\partial Q(t)}{\partial t} = 4\pi r^2 |\mathbf{J}| \tag{9.44}$$

However, for the electric field $\mathbf{E}$, spherical symmetry says that $\mathbf{E}$ is the same as the field produced by a point charge at the centre of the sphere so we have

$$\begin{aligned} \mathbf{E} &= \frac{Q(t)}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2} \\ \Rightarrow Q(t) &= 4\pi\epsilon_0 |\mathbf{E}| r^2 \end{aligned} \tag{9.45}$$

Hence combining eqs. 9.44 and 9.45 we find that

$$\begin{aligned} \frac{\partial\left(4\pi\epsilon_0 r^2 |\mathbf{E}|\right)}{\partial t} &= -4\pi r^2 |\mathbf{J}| \\ \Rightarrow \frac{\partial |\mathbf{E}|}{\partial t} &= -\frac{|\mathbf{J}|}{\epsilon_0} \\ \Rightarrow \frac{\partial \mathbf{E}}{\partial t} &= -\frac{\mathbf{J}}{\epsilon_0}, \qquad \text{since } \mathbf{E} = |\mathbf{E}|\,\hat{\mathbf{r}} \text{ and } \mathbf{J} = |\mathbf{J}|\,\hat{\mathbf{r}} \end{aligned} \tag{9.46}$$

Hence we have verified that

$$\nabla \times \mathbf{B} = \mathbf{0} \tag{9.47}$$

which is what we wanted to do.

CHAPTER X

Electromagnetic waves

§ 1. Wave solutions to Maxwell's equations

It is now time to establish the celebrated result that both **E** and **B** satisfy wave equations with the wave velocity being c—the velocity of light. In fact this is a simple consequence of Maxwell's equations so the difficult thing is to derive Maxwell's equations in the first place.

We start with the wave equation for the electric field **E**. First we set ρ and **J** to zero in Maxwell's equations giving us Maxwell's equations in vacuo, also called Maxwell's equations in free space: these are the four equations

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} &= \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \end{aligned} \tag{10.1}$$

Taking the curl of Maxwell's equation for $\nabla \times \mathbf{E}$ yields

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla \times \left(\frac{\partial \mathbf{B}}{\partial t}\right) \tag{10.2}$$

Now we point out that there is a vector identity which states that, for any **A**,

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{10.3}$$

Applied to **E** this gives, if we also use the fact that $\nabla \cdot \mathbf{E} = 0$,

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla^2 \mathbf{E} \tag{10.4}$$

But

$$\begin{aligned}\nabla \times \left(\frac{\partial \mathbf{B}}{\partial t}\right) &= \frac{\partial(\nabla \times \mathbf{B})}{\partial t} \\ &= \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}, \qquad \text{on using Maxwell's fourth equation}\end{aligned} \tag{10.5}$$

So we have now deduced that

$$\begin{aligned}-\nabla^2 \mathbf{E} &= -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \\ \Rightarrow \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E} &= 0\end{aligned} \tag{10.6}$$

But this latter is a wave equation and shows that $\mathbf{E}$ travels as a wave with velocity c where c is the velocity of light.

In an exactly similar fashion we can take the curl of Maxwell's equation for $\nabla \times \mathbf{B}$, use the same vector identity on the LHS and Maxwell's equation for $\nabla \times \mathbf{E}$. The result is the *same wave equation* for $\mathbf{B}$. Just going through these steps we obtain

$$\begin{aligned}\nabla \times (\nabla \times \mathbf{B}) &= \frac{1}{c^2}\nabla \times \left(\frac{\partial \mathbf{E}}{\partial t}\right) \\ \Rightarrow -\nabla^2 \mathbf{B} &= \frac{1}{c^2}\frac{\partial(\nabla \times \mathbf{E})}{\partial t} \\ \Rightarrow -\nabla^2 \mathbf{B} &= -\frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2}\end{aligned} \tag{10.7}$$

So $\mathbf{B}$ does indeed satisfy

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{B} = 0 \tag{10.8}$$

Maxwell gave a lecture on his work to the Royal Society of London in 1864 and his results were then published [1] in 1865. Faraday had earlier suggested [2] that light was as an electromagnetic wave in 1846; this fact was duly acknowledged by Maxwell in his paper.

