* Remember similar derivation of $\vec{\nabla}\cdot\vec{E} = 0$ from Gauss's flux theorem, using Gauss's divergence theorem.

* Magnetostatics in differential form	Electrostatics in differential form
$\vec{\nabla}\cdot\vec{B} = 0$	$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$
$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ [will be corrected]	$\vec{\nabla}\times\vec{E} = 0$ [will be corrected]

corrected forms will be Maxwell-III & Maxwell-IV

REVIEW

	Charges & steady currents	Field acts on charge	How fields are created	Integral theorems
Electro-statics	$Q = \int \rho\, dV = \int \rho\, d^3r$ $Q = \int \sigma\, dS = \int \sigma\, da$ $Q = \int \lambda\, dl$	$\vec{F} = q\vec{E}$	$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{(\widehat{\vec{r}-\vec{r}'})}{\lvert\vec{r}-\vec{r}'\rvert^2}$ NO NAME (= Coulomb's law) + superposition princip.	GAUSS's theorem (Dielectric flux thm) $\oint \vec{E}\cdot d\vec{S} = \frac{1}{\epsilon_0} Q_{enc.}$
Magneto-statics	$I = \int \vec{J}\cdot d\vec{S}$	$\vec{F} = q\vec{v}\times\vec{B}$ $d\vec{F} = I\, d\vec{l}\times\vec{B}$	$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l}\times(\widehat{\vec{r}-\vec{r}'})}{\lvert\vec{r}-\vec{r}'\rvert^2}$ Biot-Savart + superposition princip.	Ampere's theorem $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc.}$

	DIV.	CURL	Potential	
Electrostatics	$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$ (GAUSS)	$\vec{\nabla}\times\vec{E} = 0$	$\vec{E} = -\vec{\nabla} V$ (possible because $\vec{\nabla}\times\vec{E} = 0$ ⇒ will need correction)	$V = -\int_{\vec{r}_0}^{\vec{r}} \vec{E}\cdot d\vec{r}$ Depends on $\vec{r}_0$!
Magnetostatics	$\vec{\nabla}\cdot\vec{B} = 0$	$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ (AMPERE) ⇓ will need time-dependent correction	$\vec{B} = \vec{\nabla}\times\vec{A}$	$\vec{A}$ depends on "gauge" choice

possible because $\vec{\nabla}\cdot\vec{B} = 0$

V = electric potential / scalar potential

$\vec{A}$ = vector potential

* We also learned: $\vec{E} = \sigma\vec{J}$, $\vec{J} = \rho\vec{E}$, $V = RI$

where σ, ρ, R are material- & temperature dependent.

* We learned the CONTINUITY EQUATION

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$$

For <u>steady</u> current: $\frac{\partial\rho}{\partial t} = 0$, $\vec{\nabla}\cdot\vec{J} = 0$

* Is Ampere's law $\left(\vec{\nabla}\times\vec{B} = \mu_0\vec{J}\right)$ consistent with the continuity equation?

Since $\vec{J} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B}$,

$$\vec{\nabla}\cdot\vec{J} = \frac{1}{\mu_0}\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{B}\right) = 0$$

$\Rightarrow$ consistent, only for steady currents, $\rho = \text{const.}$

When $\frac{\partial\rho}{\partial t} \neq 0$,

not consistent! $\longrightarrow$ $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ has to be corrected

For consistency, one needs $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$

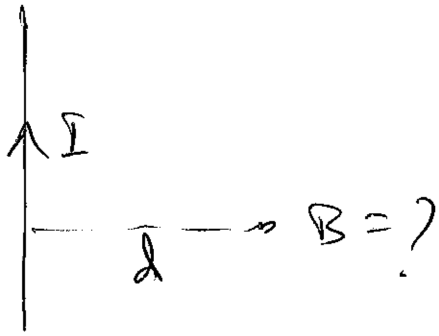
Because then $\vec{J} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t}$

so that $\vec{\nabla}\cdot\vec{J} = 0 - \epsilon_0\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{E}) = -\epsilon_0\frac{\partial}{\partial t}\frac{\rho}{\epsilon_0} = -\frac{\partial\rho}{\partial t}$

