Below is a list of important equations that we meet in our study of Electromagnetism in the MP204 module.

For your exam, you are expected to understand all of these, and to remember (or be able to derive) most of these.

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Electrostatics

- Field at $\mathbf{r}$ due to a point charge q_1 at $\mathbf{r_1}$:

$$\mathbf{E} \;=\; \frac{q_1}{4\pi\epsilon_0}\frac{\widehat{\mathbf{r}-\mathbf{r_1}}}{|\mathbf{r}-\mathbf{r_1}|^2} \;=\; \frac{q_1}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r_1}}{|\mathbf{r}-\mathbf{r_1}|^3}$$

where $\widehat{\mathbf{r}-\mathbf{r_1}}$ is a unit vector in the direction of $\mathbf{r}-\mathbf{r_1}$.

- The electric dipole moment is a vector. For a simple dipole consisting of point charges $+q$ and $-q$ at distance d apart, it has the magnitude

$$|\mathbf{p}| \;=\; qd$$

and it points from the negative charge to the positive charge.

- The electric potential:

$$\mathbf{E} \;=\; -\nabla V\;; \qquad\qquad V_{PQ} \;=\; -\int_Q^P \mathbf{E}\cdot d\mathbf{l}$$

The integral is path-independent (any path between Q and P will give the same result).

The electric potential at a point depends on the reference point. The reference point is often taken to be at infinity.

- Electric potential at $\mathbf{r}$ due to a point charge q_1 at $\mathbf{r_1}$:

$$V \;=\; \frac{q_1}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r_1}|} \qquad \left\{\begin{array}{l}\text{Reference point is taken}\\ \text{to be at infinity}\end{array}\right.$$

- Gauss' law or Gauss' dielectric flux theorem:

$$\oint_{\Sigma} \mathbf{E} \cdot d\mathbf{S} \;=\; \frac{Q_{\text{enc}}}{\epsilon_0} \qquad \begin{cases} Q_{\text{enc}} \text{ is the total charge enclosed} \\ \text{within the closed surface } \Sigma \end{cases}$$

  Note that the integral is a surface integral; I am writing it with a single integral sign and expecting that it should be clear from the context whether it is a line, surface or volume integral.

  Important to distinguish GAUSS' DIELECTRIC FLUX THEOREM of electrostatics from GAUSS' DIVERGENCE THEOREM of vector calculus.

  Gauss' law in integral form can be used to derive (using Gauss' divergence theorem) the first of Maxwell's equations ($\nabla \cdot \mathbf{E} = \rho/\epsilon_0$), which is Gauss' law in differential form.

- Poisson's law:

$$\nabla^2 V \;=\; -\frac{\rho}{\epsilon_0}$$

- Metals or good conductors in electrostatics:

  Charges sit on surface; inside the metal $E = 0$ and $V = \text{const}$

- Sphere of charge: if charge distribution is spherically symmetric, then the electric field outside is the same as that created by a point charge at the center of the sphere.

Electric Currents

- The electric current through a surface Σ is the surface integral of the current density:

$$I = \int_{\Sigma} \mathbf{J} \cdot d\mathbf{S}$$

- If current is carried by density n of carriers each having charge q and having an average velocity $\mathbf{v}$, then the current density is

$$\mathbf{J} = qn\mathbf{v} = \rho\mathbf{v}$$

  Here $\rho = qn$ is the charge density.

- Ohm's law in microscopic form:

$$\mathbf{J} = \sigma\mathbf{E} \qquad \text{or} \qquad \mathbf{E} = \rho\mathbf{J}$$

  where σ is the conductivity and ρ is the resistivity. Unfortunately, the same symbols are commonly used for surface charge density and charge density respectively, which are totally different quantities.

  The resistivity ρ is dependent on the material and the state of the material, e.g., its temperature. The equations above are the microscopic versions of the well-known circuit form of Ohm's law, $V = RI$. The resistance R depends on the material and it's state, and in addition depends on the geometry of the wire.

- The conservation of charge is encoded in the continuity equation:

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$

Magnetic Field

- Force on a charge due to magnetic field:

$$\mathbf{F} \;=\; q\mathbf{v} \times \mathbf{B} \qquad \text{(Lorentz force law)}$$

  Force due to magnetic field on an element $d\mathbf{l}$ of a current-carrying wire:

$$d\mathbf{F} \;=\; I d\mathbf{l} \times \mathbf{B}$$

  where I is the current in the wire. This equation can be derived from the Lorentz force law by regarding the carriers within the element $d\mathbf{l}$ as a point charge.