[1] Maxwell J. C., *A dynamical theory of the electromagnetic field*, Phil. Tran. Roy. Soc, **155**, 459–512, (1865).

[2] Faraday M., *Thoughts on ray vibrations*, Phil. Mag., **28**, 345–350, (1846). Faraday's *Thoughts on ray vibrations*, were actually delivered in 1846 as an off the cuff lecture to the Royal Society on the occasion of the scheduled speaker not being available.

It seems very likely that Faraday was stimulated to think along these lines by the fact that in 1845 he had carried out an experiment which showed that polarised light had its plane of polarisation rotated when it passed through a magnetic field.

An important fact about one dimensional waves travelling, say, in the x-direction with velocity c. is that if $f(x,t)$ represents any function *or vector* which is a solution to the wave equation so that

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) f(x,t) = 0 \tag{10.9}$$

then

$$f = f(x - ct) \tag{10.10}$$

is always a solution for *any*[3] f—we shall see below that $f = f(x + ct)$ is also always a solution.

Example *Standing waves*

A standing wave is a superposition of two waves travelling in opposite directions. For example, in one dimension, a wave travelling with velocity c along the x-axis towards the positive direction has x, t dependence of the form

$$f(x - ct) \tag{10.11}$$

Hence the superposition

$$f(x - ct) + g(x + ct) \tag{10.12}$$

represents two waves travelling in opposite directions and is therefore a *standing wave* (sometimes it is insisted that the functions g and f are the same for a standing wave). One should also add that it is immediate that both of the above functions are solutions to the wave equation. In other words we have

$$\begin{aligned}\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) f(x - ct) = 0\\ \left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) (f(x - ct) + g(x + ct)) = 0\end{aligned} \tag{10.13}$$

as can quickly be verified by direct differentiation.

A standing wave can often have a misleading appearance as it may be written as *a product* of two functions instead of as *a sum*. If this is so then it can be rewritten as a sum and, to elucidate this point, consider the following example. Suppose

$$f(x,t) = \cos(x)\cos(ct) \tag{10.14}$$

[3] For example the reader can try $f = \cos(x - ct)$ or $f = \exp(x - ct)$ and verify by explicit differentiation that these two functions satisfy the wave equation and so on.

It is trivial to check by differentiation that f satisfies the wave equation but to see that f is expressible as a sum we simply observe that the trigonometric formula

$$\cos(A)\cos(B) = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right) \tag{10.15}$$

from which we deduce at once that

$$f(x,t) = \frac{1}{2}\cos(x-ct) + \frac{1}{2}\cos(x+ct) \tag{10.16}$$

which is precisely what we wanted.

§ 2. Transversality of E and B and the property $\mathbf{E}\cdot\mathbf{B} = \mathbf{0}$

We now want to show that both **E** and **B** are what are called *transverse* waves. A transverse wave is one where the oscillations are perpendicular to the direction of travel so, in the current setting, we want to show that when **E** and **B** travel as waves the vectors **E** and **B** are perpendicular to the direction of travel. We shall carry out our discussion for 1-dimensional waves travelling in the x-direction—this means that we assume that there is no dependence of the wave on y and z, just a dependence on x and t.