Using Ampere's law/theorem

Example 1a

Long thin wire



Top view

(We know already $B = \frac{\mu_0 I}{2\pi d}$)

Amperean loop/curve

By symmetry, $\vec{B}$ points along curve and is the same at all points on curve

$$\oint \vec{B} \cdot \vec{dl}$$

$$= \oint B\,dl = B \oint dl = B \cdot 2\pi d$$

Ampere's law gives $\quad B \cdot 2\pi d = \mu_0 I$

$$\Rightarrow B = \frac{\mu_0 I}{2\pi d}$$

Example 1b

Long thick wire, cylindrically symmetric current density

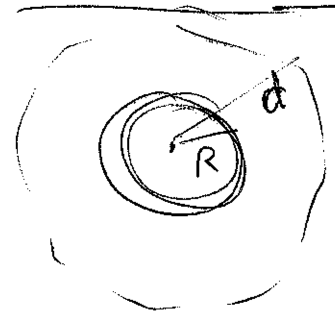
J depends only on ~~d~~ r

(44)

$$I = \int \vec{J} \cdot \vec{dA} = 2\pi \int_0^R r J(r) \, dr$$

$$(|dS| = 2\pi r \, dr)$$

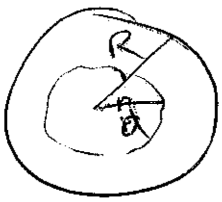
<u>Outside wire</u> $(d > R)$



$$B \cdot 2\pi d = \mu_0 I$$

$$B(d) = \frac{\mu_0 I}{2\pi d}$$ Exactly as for thin wire

<u>Inside wire</u> $(d < R)$



$$B \cdot 2\pi d = \mu_0 I_{enc.}$$

$$= \mu_0 \cdot 2\pi \int_0^d r J(r) \, dr$$

$$B = \frac{\mu_0}{d} \int_0^d r J(r) \, dr$$

If $J(r) =$ constant, then $J = \frac{I}{\pi R^2}$

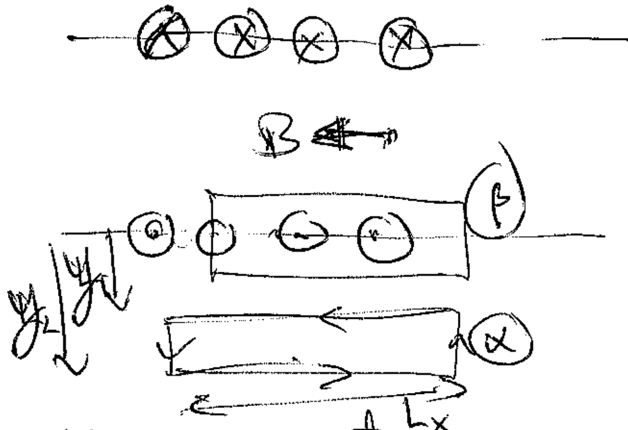
then $$B(d) = \frac{\mu_0}{d} \cdot \frac{I}{\pi R^2} \cdot \frac{d^2}{2} = \frac{\mu_0 I}{2\pi} \frac{d}{R^2}$$

<u>Exercise</u>: Show that inside & outside formulae match at $d = R$.

Infinitely long solenoid

We found, on axis only, using B-S law,

$$B = \mu_0 n I$$



n = turns per unit length

Using symmetry arguments, $\vec{B}$ must point parallel to axis everywhere, leftward inside, rightward outside.

Consider loop outside (α): $\oint \vec{B}\cdot d\vec{l} = B(y_1)L_x - B(y_2)L_x$

$$\Rightarrow B(y_1) = B(y_2) = B(\infty) = 0$$

⇒ Field outside is ZERO!

