* Maxwell's eqns, differential form, SI units

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}, \quad \vec{\nabla}\cdot\vec{B} = 0, \quad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

Maxwell didn't have $\vec{\nabla}$-notation. (8 equations)

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$\vec{J}_D = \epsilon_0\frac{\partial\vec{E}}{\partial t}$, DISPL. CURRENT DENSITY

In Gaussian (CGS) units: $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$, $\vec{\nabla}\cdot\vec{B} = 0$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}, \quad \vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J}$$

→ highlights symmetry bet^n $\vec{E}$ & $\vec{B}$

→ choice of units is a matter of tradition/ convention / (in)convenience

→ $\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$ comes from CONTINUITY EQ. (Maxwell's correction)

{ $\epsilon_0\frac{\partial\vec{E}}{\partial t}$ is called DISPLACEMENT CURRENT ~~DENSITY~~ }

* Maxwell's eqns in integral form

$$\oint_{\Sigma}\vec{E}\cdot d\vec{S} = \frac{1}{\epsilon_0}Q_{encl.} = \frac{1}{\epsilon_0}\int_V \rho\, d^3r$$

[closed surface Σ encloses volume V]

$$\oint_{\Sigma}\vec{B}\cdot d\vec{S} = 0$$

$$\oint_C \vec{E}\cdot d\vec{l} = \int_{\Sigma'}\left(-\frac{\partial\vec{B}}{\partial t}\right)\cdot d\vec{S} = -\frac{d}{dt}\int_{\Sigma'}\vec{B}\cdot d\vec{S}$$

[closed path C encloses surface Σ']

G2

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I_{encl.} + \mu_0\epsilon_0 \int_{\Sigma'} \left(\frac{\partial \vec{E}}{\partial t}\right)\cdot d\vec{S}$$

$$= \mu_0 \int_{\Sigma'} \vec{J}\cdot d\vec{S} + \mu_0\epsilon_0 \int_{\Sigma'} \left(\frac{\partial \vec{E}}{\partial t}\right) d\vec{S}$$

$$= \mu_0 \int_{\Sigma'} \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)\cdot d\vec{S} = \mu_0 \int_{\Sigma'} \left(\vec{J} + \vec{J}_D\right)\cdot d\vec{S}$$

→ Deriving differential forms from integral forms: use Gauss's divergence theorem or Stoke's theorem

$$\oint_{\Sigma} \vec{V}\cdot d\vec{S} = \int_{V} \left(\vec{\nabla}\cdot\vec{V}\right) d^3r$$

$$\oint_C \vec{V}\cdot d\vec{l} = \int_{\Sigma'} \left(\vec{\nabla}\times\vec{V}\right)\cdot d\vec{S}$$

* $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \left(\vec{J} + \vec{J}_D \right)$

$\vec{J}_D = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ is called the <u>DISPLACEMENT CURRENT</u> DENSITY (as opposed to $\vec{J}$ = conduction current density)

↑
Historical name, from Maxwell

* SCALAR & VECTOR POTENTIALS

We learned in electrostatics: $\vec{E} = -\frac{\partial V}{\partial t}$

because $\vec{\nabla} \times \vec{E} = 0$,

In magnetostatics: $\vec{B} = \vec{\nabla} \times \vec{A}$ because $\vec{\nabla} \cdot \vec{B} = 0$

Now we are doing electrodynamics.

$\vec{\nabla} \cdot \vec{B} = 0$ is still true $\Rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$

But $\vec{\nabla} \times \vec{E} \neq 0$! Let's correct $\vec{E} = -\frac{\partial V}{\partial t}$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{A}\right) = -\vec{\nabla} \times \left(\frac{\partial \vec{A}}{\partial t}\right)$$

$$\Rightarrow \vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0$$

$\Rightarrow$ We can define $\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} V$

Thus
$$\boxed{\begin{aligned} \vec{E} &= -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \\ \vec{B} &= \vec{\nabla} \times \vec{A} \end{aligned}}$$

* GAUGE TRANSFORMATIONS

Certain Transformations ~~keep~~ of $(V, \vec{A})$ keep the physical fields $(\vec{E}, \vec{B})$ unchanged (invariant).

For any scalar field $\lambda(\vec{r}, t)$, the replacement

$$\boxed{\begin{aligned} \vec{A} &\longrightarrow \vec{A} + \vec{\nabla}\lambda \\ V &\longrightarrow V - \frac{\partial \lambda}{\partial t} \end{aligned}}$$

leave the fields $(\vec{E}, \vec{B})$ invariant

$\boxed{\text{Exercise : Show}}$

CULTURE : In the Special Theory of Relativity, Electricity & Magnetism are Unified even further.
$(V, \vec{A})$ is combined to a single object (a 4-vector)
$(\rho, \vec{J})$ is " " " " 4-vector
$(\vec{E}, \vec{B})$ is combined to a single 4-tensor.