Since $\nabla\cdot\mathbf{E} = \mathbf{0}$ we have

$$\begin{aligned}
&\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \\
&\qquad\qquad \Rightarrow \frac{\partial E_x}{\partial x} = 0, \quad \text{since } \mathbf{E} \text{ is independent of } y \text{ and } z \\
&\qquad\qquad \Rightarrow E_x = C, \qquad \text{with } C \text{ a constant}
\end{aligned} \tag{10.17}$$

However, on physical grounds, we require

$$C = 0 \tag{10.18}$$

because we wish all waves to die away at infinity and a constant field does not do this. So we have

$$\begin{aligned}
&E_x = 0 \\
&\Rightarrow \mathbf{E} = E_y\,\mathbf{j} + E_z\,\mathbf{k} \\
&\Rightarrow \mathbf{E}\cdot\mathbf{i} = 0 \\
&\Rightarrow \mathbf{E} \quad \text{is transverse}
\end{aligned} \tag{10.19}$$

But now we *rotate* our coordinate system about the x-axis so that the y-axis coincides with the direction of **E**, this makes $E_z = 0$ so that we have, finally

$$\mathbf{E} = E_y\,\mathbf{j} \tag{10.20}$$

Next we turn our attention to $\mathbf{B}$. We also have $\nabla \cdot \mathbf{B} = \mathbf{0}$ so, as we just did for $\mathbf{E}$, we can deduce that

$$B_x = C, \quad C \text{ a constant} \tag{10.21}$$

and C must be zero for the same reason as we gave for $\mathbf{E}$. Hence

$$\begin{aligned} \mathbf{B} &= B_y\,\mathbf{j} + B_z\,\mathbf{k} \\ \Rightarrow \mathbf{B} \cdot \mathbf{i} &= 0, \qquad \text{so } \mathbf{B} \text{ is transverse} \end{aligned} \tag{10.22}$$

Finally we want to prove that $\mathbf{E}$ is perpendicular to $\mathbf{B}$. To do this consider the Maxwell equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{10.23}$$

Using $\mathbf{E} = E_y\,\mathbf{j}$ and $\mathbf{B} = B_y\,\mathbf{j} + B_z\,\mathbf{k}$ we obtain

$$\begin{aligned} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & E_y & 0 \end{vmatrix} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \Rightarrow \frac{\partial E_y}{\partial x}\,\mathbf{k} &= -\frac{\partial B_y}{\partial t}\mathbf{j} - \frac{\partial B_z}{\partial t}\mathbf{k} \end{aligned} \tag{10.24}$$

and so we must have

$$\frac{\partial B_y}{\partial t} = 0 \tag{10.25}$$

and this allows only a constant B_y which we reject as before. Hence we have found that

$$\mathbf{B} = B_z\,\mathbf{k} \tag{10.26}$$

so that $\mathbf{E}$ and $\mathbf{B}$ are indeed perpendicular.

Summarising the various properties of wave solutions to Maxwell's equations we have found that both $\mathbf{E}$ and $\mathbf{B}$ are transverse and that $\mathbf{E} \cdot \mathbf{B} = \mathbf{0}$. This then means that if a one dimensional wave travels in the x-direction, with $\mathbf{E}$ along the y-axis, then we have

$$\mathbf{E} = E_y\,\mathbf{j}, \quad \mathbf{B} = B_z\,\mathbf{k} \tag{10.27}$$

giving automatically

$$\mathbf{E} \cdot \mathbf{B} = \mathbf{0} \tag{10.28}$$

Example *A simple electric wave*

Suppose $\mathcal{E}$ is a constant and

$$\mathbf{E} = \mathcal{E}\cos(kx - \omega t)\,\mathbf{j}, \qquad \text{with } k = \frac{2\pi}{\lambda} \qquad \text{and } \omega = 2\pi\nu \tag{10.29}$$

then $\mathbf{E}$ is a wave provided

$$\lambda\nu = c \tag{10.30}$$

The quantities λ and ν are called the *wavelength* and *frequency* of the wave respectively. To verify this we simply substitute $\mathbf{E}$ into the wave equation 10.9 and obtain

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\cos(kx - \omega t)\,\mathbf{j} = (-k^2 + \frac{\omega^2}{c^2})\cos(kx - \omega t)\,\mathbf{j} \tag{10.31}$$

Hence, for $\mathbf{E}$ to be a solution,. we must have

$$\begin{aligned} k^2 &= \frac{\omega^2}{c^2} \\ \Rightarrow \frac{(2\pi)^2}{\lambda^2} &= \frac{(2\pi\nu^2)}{c^2} \\ \Rightarrow \lambda^2\nu^2 &= c^2 \\ \text{or} \qquad \lambda\nu &= c \end{aligned} \tag{10.32}$$

as claimed.