- Cyclotron motion:

  A charge q having speed v perpendicular to uniform magnetic field of strength B will undergo cyclotron motion. Equating the centripetal force with the magnetic force yields the cyclotron radius:

$$qvB \;=\; \frac{mv^2}{R} \qquad \Longrightarrow \qquad R \;=\; \frac{mv}{qB}$$

  The angular frequency of rotation is called the cyclotron frequency:

$$\omega \;=\; \frac{qB}{m}$$

- Currents create magnetic fields. According to the Biot-Savart law, the field element created at $\mathbf{r}$ due to the line elelment $d\mathbf{l}'$ at position $\mathbf{r}'$ is given by

$$d\mathbf{B} \;=\; \left(\frac{\mu_0 I}{4\pi}\right) \; \frac{d\mathbf{l}' \times \left(\widehat{\mathbf{r} - \mathbf{r}'}\right)}{|\mathbf{r} - \mathbf{r}'|^2}$$

- Ampere's law for steady-state situations:

$$\int_C \mathbf{B} \cdot d\mathbf{l} \;=\; \mu_0 I_{\text{enc.}}$$

  where $I_{\text{enc.}}$ is the current flowing through any surface enclosed by the closed curve C.

- Using Stokes' theorem, Ampere's law can be put into differential form:

$$\nabla \times \mathbf{B} \;=\; \mu_0 \mathbf{J}$$

  In time-dependent situations, Ampere's law needs to be corrected (Maxwell's correction).

- Magnetic field due to infinite straight wire carrying current I has magnitude

$$B \;=\; \frac{\mu_0 I}{2\pi d}$$

  at a distancce d from the wire. The field direction 'curls' around the wire: please learn a right-hand rule to determine the direction of the field.

  The equation above can be derived using the Biot-Savart law (integration required). It can also be derived using Ampere's law (much easier).

- Magnetic field due to a circular loop of wire carrying current I has magnitude

$$B \;=\; \frac{\mu_0 I}{2R}$$

  where R is the radius of the loop. The field is perpendicular to the plane of the loop.

  A current-carrying loop functions as a magnetic dipole. The dipole moment has magnitude

$$|\mathbf{m}| \;=\; I \times (\text{area of loop})$$

Electromagnetic induction; Faraday's law

- The 'electromotive force' (EMF or e.m.f.) is not a force; it has the same units as a potential difference or a voltage.

- EMF generated by a changing magnetic flux:

$$\mathcal{E} \;=\; -\frac{d\Phi_B}{dt} \qquad \text{(Faraday's law)}$$

  The magnetic flux can be changing because of a change in the magnetic field, or because of a change in location or shape or orientation of the wire loop.

- Faraday's law can be written as

$$\oint_C \mathbf{E}\cdot d\mathbf{l} \;=\; -\frac{d}{dt}\Phi_B \;=\; -\frac{d}{dt}\int_\sigma \mathbf{B}\cdot d\mathbf{S}$$

  for any closed curve C, even when there is no wire loop present along this curve. Here Σ is any surface which has C as boundary.

- Lenz's law gives the direction of induced EMF. It corresponds to the minus sign of Faraday's law. According to Lenz's law, the induced current creates a magnetic field which opposes the change of flux responsible for the induction.

- Using Stokes' theorem, Ampere's law can be put into differential form:

$$\nabla\times\mathbf{E} \;=\; -\frac{\partial\mathbf{B}}{\partial t}$$

  This is Maxwell's third equation.

Scalar and Vector Potentials

- Given the scalar potential V and the vector potential $\mathbf{A}$, the fields are

$$\mathbf{E} \;=\; -\nabla V \;-\; \frac{\partial \mathbf{A}}{\partial t} \qquad \text{and} \qquad \mathbf{B} \;=\; \nabla \times \mathbf{A}$$

- In electrostatics, $\mathbf{E} = -\nabla V$. A correction is required for time-dependent situations, as above.

  The relation $\mathbf{B} \;=\; \nabla \times \mathbf{A}$ is correct both for magnetostatics and in dynamic situations.