* Maxwell's Eqs. in terms of Potentials:

② & ③ automatically satisfied:

①: $$\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\frac{\rho}{\epsilon_0}$$

④: $$\left(\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\right) - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\vec{J}$$

Used $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$

$\boxed{\text{Exercise : Show}}$

* Coulomb gauge Pick $\vec{\nabla}\cdot\vec{A} = 0$

Maxwell's Eqs. become:

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}, \quad \left(\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\right) - \mu_0\epsilon_0\vec{\nabla}\left(\frac{\partial V}{\partial t}\right) = -\mu_0\vec{J}$$

(Eq. for $\vec{A}$ not simple, if not in static situation.)

* Lorentz gauge $\vec{\nabla}\cdot\vec{A} = -\mu_0\epsilon_0\frac{\partial V}{\partial t}$

(same as Coulomb gauge in case of statics)

Maxwell's Eqs.:
$$\left.\begin{aligned}\nabla^2 V - \mu_0\epsilon_0\frac{\partial^2 V}{\partial t^2} &= -\frac{\rho}{\epsilon_0}\\ \nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} &= -\mu_0\vec{J}\end{aligned}\right\}\ \text{symmetric treatment}$$

6

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0}, \quad \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) = \Box^2,$$ the d'Alembertian operator

$$\Box^2 V = -\frac{\rho}{\epsilon_0}, \quad \Box^2 \vec{A} = -\mu_0 \vec{J}$$

---

* The Displacement Current

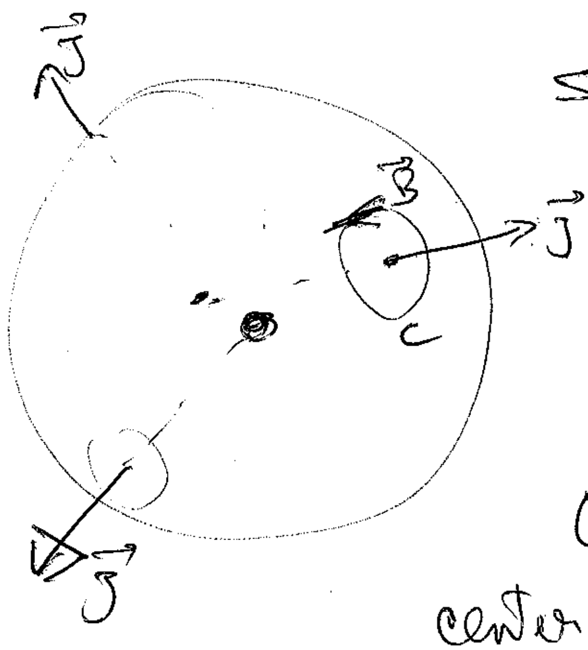
(Maxwell's new term)

$$\vec{J}_D = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

corrects $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$ (Ampere's law)

$\rightarrow \vec{\nabla}\times\vec{B} = \mu_0(\vec{J}+\vec{J}_D)$

Example 1



Imagine charge being emitted, spherically symmetrically, from a source.

(Could be a radioactive source)

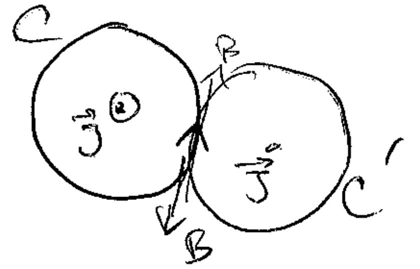
Consider distance R from center of source, imagine sphere.

Using Ampere's law, $\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I_{enc.}$,

At this distance, $J(R) = \frac{1}{4\pi R^2}\frac{dQ}{dt}$

we predict $\vec{B}$-field along C.

$\Rightarrow$ But this breaks symmetry. E.g., choosing C', you find $\vec{B}$ in opposite direction! $\Rightarrow$ $\vec{B}$ must be zero.



Q. Current does not produce $\vec{B}$? $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$ violated?

A. In this case, $\vec{J}$ is balanced by $\vec{J}_D = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$. The charge inside sphere creates field $E = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}$ (Gauss's law)

$$\Rightarrow J_D = \epsilon_0 \frac{\partial E}{\partial t} = \frac{\epsilon_0}{4\pi\epsilon_0 R^2}\frac{dQ}{dt} = \frac{1}{4\pi R^2}\left(-4\pi R^2 J\right)$$

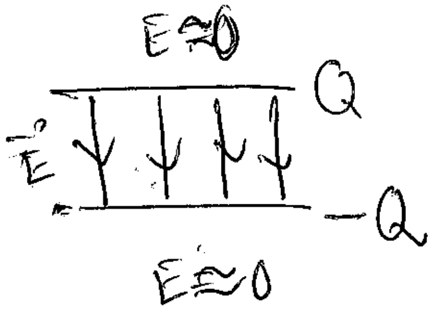
$$= -J$$

so that $\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \vec{J}_D\right) = 0$