We can also write $\mathbf{E}$ in the form $f(x - ct)$ for note that we have

$$\begin{aligned} \mathbf{E} &= \mathcal{E}\cos\left(\frac{2\pi}{\lambda}x - 2\pi\nu t\right)\mathbf{j} \\ &= \mathcal{E}\cos\left\{\frac{2\pi}{\lambda}(x - ct)\right\}\mathbf{j} \\ &= f(x - ct) \\ \text{with} \qquad f(x - ct) &= \mathcal{E}\cos\left\{\frac{2\pi}{\lambda}(x - ct)\right\}\mathbf{j} \end{aligned} \tag{10.33}$$

§ 3. The flow of energy for electromagnetic waves

We wish, in this section, to pin down the energy properties of electromagnetic waves and discover how they can flow through a vacuum as radiation. We shall find that Maxwell's equations do, more or less, all this for us.

We begin with Maxwell's equation

$$\nabla \times \mathbf{B} = \frac{1}{\epsilon_0 c^2}\mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{10.34}$$

which we rewrite as

$$\mathbf{J} = \epsilon_0 c^2 \nabla \times \mathbf{B} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{10.35}$$

Dotting with $\mathbf{E}$ gives

$$\begin{aligned} \mathbf{E} \cdot \mathbf{J} &= \epsilon_0 c^2 \mathbf{E} \cdot \nabla \times \mathbf{B} - \epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} \\ &= \epsilon_0 c^2 \mathbf{E} \cdot \nabla \times \mathbf{B} - -\frac{\epsilon_0}{2} \frac{\partial}{\partial t} (\mathbf{E}^2) \end{aligned} \tag{10.36}$$

But using the vector identity

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B} \tag{10.37}$$

on $\mathbf{E}$ and $\mathbf{B}$ in the form

$$\mathbf{E} \cdot \nabla \times \mathbf{B} = -\nabla \cdot (\mathbf{E} \times \mathbf{B}) + \mathbf{B} \cdot \nabla \times \mathbf{E} \tag{10.38}$$

we get

$$\mathbf{E} \cdot \mathbf{J} = -\epsilon_0 c^2 \nabla \cdot (\mathbf{E} \times \mathbf{B}) + \epsilon_0 c^2 \mathbf{B} \cdot \nabla \times \mathbf{E} - \frac{\epsilon_0}{2} \frac{\partial}{\partial t} (\mathbf{E}^2) \tag{10.39}$$

However

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{10.40}$$

and making this substitution on the RHS of the expression for $\mathbf{E} \cdot \mathbf{J}$ we find that

$$\begin{aligned} \mathbf{E} \cdot \mathbf{J} &= -\epsilon_0 c^2 \nabla \cdot (\mathbf{E} \times \mathbf{B}) - \epsilon_0 c^2 \mathbf{B} \cdot \frac{\partial}{\partial t} (\mathbf{B}) - \frac{\epsilon_0}{2} \frac{\partial}{\partial t} (\mathbf{E}^2) \\ &= -\epsilon_0 c^2 \nabla \cdot (\mathbf{E} \times \mathbf{B}) - \frac{\epsilon_0 c^2}{2} \cdot \frac{\partial}{\partial t} (\mathbf{B}^2) - \frac{\epsilon_0}{2} \frac{\partial}{\partial t} (\mathbf{E}^2) \\ &= -\epsilon_0 c^2 \nabla \cdot (\mathbf{E} \times \mathbf{B}) - \frac{\epsilon_0}{2} \frac{\partial}{\partial t} \left(\mathbf{E}^2 + c^2 \mathbf{B}^2 \right) \end{aligned} \tag{10.41}$$