Consider loop straddling the coil surface (β):

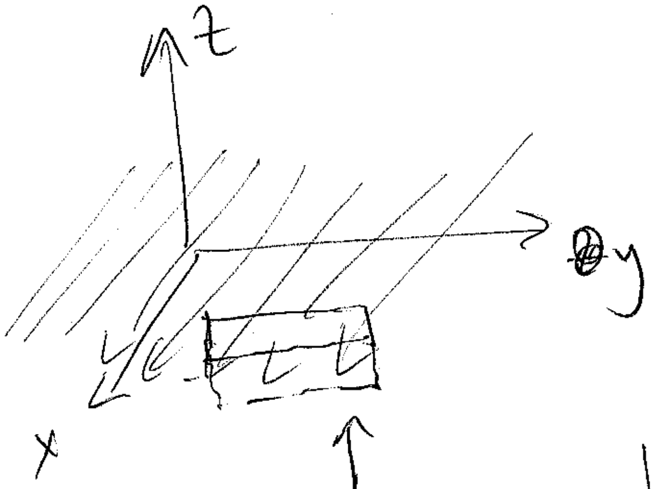
$$B \cdot L_x = \mu_0 N I$$

$$\Rightarrow B = \mu_0 n I$$ EVERYWHERE INSIDE SOLENOID!

⇒ Solenoid can be used to produce nearly *uniform* field.

(46)

* Infinite surface of current — optional



$$\vec{B} = \frac{\mu_0}{2} K \hat{j} \quad z<0$$

$$- \frac{\mu_0}{2} K \hat{j} \quad \text{for } z>0$$

Can prove using such Amperian loop

$$B \cdot L_y + B \cdot L_y = \mu_0 K L_y$$

$$\Rightarrow B = \frac{\mu_0}{2} K$$

Surface current not so strange. Simply think of many parallel wires packed next to each other. If n wires per unit length, $K = nI$

The Magnetic Vector Potential

$\vec{\nabla} \times \vec{E} = 0$ allows us to write $\vec{E} = -\vec{\nabla} V$ in electrostatics.

V is defined upto a constant.

$\vec{\nabla} \cdot \vec{B} = 0$ allows us to write $\vec{B} = \vec{\nabla} \times \vec{A}$

If a vector with zero curl is added to $\vec{A}$, that vector potential will give the same $\vec{B}$-field. $\Rightarrow$ Large freedom in choice of $\vec{A}$. Often called "gauge" freedom.

$$\vec{A} \longrightarrow \vec{A} + \vec{\nabla} f \quad \text{is a "gauge" transformat}^{n}.$$

Notice $\vec{\nabla} \times \vec{A} = \vec{\nabla} \times (\vec{A} + \vec{\nabla} f)$ b/c $\vec{\nabla} \times \vec{\nabla} f = 0$

A common choice : impose $\boxed{\vec{\nabla} \cdot \vec{A} = 0}$

Coulomb condition/gauge

$\Rightarrow$ even with this condition, many choices of $\vec{A}$ will produce same $\vec{B}$-field.

48 $\vec{E}$ and $\vec{B}$ fields are "physical" (measurable).
V and $\vec{A}$ potentials are mathematical constructs.
(Not measurable but very important.)

With the Coulomb condition, ($\underbrace{\vec{\nabla}\cdot\vec{A}}_{=0}$), we get

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = -\nabla^2\vec{A}$$

So Ampere's law becomes

$$\boxed{\nabla^2\vec{A} = -\mu_0\vec{J}}$$

Ampere's law with Coulomb condition.

* <u>Calculating electric and magnetic potentials.</u>

Electric potentials are determined by

$$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}}$$

Poisson's Equation.