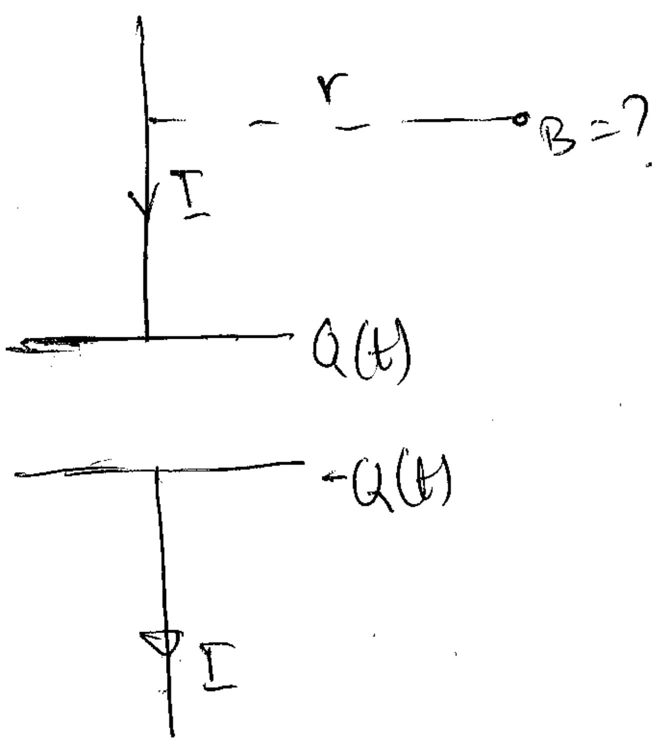
optional

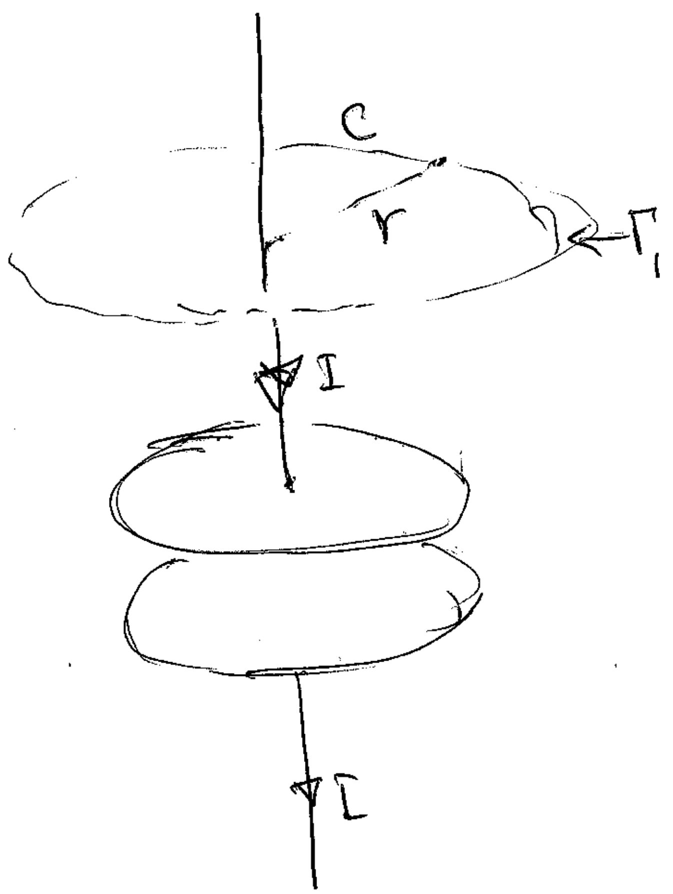
Look up Feynman chapter 18

Example 2 Charging a parallel-plate capacitor (a.k.a. condenser)



capacitor

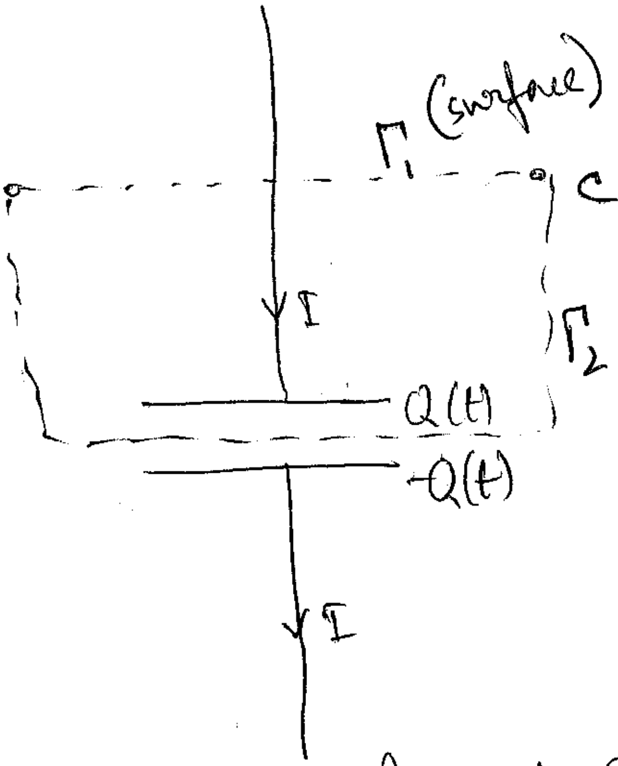




Ampere's law gives

$$\oint_C \vec{B}\cdot d\vec{l} = B.2\pi r = \mu_0 I \quad \Rightarrow \quad B = \frac{\mu_0 I}{2\pi r}$$

I = current thru Γ_1



Surface Γ_2 also has C as boundary. No current through $\Gamma_2 \Rightarrow$ Ampere's law gives zero field at same point P.