Hence if we define the vector $\mathbf{S}$ and the scalar u by writing

$$\mathbf{S} = \epsilon_0 c^2 (\mathbf{E} \times \mathbf{B}), \quad u = \frac{\epsilon_0}{2} \left(\mathbf{E}^2 + c^2 \mathbf{B}^2 \right) \tag{10.42}$$

then $\mathbf{E} \cdot \mathbf{J}$ satisfies the equation

$$\mathbf{E} \cdot \mathbf{J} = -\nabla \cdot \mathbf{S} - \frac{\partial u}{\partial t} \tag{10.43}$$

To understand the physical meaning of this result we integrate both sides over an arbitrary volume V obtaining

$$\int_V \mathbf{E}\cdot\mathbf{J}\,dV = -\int_V \nabla\cdot\mathbf{S}\,dV - \frac{\partial}{\partial t}\int_V u\,dV \tag{10.44}$$

Now u has the dimension of energy per unit volume—i.e. it has the dimensions of an *energy density*—so we write this equation as

$$-\frac{\partial}{\partial t}\int_V u\,dV = \int_S \mathbf{S}\cdot\mathbf{ds} + \int_V \mathbf{E}\cdot\mathbf{J}\,dV \tag{10.45}$$

where we also used Gauss's divergence theorem on the $\mathbf{S}$ integral. But the LHS of 10.45 is identifiable as the decrease per unit time of the *total* energy in V and $\int_V \mathbf{E}\cdot\mathbf{J}\,dV$ is the work done on any charges inside V. Hence, conservation of energy means that the term

$$\int_S \mathbf{S}\cdot\mathbf{ds} \tag{10.46}$$

must represent the flow inwards or outwards of any energy entering or leaving V across its surface.

In fact u is the energy density of the radiation of the electromagnetic field and $\mathbf{S}$ is called *Poynting's vector*. The physical interpretation of Poynting's vector [4] is that the flow of energy per unit area, per unit time, in a direction $\mathbf{n}$, where $\mathbf{n}$ is a unit vector is given by

$$\mathbf{S}\cdot\mathbf{n} \tag{10.47}$$

Note that $\mathbf{S}$ points in the direction of travel of the wave so that the flow of energy is in the right direction.

The reader should also be able to see by now that an electromagnetic wave is *a pair* of waves, one electric and one magnetic, simultaneously travelling with the velocity of light; further both waves are transverse and one is perpendicular to the other. What we have really done here—and this is really very important and a great triumph of Maxwell's equations—is to elucidate fully the electromagnetic nature of *any* light wave.

[4] Poynting's vector is also written in many experimental physics texts as

$$\mathbf{S} = \mathbf{E}\times\mathbf{H}$$

where $\mathbf{H} = \epsilon_0 c^2 \mathbf{B}$ but we cannot go into the reason for that here.

In sum then the energy properties of the waves making up **E** and **B** are

$$\begin{aligned} u &= \frac{\epsilon_0}{2}\left(\mathbf{E}^2 + c^2\mathbf{B}^2\right), \quad \text{(the energy density)} \\ &\mathbf{S}\cdot\mathbf{n}, \quad \text{(the flow of energy}/m^2/s \text{ in direction } \mathbf{n}\ (\mathbf{n}^2 = 1)) \end{aligned} \tag{10.48}$$

where $\mathbf{S} = \epsilon_0 c^2(\mathbf{E}\times\mathbf{B})$

Example *The expressions u and $\mathbf{S}$ for a simple wave*

We shall calculate here the quantities u and $\mathbf{S}$ for a simple wave. Let $\mathcal{E}$ and $\mathcal{B}$ be constants and k and ω have their usual meaning. Then consider the electromagnetic wave

$$\begin{aligned} \mathbf{E} &= \mathcal{E}\cos(kx-\omega t)\,\mathbf{j} \\ \mathbf{B} &= \mathcal{B}\cos(kx-\omega t)\,\mathbf{k} \end{aligned} \tag{10.49}$$

then for u and $\mathbf{S}$ we find the expressions

$$\begin{aligned} u &= \frac{\epsilon_0}{2}\left(\mathcal{E}^2 + c^2\mathcal{B}^2\right)\cos^2(kx-\omega t) \\ \mathbf{S} &= \epsilon_0 c^2\mathcal{E}\mathcal{B}\cos^2(kx-\omega t)\,\mathbf{i} \end{aligned} \tag{10.50}$$