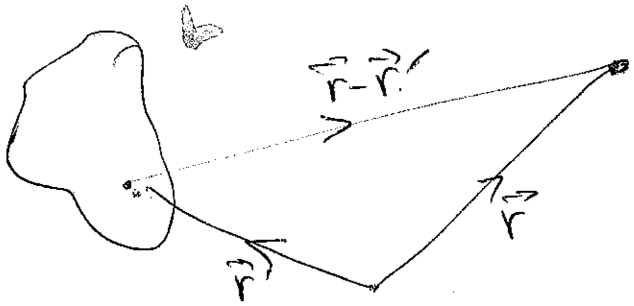
Obtained by combining $\vec{E} = -\vec{\nabla}V$ and $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$
and $\vec{\nabla}\cdot(\vec{\nabla}f) = \nabla^2 f$

In free space ($\rho = 0$), $\boxed{\nabla^2 V = 0}$ Laplace's Equation.

* Solving Poisson's equation [obtaining $V(\vec{r})$ for a given $\rho(\vec{r})$] is a rich topic — many methods exist. General solution:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i(\vec{r}_i')}{|\vec{r}-\vec{r}_i'|} \quad \text{for point charges}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} \, d^3\vec{r}'$$



Many approximations and tricks exist.
[MULTIPOLE EXPANSION, METHOD OF IMAGES, ...]

* Ampere's law with Coulomb condition:

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

Each Cartesian component is a Poisson's equatn!

$$\nabla^2 A_x = -\mu_0 J_x, \quad \nabla^2 A_y = -\mu_0 J_y, \quad \nabla^2 A_z = -\mu_0 J_z$$

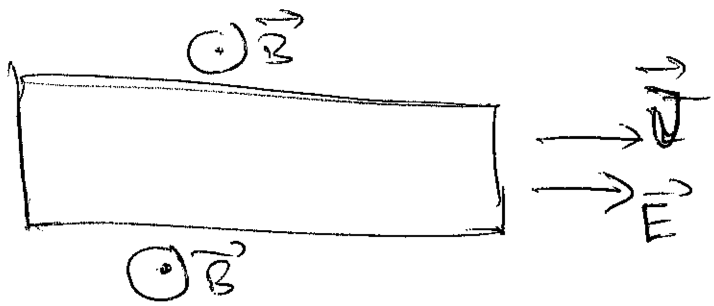
$$\Rightarrow \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} \, d^3\vec{r}'$$ Same methods can be used.

50

The HALL effect

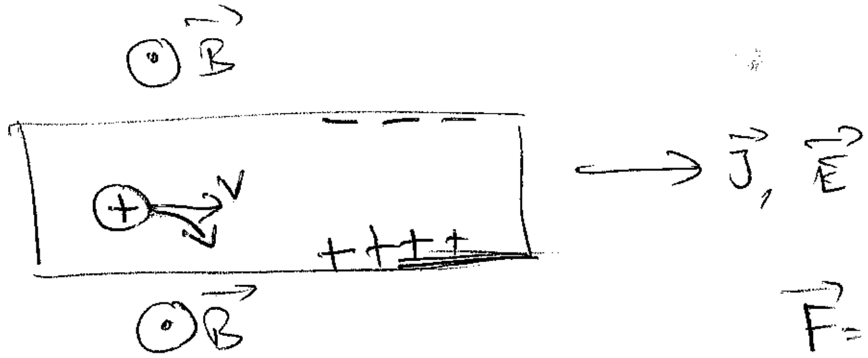
optional

Consider conducting slab, in perpend. magnetic field:



The CARRIERS are then pushed in third perpendicular direction.. (Lorentz force law).

If CARRIERS are positive:

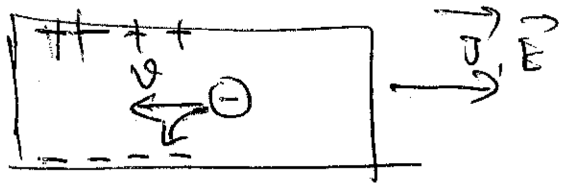


$\vec{F} = q \vec{v} \times \vec{B}$

Charges accumulate at edges

⇒ additional $\vec{E}$-field (Voltage) created in transverse direction.

If CARRIERS ARE negative:



$\vec{F} = (-e) \vec{v} \times \vec{B}$

Negative carriers deflected in the same direction.