Ampere: $\oint_C \vec{B}\cdot d\vec{l} = B\cdot 2\pi r = \int_{\Gamma_2} \vec{J}\cdot d\vec{S} = I_{2,\text{enc}} = 0$

Solution: Maxwell's term. If considering Γ_2 the displacement current generates B-field.

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \int_{\Gamma_2} \vec{J}\cdot d\vec{S} + \mu_0 \epsilon_0 \int_{\Gamma_i} \frac{\partial \vec{E}}{\partial t}\cdot d\vec{S} \qquad \left(\Gamma_i = \Gamma_1 \text{ or } \Gamma_2\right)$$

If surface is Γ_1,

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I + 0 \Rightarrow B = \frac{\mu_0 I}{2\pi r}$$

If surface is Γ_2,

$$\oint \vec{B}\cdot d\vec{l} = 0 + \mu_0\epsilon_0 \frac{d}{dt}\int_{\Gamma_2} \vec{E}\cdot d\vec{S}$$

$$= \mu_0\epsilon_0 \frac{d}{dt}\left(\frac{Q}{\epsilon_0}\right) \quad \text{Gauss's law}$$

$$= \mu_0 \frac{\partial Q}{dt} = \mu_0 I$$

$\Rightarrow$ Same answer, $B = \dfrac{\mu_0 I}{2\pi r}$, whichever surface is used.

★ Maxwell's Eqs in Free Space/Vacuum

$\rightarrow$ Electromagnetic waves

$\boxed{\rho=0,\ \vec{J}=0 \ \text{(no charge or current)}}$ (wave solution of Max. Eqs.)

$$\vec{\nabla}\cdot\vec{E} = 0, \quad \vec{\nabla}\cdot\vec{B} = 0, \quad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{dt},$$

$$\vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\frac{\partial\vec{E}}{dt} = \frac{1}{c^2}\frac{\partial\vec{E}}{dt}$$

The curl equations, indicate, $\vec{E}$ & $\vec{B}$ fields could "keep going" without charges or currents, once initiated.

More symmetric:

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial(c\vec{B})}{dt}$$

$$\vec{\nabla}\times(c\vec{B}) = \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

in Gaussian units $c\vec{B}$ is called $\vec{B}$.

* WAVE EQUATIONS FROM MAXWELL'S EQUATIONS

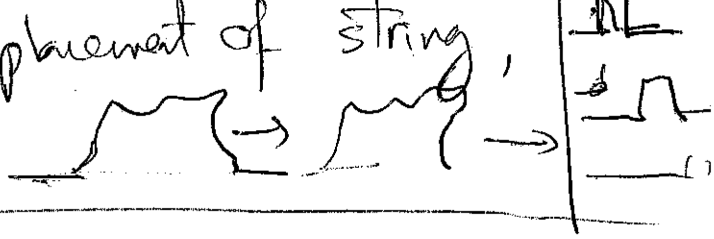
Reminder: — $\frac{\partial^2 f}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0$

is a "wave equation". Any function of the form $f(x,t) = \phi(x - vt)$ or $\phi(x + vt)$ is a solution. are traveling wave solutions. E.g., traveling with speed v.

$$f(x,t) = f_0 \sin(x \pm vt), \quad f(x,t) = f_0 \cos(x \pm vt)$$

are solutions of the wave equation.

f could be: displacement of string, water surface, etc.



* Now, wave equations for $\vec{E}$ & $\vec{B}$:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$\Rightarrow \underbrace{\vec{\nabla}(\vec{\nabla} \cdot \vec{E})}_{=0} - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}\underbrace{(\vec{\nabla} \times \vec{B})}_{= +\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}}$$

$$\Rightarrow \boxed{\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}}$$

This is a 3D version of the wave equation:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$ (propagation can be in any direction)

The object propagating is a vector.