As an exercise the reader should use Maxwell's equations to show that the constants $\mathcal{E}$ and $\mathcal{B}$ are related and that in fact

$$\mathcal{B} = \frac{\mathcal{E}}{c} \tag{10.51}$$

thus one of $\mathcal{E}$ or $\mathcal{B}$ can be eliminated in the expressions above for u and $\mathbf{S}$.

Example *The average energy per cycle*

Since a wave, in general, repeats itself every cycle, i.e. every $1/\nu$ seconds, it is more useful when quoting $\mathbf{S}\cdot\mathbf{n}$ to give its *average* over one compete cycle. Let us calculate this average for the wave of the previous example.

First we denote the average of $\mathbf{S}\cdot\mathbf{n}$ over one cycle by $<\mathbf{S}\cdot\mathbf{n}>$ where $<\mathbf{S}\cdot\mathbf{n}>$ is defined by

$$<\mathbf{S}\cdot\mathbf{n}> = \frac{\int_0^T \mathbf{S}\cdot\mathbf{n}\,dt}{T}, \qquad \text{where } T = \frac{1}{\nu} = \frac{2\pi}{\omega}, \quad \text{since } \omega = 2\pi\nu \tag{10.52}$$

Now let us choose, for simplicity, $\mathbf{n} = \mathbf{i}$ so that the expressions from the previous example give us

$$\begin{aligned} \mathbf{S}\cdot\mathbf{i} &= \epsilon_0 c^2\mathcal{E}\mathcal{B}\cos^2(kx-\omega t) \\ \Rightarrow <\mathbf{S}\cdot\mathbf{i}> &= \epsilon_0 c^2\mathcal{E}\mathcal{B}\frac{1}{T}\int_0^T \cos^2(kx-\omega t)\,dt, \quad T = \frac{2\pi}{\omega} \end{aligned} \tag{10.53}$$

To evaluate the integral[5] note that T is just a complete period and so

$$\begin{aligned}\frac{1}{T}\int_0^T \cos^2(kx-\omega t)\,dt &= \frac{1}{2\pi}\int_0^{2\pi}\cos^2(\theta)\,d\theta \\ &= \frac{1}{2}\end{aligned} \tag{10.56}$$

Now we can substitute for the integral in the expression for $<\mathbf{S}\cdot\mathbf{i}>$ and obtain the final result which is

$$\begin{aligned}<\mathbf{S}\cdot\mathbf{i}> &= \frac{\epsilon_0 c^2 \mathcal{E}\mathcal{B}}{2} \\ &= \frac{1}{2}\epsilon_0 c\mathcal{E}^2, \qquad \text{if we use the result } \mathcal{B} = \frac{\mathcal{E}}{c}\end{aligned} \tag{10.57}$$

Example *Poynting's vector* **S** *in practice: power dissipation in a wire*

Let us take a conducting wire of resistance R and apply a potential difference V to it producing a current I. Then we immediately have an electric field $\mathbf{E}$ inside the wire related to V by

$$V = |\mathbf{E}|L \tag{10.58}$$

[5] If the reader wants the details spelled out we give them in this footnote; however we do not require them for this course. Setting $\theta = kx - \omega t$ we find that

$$\frac{1}{T}\int_0^T \cos^2(kx-\omega t)\,dt = -\frac{1}{\omega T}\int_{kx}^{(kx-\omega T)}\cos^2(\theta)\,d\theta$$