$\Rightarrow$ Transverse voltage / $\vec{E}$-field created in <u>opposite</u> direction.

$\Rightarrow$ Hall experiment can identify sign of carriers. (negative for most but not all materials)

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* ELECTROMAGNETIC INDUCTION, ~~FARADAY'S LAW~~

* Electromotive force (EMF):
  - not a force, more like a voltage or potential difference.
  - In static or steady-state situations, the potential difference (voltage) drives a current through a wire or loop/circuit.
  - EMF is the generalization, for static + dynamic situations.

$$\mathcal{E} = \oint \vec{f} \cdot \vec{dl}$$

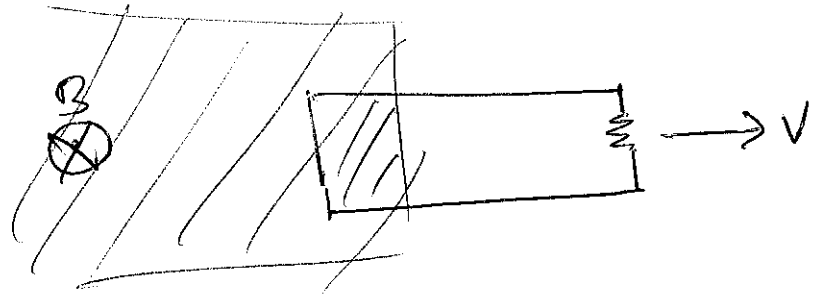
EMF around a circuit.

where $\vec{f}$ is the force per unit charge driving the current = Electric field

* Motional EMF

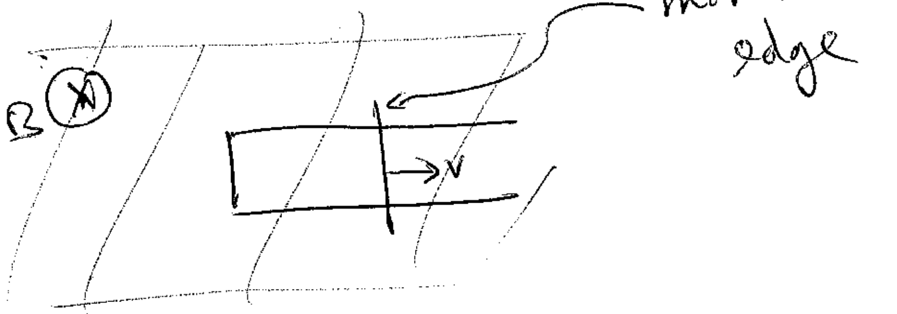
If the magnetic flux through a circuit changes, an EMF is created in the circuit.

Ex. 1



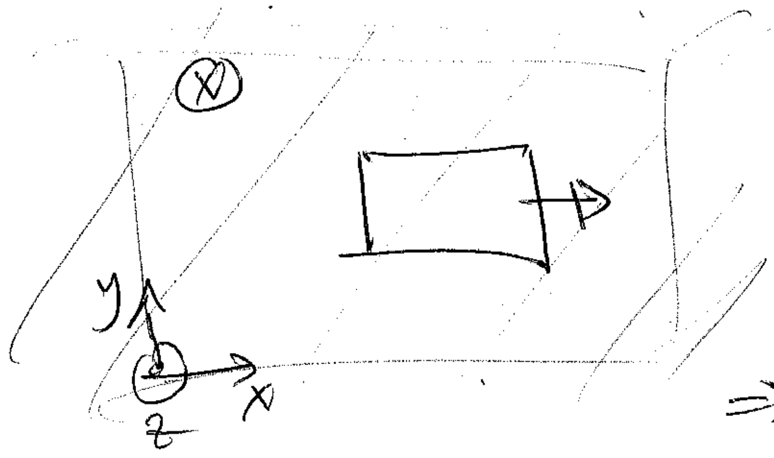
Flux decreases, magnetic force causes charges to move, drives current. (EMF created)

Ex. 2



Flux increases. EMF will be created

Ex. 3 $\quad \mathbf{B} = -(\lambda x)\hat{k}$



Flux changes because $\vec{B}$-field is not uniform

$\Rightarrow$ EMF created.