- The potentials are constructed to be consistent with the two of Maxwell's equations which do not contain charges or currents and only involves fields $\mathbf{E}$ and $\mathbf{B}$, i.e.,

  the 2nd equation $\nabla \cdot \mathbf{B} = 0$ and the 3rd equation $\nabla \times \mathbf{E} + \dfrac{\partial \mathbf{B}}{\partial t} = 0.$

  Thus, if fields are calculated from V and $\mathbf{A}$, they will automatically satisfy the second and third equations.

- Fields are physical. Potentials are not.

  There is considerable freedom in choosing the potentials: different potentials might give the same physical fields. This is called gauge freedom. For any scalar function $\lambda(\mathbf{r})$, the pair

$$V \;-\; \frac{\partial \lambda}{\partial t}\,, \qquad \mathbf{A} \;+\; \nabla \lambda$$

  produces the same fields as the pair V, $\mathbf{A}$. Thus one has an infinite number of potential pairs corresponding to the same physical fields.

  In other words, the gauge transformation

$$\mathbf{A} \longrightarrow \mathbf{A} \;+\; \nabla \lambda\,, \qquad V \longrightarrow V \;-\; \frac{\partial \lambda}{\partial t}$$

  leaves the fields $(\mathbf{E}, \mathbf{B})$ invariant.

- Coulomb gauge: $\quad \nabla \cdot \mathbf{A} \;=\; 0.$

- Lorentz gauge: $\quad \nabla \cdot \mathbf{A} \;=\; -\,\mu_0 \epsilon_0 \dfrac{\partial V}{\partial t}.$

Maxwell's Equations + Summarizing Electromagnetism

- Maxwell's Equations:

$$\nabla \cdot \mathbf{E} \;=\; \frac{\rho}{\epsilon_0}$$
$$\nabla \cdot \mathbf{B} \;=\; 0$$
$$\nabla \times \mathbf{E} \;=\; -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{B} \;=\; \mu_0 \mathbf{J} \;+\; \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \;=\; \mu_0 \left(\mathbf{J} \;+\; \mathbf{J_D} \right)$$

  Here $\mathbf{J_D} \;=\; \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}$ is the displacement current density.

- Speed of light:

$$\mu_0 \epsilon_0 \;=\; \frac{1}{c^2} \qquad c \;=\; \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

- The continuity equation (conservation of charge):

$$\nabla \cdot \mathbf{J} \;+\; \frac{\partial \rho}{\partial t} \;=\; 0$$

- Force on a charge:

$$\mathbf{F} \;=\; q\mathbf{E} \;+\; q\mathbf{v} \times \mathbf{B}$$

- Fields from potentials:

$$\mathbf{E} \;=\; -\nabla V \;-\; \frac{\partial \mathbf{A}}{\partial t} \quad ; \qquad \mathbf{B} \;=\; \nabla \times \mathbf{A}$$

Electromagnetic waves; Energy

- Maxwell's equations in free space ($\rho = 0$, $\mathbf{J} = 0$) can be used to show
$$\nabla^2 \mathbf{E} \;=\; \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}\;; \qquad \nabla^2 \mathbf{B} \;=\; \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2}$$
i.e., the fields satisfy wave equations, and hence must have wave solutions.

- An example solution:
$$\mathbf{E} \;=\; \hat{\jmath} E_0 \sin(x - ct)\;; \qquad \mathbf{B} \;=\; \hat{\jmath} B_0 \sin(x - ct)\;; \qquad B_0 = E_0/c$$
The two fields are perpendicular to each other and the wave propagates in the direction $\mathbf{E} \times \mathbf{B}$. The amplitudes E_0 and B_0 are not independent but related as $E_0 = cB_0$.

- Traveling waves in opposite directions can be combined to form *standing waves.*

- The energy density due to electromagnetic fields is
$$u \;=\; \frac{1}{2}\epsilon_0 E^2 \;+\; \frac{1}{2\mu_0} B^2$$

- The Poynting vector
$$\mathbf{S} \;=\; \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$$
serves as an energy current density.

- In free space, conservation of electromagnetic energy is encoded in an analog of the continuity equation:
$$\nabla \cdot \mathbf{S} \;+\; \frac{\partial u}{\partial t} \;=\; 0$$
In the presence of currents this is modified to
$$\nabla \cdot \mathbf{S} \;+\; \frac{\partial u}{\partial t} \;+\; \mathbf{E} \cdot \mathbf{J} \;=\; 0$$