$\vec{B}$-field obeys same equation in free space:

$$\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\times\left(\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}\right)$$

$$\Rightarrow \vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{B}}_{=0}) - \nabla^2\vec{B} = \frac{1}{c^2}\frac{\partial}{\partial t}(\underbrace{\vec{\nabla}\times\vec{E}}_{= -\frac{\partial \vec{B}}{\partial t}})$$

$$\Rightarrow \boxed{\nabla^2\vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}}$$

* Thus $\vec{E}$ and $\vec{B}$ in free space satisfy

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0, \quad \nabla^2\vec{B} - \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

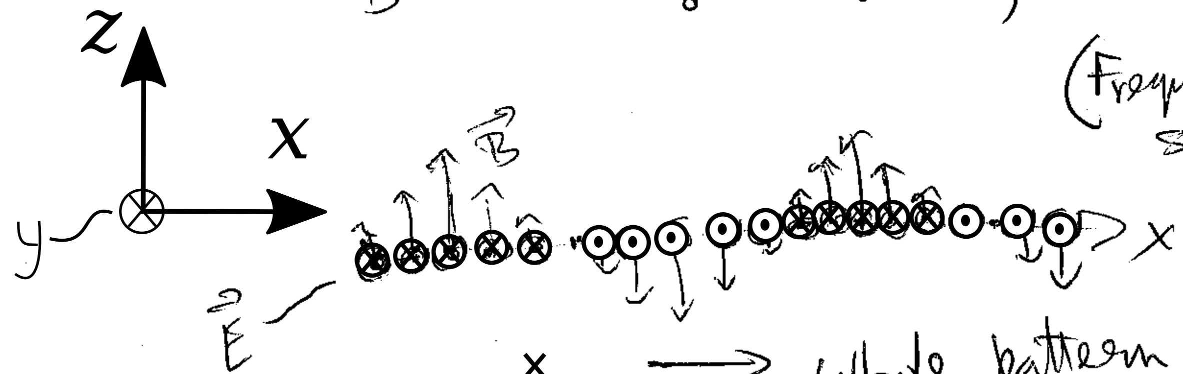
* In Lorentz gauge, potentials in free space ($\rho = 0$, $\vec{J} = 0$) also satisfy

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = 0, \quad \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = 0$$

* An example solution:

$$\vec{E} = \hat{j}\, E_0 \sin(x - ct)$$

$$\vec{B} = \hat{k}\, B_0 \sin(x - ct)$$



(Frequency/wavelength specially chosen)

$x \longrightarrow$ whole pattern moves forward with speed c.

The form above is a solution of Maxwell's eqs: $\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\vec{B} = 0$

$$\vec{\nabla}\times\vec{E} = \hat{k}\, \frac{\partial E_y}{\partial x} = \hat{k}\, E_0 \cos(x - ct)$$

$$\frac{\partial \vec{B}}{\partial t} = -\hat{k}\, cB_0 \cos(x - ct)$$

only if $E_0 = cB_0$.

$$\vec{E} = \hat{j}\, cB_0 \sin(x - ct)$$

$$\vec{B} = \hat{k}\, B_0 \sin(x - ct)$$

(74) In our solution,

$\vec{E} \perp \vec{B}$, both are perpendicular to direction of propagation.

$\Rightarrow$ general ~~[illegible]~~ features of traveling E.M. waves (can be shown).

* Field pattern travels with speed c.

$c =$ speed of light

$\Rightarrow$ Light is an electromagnetic wave.

~~[illegible]~~

The E.M. wave has a FREQUENCY f and a wavelength $\lambda = \frac{c}{f}$

Generalize our simple solution!

$$\vec{E} = \hat{j}\, cB_0 \sin\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{\lambda}ct\right)$$

$$= \hat{j}\, cB_0 \sin\left(\frac{2\pi}{\lambda}(x - ct)\right) = \hat{j}\, cB_0 \sin(kx - \omega t)$$

$$\vec{B} = \hat{k}\, B_0 \sin\left(\frac{2\pi}{\lambda}(x - ct)\right) = \hat{k}\, B_0 \sin(kx - \omega t)$$

* Depending on frequency, an electromagnetic wave can be a

γ-ray, X-ray, ultraviolet wave, visible light, infrared wave, microwave, radio wave

---

* Spatial form of a traveling wave does not have to be oscillatory. Example of localized traveling pulse :

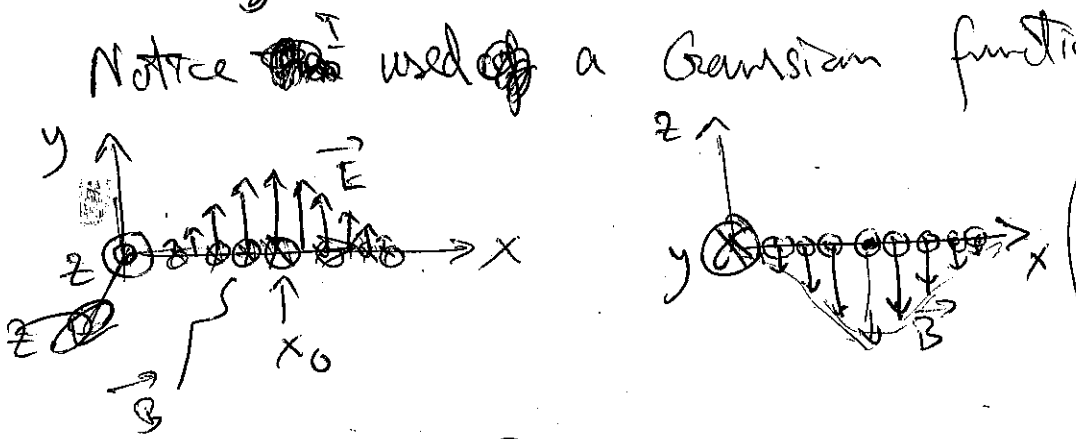
$$\vec{E} = E_0 e^{-\left(\frac{x+ct}{\lambda_0}\right)^2} \hat{j}$$

$$\vec{B} = -\left(\frac{E_0}{c}\right) e^{-\left(\frac{x+ct}{\lambda_0}\right)^2} \hat{k}$$

Traveling in -ve x direction. Notice relative minus sign.