$\boxed{\text{MORE EXAMPLES LATER}}$

* Faraday's law

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{S}$$

→ Found experimentally by Faraday (reported 1831)

→ Can be derived using the Lorentz force law.

→ Leads to Maxwell's Third equation:

$$\mathcal{E} = \oint \vec{E}\cdot d\vec{l} = \int (\vec{\nabla}\times\vec{E})\cdot d\vec{S} \quad \text{(Stokes theorem)}$$

$$-\frac{d}{dt}\int \vec{B}\cdot d\vec{S} = \int\left(-\frac{\partial \vec{B}}{\partial t}\right)\cdot d\vec{S}$$

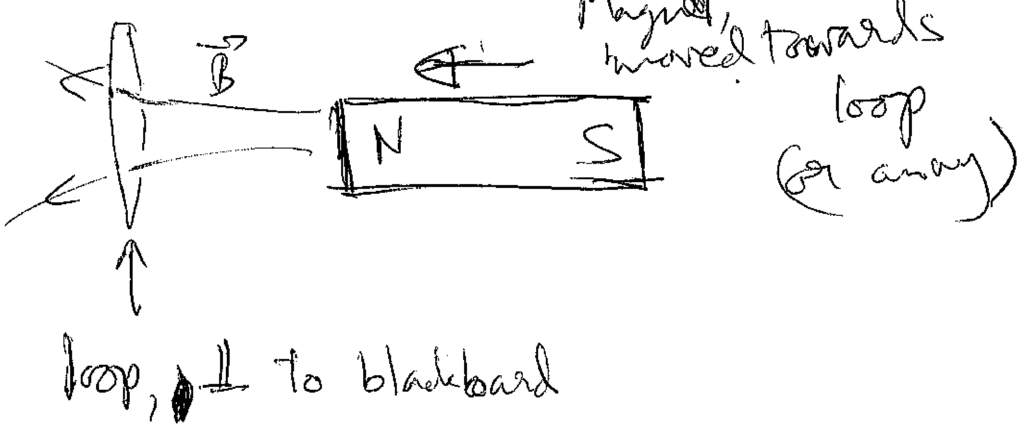
Thus $\int (\vec{\nabla}\times\vec{E})\cdot d\vec{S} = \int\left(-\frac{\partial \vec{B}}{\partial t}\right)\cdot d\vec{S} \Rightarrow \boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$

Faraday's

MAXWELL'S EQNS COMPLETE!

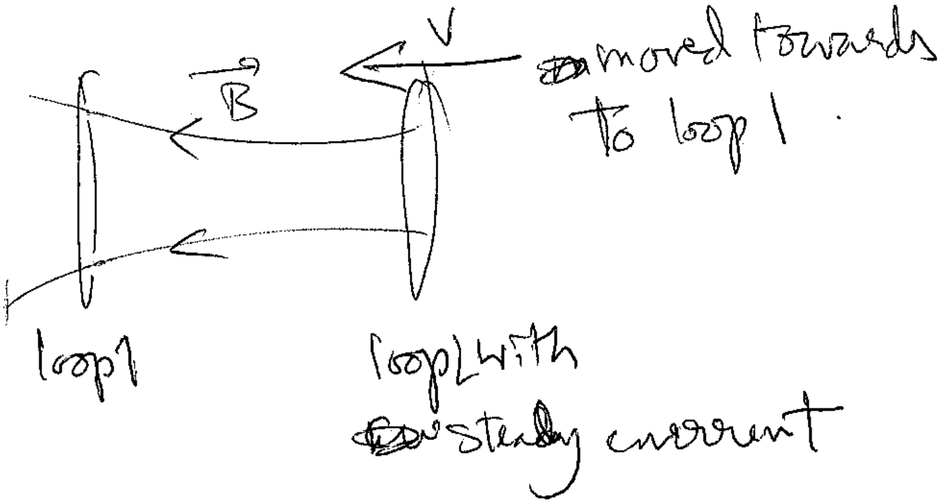
EXAMPLES, INDUCTION, CONT'D

Ex. 4



Flux changes $\Rightarrow$ EMF created.