But is easy to verify that

$$\begin{aligned}&\int \cos^2(\theta)\,d\theta = \{\sin(\theta)\cos(\theta)+\theta\} \\ \Rightarrow -\frac{1}{\omega T}&\int_{kx}^{(kx-\omega T)}\cos^2(\theta)\,d\theta = -\frac{1}{2\omega T}\left[\sin(\theta)\cos(\theta)+\theta\right]_{kx}^{(kx-\omega T)}\end{aligned} \tag{10.54}$$

and when we compute the RHS of 10.54 we obtain

$$-\frac{1}{2\omega T}\{\sin(kx-\omega T)\cos(kx-\omega T) - \sin(kx)\cos(kx) - \omega T\} = \frac{1}{2} \tag{10.55}$$

where the cos and sin terms completely cancel with one another because $\omega T = 2\pi\nu\nu^{-1} = 2\pi$ and both are periodic with period 2π.

and a magnetic field $\mathbf{B}$ which at a distance r from the wire we know is given by the expression

$$\begin{aligned} \mathbf{B} &= \frac{\mu_0 I}{2\pi r}\,\mathbf{e} \\ &= \frac{I}{2\pi\epsilon_0 c^2 r}\,\mathbf{e}, \qquad \text{using } \mu_0 = \frac{1}{\epsilon_0 c^2} \end{aligned} \tag{10.59}$$

where $\mathbf{e}$ is a unit tangent vector to the line of force of radius r. Now take a cylinder of length L and radius r that encloses the wire. Since $\mathbf{S} = \epsilon_0 c^2 \mathbf{E} \times \mathbf{B}$ is radial then the energy escaping per unit time from the wire is expressible as an integral over the curved surface S of this cylinder. If we denote this quantity by P, since it is actually the power dissipated by the wire, then we have

$$P = \int_S \mathbf{S} \cdot \hat{\mathbf{r}} = \epsilon_0 c^2 |\mathbf{E} \times \mathbf{B}| 2\pi r L, \qquad \text{since } \mathbf{E} \times \mathbf{B} \text{ is constant on } S \tag{10.60}$$

But, if $\mathbf{n}$ denotes a unit vector along the wire, then

$$\mathbf{E} = |\mathbf{E}|\mathbf{n} \tag{10.61}$$

so that, if we use the expressions for $\mathbf{E}$ and $\mathbf{B}$ and the fact that $\mathbf{E}$ is perpendicular to $\mathbf{B}$ then we can compute that

$$\begin{aligned} P &= \epsilon_0 c^2 |\mathbf{E}| \frac{I}{2\pi\epsilon_0 c^2 r} 2\pi r L \\ &= |\mathbf{E}| I L \end{aligned} \tag{10.62}$$

But $V = |\mathbf{E}|L$ so we obtain the result

$$P = VI = I^2 R, \quad \text{using } V = RI \tag{10.63}$$

Thus we see that the Poynting vector has recovered for us the expression we already calculated by a more pedestrian method, cf. 7.27.

Example *The energy density u in practice: energy storage by a capacitor*

Now we shall use the energy density

$$u = \frac{\epsilon_0}{2}\left(\mathbf{E}^2 + c^2 \mathbf{B}^2\right) \tag{10.64}$$

to calculate the energy stored in a standard parallel plate capacitor with plates of area A separated by a distance d.

First of all there is no magnetic field inside the capacitor so we have

$$u = \frac{\epsilon_0}{2}\mathbf{E}^2 \tag{10.65}$$

The energy U stored inside is got by integrating u over the internal volume of the capacitor; thus

$$U = \int_V u\,dV = \frac{\epsilon_0}{2}\mathbf{E}^2 Ad, \quad \text{since } |\mathbf{E}| \text{ is constant} \tag{10.66}$$

But we know that the voltage V across the capacitor is related to $\mathbf{E}$ by the equation

$$|\mathbf{E}| = \frac{V}{d} \tag{10.67}$$

Hence

$$U = \frac{\epsilon_0}{2}\frac{V^2}{d^2}Ad = \frac{1}{2}\frac{\epsilon_0 A}{d}V^2 \tag{10.68}$$

But we know that

$$C = \frac{A\epsilon_0}{d} \tag{10.69}$$

so we immediately have the result that

$$U = \frac{1}{2}CV^2 \tag{10.70}$$

and we are in agreement with our previous result, cf. 6.23.

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