* Satisfies Maxwell's equations. (check)

Notice I used a Gaussian function.



Snapshot at $ct = -x_0$

Blob of $\vec{E}$ & $\vec{B}$ fields travels leftward with speed c.

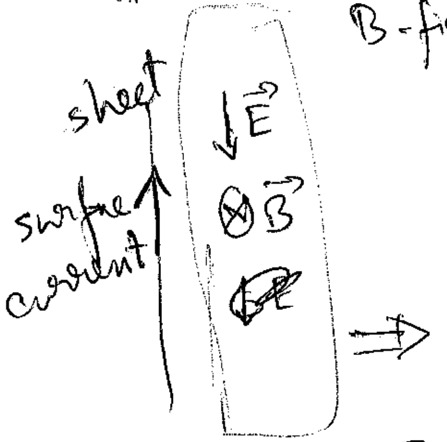
* Another example : Feynman lectures II, chs. 18 & 20

Sheet of current, turned rapidly on. ~~and off~~

Creates $\vec{B}$-field, (increasing, larger nearer the sheet)

$\vec{B}$-field creates $\vec{E}$-field.

And so on....



$\vec{E}$ & $\vec{B}$ fields travel off.

⟶ ACCELERATING CHARGES CREATE E.M. radiation.

* STANDING WAVES

* Maxwell's Eqs. in empty space are LINEAR.

⇒ Sum of any two solutions is also a solution.

* $\vec{E}_1 = \hat{j}\, E_0 \sin\left(\frac{2\pi}{\lambda}(x-ct)\right)$

$\vec{B}_1 = \hat{k}\left(E_0/c\right) \sin\left(\frac{2\pi}{\lambda}(x-ct)\right)$

Travels rightward

* $\vec{E}_2 = \hat{j}\, E_0 \sin\left(\frac{2\pi}{\lambda}(x+ct)\right)$

$\vec{B}_2 = (-)\hat{k}\left(E_0/c\right) \sin\left(\frac{2\pi}{\lambda}(x+ct)\right)$

Travels leftward

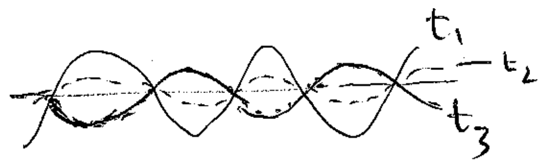
The sum will also be a solution:

$$\vec{E} = \vec{E}_1 + \vec{E}_2 , \quad \vec{B} = \vec{B}_1 + \vec{B}_2$$

Use
$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2 \sin\alpha \cos\beta$$
$$\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2 \cos\alpha \sin\beta$$

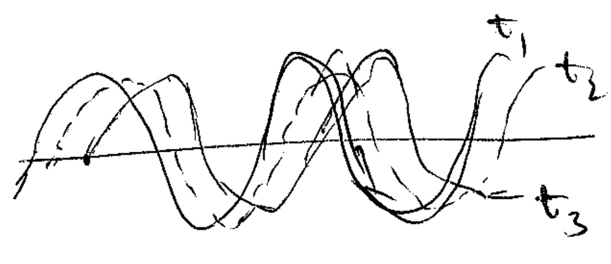
$$\Rightarrow \quad \vec{E} = 2 \hat{j}\, E_0 \sin\left(\frac{2\pi x}{\lambda}\right) \cos\left(\frac{2\pi c}{\lambda} t\right)$$
$$\vec{B} = -2 \hat{k} \left(E_0/c\right) \cos\left(\frac{2\pi x}{\lambda}\right) \sin\left(\frac{2\pi c}{\lambda} t\right)$$

Standing



$t_1 < t_2 < t_3$

Traveling



$t_1 < t_2 < t_3$

* MORE ON EM WAVES, p. 80a

INTERLUDE

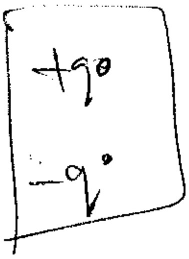
Electric Dipoles & Magnetic Dipoles

Electric "monopole" = Electric charge.

Electric dipole $\simeq$ ~~separated~~
configuration with separated positive & negative charges.

Simplest example: two point charges $+q$ & $-q$, separated by distance d.

Dipole moment: $|\vec{p}| = qd$, direction = $-$ve to $+$ve.