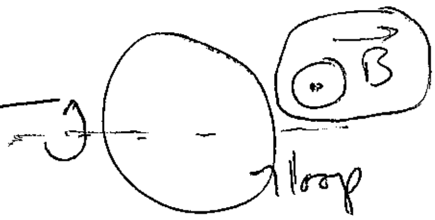
Ex. 5



Magn. flux thru loop 1 changes

$\Rightarrow$ EMF created

Ex. 6



Rotating loop : flux changes. If rotation constant, $\Phi_B = BA\cos\omega t$

$\Rightarrow \mathcal{E} = BA\omega\sin\omega t$

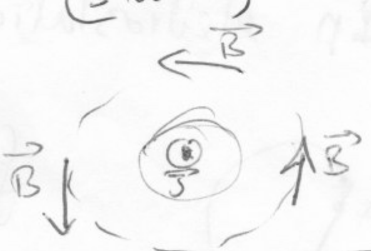
* FARADAY'S LAW IN INTEGRAL FORM

$$\oint_C \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_\Sigma \vec{B}\cdot d\vec{S} = -\int_\Sigma \left(\frac{\partial \vec{B}}{\partial t}\right)\cdot d\vec{S}$$

Where the curve C encloses the surface Σ.

$\Rightarrow$ Comparison with Ampere's law for magnetic fields! (static)

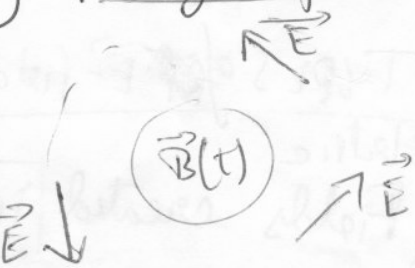
$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I_{encl.} = \int_\Sigma (\mu_0 \vec{J})\cdot d\vec{S}$$



$\Downarrow$

Piercing/enclosed current causes "curly" $\boxed{\text{magn. field}}$

$$\oint \vec{E}\cdot d\vec{l} = \int_\Sigma \left(-\frac{\partial \vec{B}}{\partial t}\right)\cdot d\vec{S}$$



$\Downarrow$

Enclosed/~~piercing~~ time-dependent magnetic field causes "curly" $\boxed{\text{electric field}}$.

* TWO TYPES OF ELECTRIC FIELDS

EMF $\mathcal{E} = \oint \vec{E}\cdot d\vec{l}$

Q. Why didn't we just say voltage = "potential difference"

5C

A: Because potential is not defined.

$\Rightarrow$ Definition $\vec{E} = -\vec{\nabla} V$ requires $\vec{\nabla} \times \vec{E} = 0$

Now we have $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$;

(studying physics beyond electrostatics.)

In electrostatics, $\oint \vec{E} \cdot d\vec{l} = 0$, because

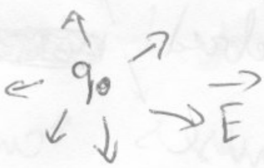
$$\oint \vec{E} \cdot d\vec{l} = V_A - V_A = 0$$

(potential difference of a point with itself.)

2 TYPES of E-fields

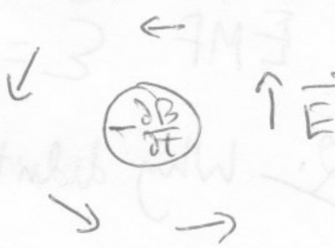
Electric Fields created $\boxed{\text{in electrostatics}}$:

- field lines end or start at charges.
- zero curl.



Electric fields $\boxed{\text{created by induction}}$ (changing $\vec{B}$):

- field lines loop around onto themselves
- no divergence.



* Maxwell's 4 equations

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}, \quad \vec{\nabla}\cdot\vec{B} = 0$$

$$\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}} \quad \text{electromagnetic induction}$$

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

* We derived the term $\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$ by imposing the continuity equation. This quantity is called the DISPLACEMENT CURRENT

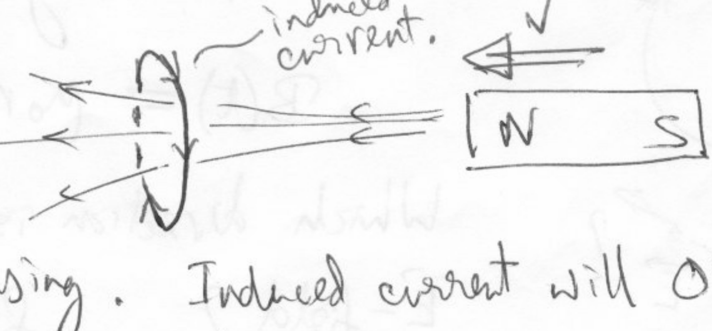
---

* LENZ'S LAW (fixes the direction of induced EMF/current)

Currents induced by change in magnetic flux

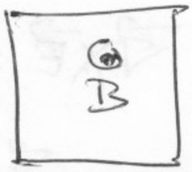
OPPOSE the change, through the $\vec{B}$-field created by the induced current.

Example I:



Flux increasing. Induced current will OPPOSE increase.
$\Rightarrow$ Will point rightward.

Example 2a

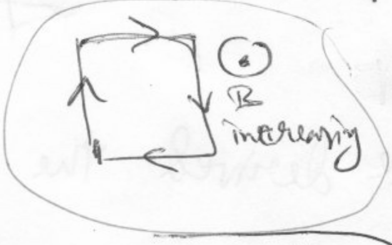


$|\vec{B}| = \alpha t$

increasing $|\vec{B}|$

induced current opposes increase

$\Downarrow$



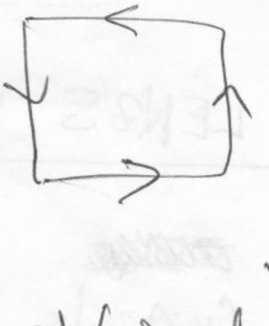
$\therefore$ induced field points INWARD

Example 2b

decreasing $|\vec{B}|$, e.g. $|\vec{B}| \propto B_0 e^{-\alpha t}$

induced current opposes decrease

$\Rightarrow$ field due to induced current points UP.

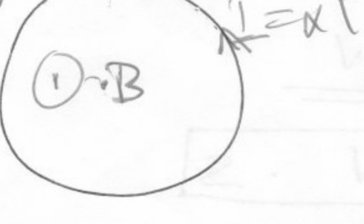


Example 3

(Without a circuit/wire!)

Changing current in solenoid

CROSS-SECTION OF SOLENOID



Increasing solenoid current

$\Rightarrow$ increasing $\vec{B}$-field / flux

$B(t) = \mu_0 n I(t) = \mu_0 n \alpha t$

? $\vec{E}$?

Which direction is the induced $\vec{E}$-field?

$$\oint \vec{E}\cdot d\vec{l} = \int\left(-\frac{\partial \vec{B}}{\partial t}\right)\cdot d\vec{S} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{S}$$

Answer: imagine a wire loop circling around solenoid. Induced EMF should OPPOSE INCREASE of flux ⇒ imagined induced current should create $\vec{B}$-field downward.

⇒ induced $\vec{E}$-field points CLOCKWISE, opposite direction of increasing current in solenoid.

* Lenz's law corresponds to negative SIGN in Faraday's law.

→ Lenz's law makes sense energetically, because other sign would lead to perpetual energy production.

* EXERCISE:

Loop is being deformed so that loop area DECREASES. $\vec{B}$ is constant and $\perp$ to plane.

Find the direction of the induced current.