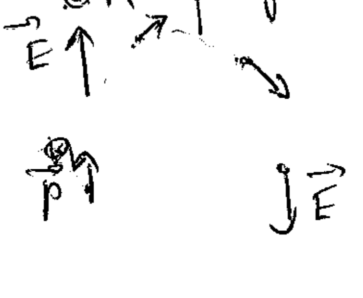


Field far away:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left(\frac{3(\vec{p}\cdot\vec{r})\vec{r}}{r^5} - \frac{\vec{p}}{r^3}\right)$$

On dipole axis: $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{2\vec{p}}{r^3}$ $\Big\{$ WORK OUT explicitly

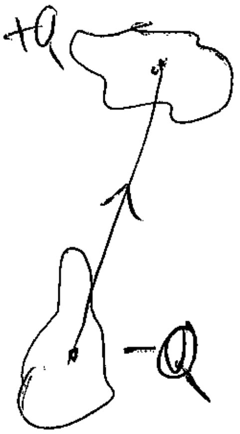
On perpendicular direction: $(\vec{p}\perp\vec{r})$:



$$\vec{E} = -\frac{1}{4\pi\epsilon_0}\frac{\vec{p}}{r^3}$$

At moderate distances, $\vec{E}$ will deviate from far-field expression.

→ TORQUE on dipole due to $\vec{E}$-field: $\vec{\tau} = \vec{p} \times \vec{E}$



More involved electric dipole.

$$p = Qd$$

d = distance between "centres" of two charge clusters.

* Electric quadrupole: $\boxed{\begin{matrix} +q & -q \\ -q & +q \end{matrix}}$ OR $\boxed{\begin{matrix} +q \\ -q \\ +q \\ -q \end{matrix}}$

Quadrupole moment IS a TENSOR

Long-distance electric fields:

$$E \propto \frac{1}{r^2} \text{ (monopole)}, \quad E \propto \frac{1}{r^3} \text{ (dipole)}$$

$$E \propto \frac{1}{r^4} \text{ (quadrupole)}, \ \ldots \text{ etc.}$$

r is distance to charge distn

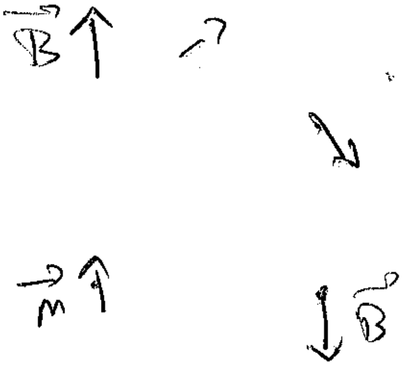
→ $V(\vec{r}) \propto \frac{1}{r}$ (monopole) ... Expanding $V(r)$ in powers of r! $\boxed{\text{MULTIPOLE EXPANSION}}$

$$V(r) = \frac{\alpha_1}{r} + \frac{\alpha_2}{r^2} + \frac{\alpha_3}{r^3} + \cdots$$

* Magnetic dipole

Object that creates far magnetic field:

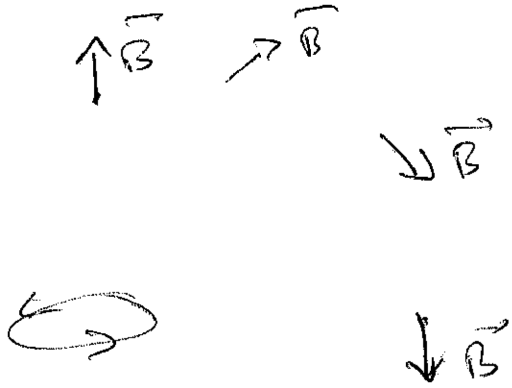
$$\vec{B} = \frac{\mu_0}{4\pi}\left(\frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^5} - \frac{\vec{m}}{r^3}\right)$$



Similar profile of fields.

* Simplest example: current loop.

$|\vec{m}|$ = current $\times$ area of loop



* Reminiscent of field from a permanent magnet. $\Rightarrow$ permanent magnets ~~consist~~ contain current loops at atomic level.

End Interlude

E.M. waves continued

$$\vec{E} = E_0 \sin(k\cancel{z} - \omega t)\,\hat{i}$$
$$\vec{B} = \left(E_0/c\right) \sin(k\cancel{z} - \omega t)\,\hat{j}$$

wave propagating in $\hat{k}$ direction (direction of $\vec{E} \times \vec{B}$)

$k = \frac{2\pi}{\lambda}$ is the wavenumber

$\omega = 2\pi f$ is the angular frequency

$$k = \frac{2\pi}{c/\cancel{f}} = \frac{\omega}{c} \quad , \quad \lambda f = c$$

λ = wavelength
f = frequency

$$\frac{k}{\omega} = c$$

$$\vec{E} = E_0 \sin\left(\frac{\omega}{c}\cancel{z} - \omega t\right)\hat{i}$$
$$\vec{B} = \left(\frac{E_0}{c}\right) \sin\left(\frac{\omega}{c}\cancel{z} - \omega t\right)\hat{j}$$

* Sine or cosine? only difference in

PHASE!

$$\sin\left(u + \frac{\pi}{2}\right) = \cos u$$

$$\sin\left(k\cancel{z} - \omega t + \frac{\pi}{2}\right) = \cos(k\cancel{z} - \omega t)$$

$\rightarrow$ a shift of position / time.

80b

* $\vec{E}$, $\vec{B}$ and propagation direction are perpendicular.

→ EM waves are TRANSVERSE waves.

Contrast: sound waves are LONGITUDINAL waves.

* If wave travels in x-direction, $\vec{E}$-field can be in either y- or z-direction → POLARIZATION.

If both transverse directions are present, wave is said to be UNPOLARIZED.

Ex. Sunlight is unpolarized.

A "polarizer" cuts off one of the transverse directions.

ENERGY & ENERGY FLOW IN ELECTRO~~DYNAMICS~~ MAGNETISM

Energy density due to an $\vec{E}$-field : $\frac{1}{2}\epsilon_0 E^2$

Due to $\vec{B}$-field : $\cancel{\frac{1}{2}\frac{1}{\mu_0}B^2}$ $\frac{1}{2\mu_0}B^2$

$$u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2 = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)$$

(Often derived in the setting of CAPACITORS and INDUCTORS. We skipped these topics.)

Fields carry their own energy, even in free space! So EM waves should transport energy on its own.

Define Poynting vector:

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \epsilon_0 c^2\, \vec{E}\times\vec{B}$$

$\Rightarrow$ Acts as a "current" of energy.

In free space, can derive from Maxwell's eqs.:

$$\boxed{\vec{\nabla}\cdot\vec{S} + \frac{\partial u}{\partial t} = 0}$$

(82)

→ Continuity equation for energy of EM field in free space

Derivation from Maxwell's equations. →

problem set 4

* In the presence of currents (not in free space),

$$\boxed{\vec{\nabla}\cdot\vec{S} + \frac{\partial u}{\partial t} + \vec{E}\cdot\vec{J} = 0}$$

Use the identity $\vec{\nabla}\cdot(\vec{C}\times\vec{D}) = \vec{D}\cdot(\vec{\nabla}\times\vec{C}) - \vec{C}\cdot(\vec{\nabla}\times\vec{D})$

OPTIONAL

* Proof: $\vec{\nabla}\cdot\vec{S} = \frac{1}{\mu_0}\vec{\nabla}\cdot(\vec{E}\times\vec{B})$

$$= \frac{1}{\mu_0}\left[\vec{B}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{B})\right]$$

$$= \frac{1}{\mu_0}\left[\vec{B}\cdot\left(-\frac{\partial\vec{B}}{\partial t}\right) - \vec{E}\cdot\left\{\mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}\right\}\right]$$

$$= -\frac{1}{\mu_0}\left(\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}\right) - \vec{E}\cdot\vec{J} - \epsilon_0\left(\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}\right)$$

$$= -\frac{1}{\mu_0}\frac{1}{2}\frac{\partial}{\partial t}(B^2) - \frac{\epsilon_0}{2}\frac{\partial}{\partial t}(E^2) - \vec{E}\cdot\vec{J}$$

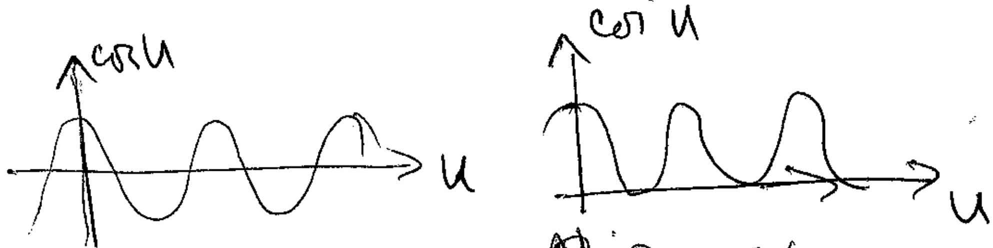
$$= -\frac{\partial u}{\partial t} - \vec{E}\cdot\vec{J}$$

* ENERGY in TRAVELING WAVES

$$\vec{E} = E_0 \cos\left(\frac{\omega}{c}z - \omega t\right)\hat{i}, \quad \vec{B} = \left(\frac{E_0}{c}\right)\cos\left(\frac{\omega}{c}z - \omega t\right)\hat{j}$$

Propagation in $\hat{k}$ direction.

$$\vec{S} = \frac{1}{\mu_0}\frac{E_0^2}{c}\cos^2\left(\frac{\omega}{c}z - \omega t\right)\hat{k}$$



SIMILAR for sin u & $\sin^2 u$ → just shifted

$\cos^2 u$ is ALWAYS POSITIVE

$\vec{S}$ is oscillatory but always points in direction of $\hat{k}$: direction of propagation of wave.

⇒ Traveling Waves carry energy in definite direction.

* ENERGY in STANDING WAVES

$\vec{S}$ oscillates direction. (cos or sin, not $\cos^2$ or $\sin^2$). Energy oscillates locally but does not propagate.