Electromagnetic Induction, Faraday's law

This writeup

We discuss electromagnetic induction — how electric currents and electric fields are generated when the magnetic flux changes.

Electromotive force (EMF)

- This is an unfortunate historical name. It's not a 'force' at all. Instead, it has the dimensions of a potential difference, or a voltage. Unfortunately we are stuck with this name. We will usually use the abbreviation EMF.
- Imagine a closed loop Γ. This loop might be a physical wire loop (a 'circuit'), or an imagined line looping back onto itself. The EMF along this loop/circuit is defined as

  $$\mathcal{E} = \oint_\Gamma \mathbf{E} \cdot d\mathbf{l} \tag{1}$$

  i.e., it is the line integral of the electric field along the closed loop.

  Of course, if you are writing this on paper, you would write it as

  $$\mathcal{E} = \oint_\Gamma \vec{E} \cdot \vec{dl}$$

  The conventional symbol for EMF is the letter E written calligraphically, $\mathcal{E}$. Please practice writing it a few times, so that it is well-distinguished from E.
- Eq. (1) should show that the EMF has the same dimensions (units) as a potential difference.
- But the potential difference was also a line integral of the electric field! So isn't Eq. (1) another expression for a potential difference?

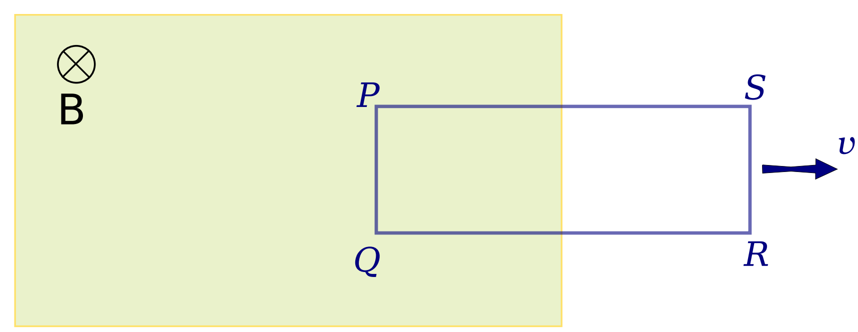
  The answer is no. For an electric field in **electrostatics**, the potential difference along a closed loop is always zero. If the loop integral starts at point P and ends at the same point P, then clearly the potential difference is $V_P - V_P = 0$.

  Eq. (1) and the concept of EMF is not meaningful in electrostatics. In the following, we will talk about changing magnetic fields, moving cricuits, etc. In fact, we will find that the definition of electric potential has to be modified, now that we are beyond electrostatics.

Motional EMF

If we take a loop/circuit of wire and move it in a magnetic field, we might be able to create an EMF and hence induce a current in the loop.

Example 1



- In Example 1, there is a magnetic field pointing inward (into the paper/screen/board) only in the shaded region. (No field outside this region.) A wire loop $PQRS$ is being dragged away from this region.

- The conduction charges in the wire will feel a force due to their motion through a magnetic field. (Remember the Lorentz force law, $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$.)

  Assuming the charges to be positive, find the direction of the force.

- In the segment PQ of the wire loop, the charges feel a force along the wire, and hence are driven across the wire!

- You can calculate the EMF $\oint \mathbf{E} \cdot d\mathbf{l}$ along the loop. You can interpret $\mathbf{E}$ as $\mathbf{F}/q$, where $\mathbf{F}$ is the force due to the Lorentz force law.

  The only contribution comes from the segment PQ. (Why?)

- EMF thus generated, due to motion, is called motional EMF.

- Consider the magnetic flux through the area enclosed by the wire loop. It is B times the area within the loop which is shaded, i.e., which is within the magnetic field region. The area is decreasing as the loop moves out of the region supporting the magnetic field. So the magnetic flux magnitude is decreasing.

- Exercise: calculate the rate of change of the magnetic flux.

- More involved exercise: you can show that the EMF generated in the wire has the same magnitude as the rate of change of magnetic flux.

From motional EMF to Faraday's law

In the example above, an EMF is generated in an wire loop due to motion, while at the same time, the magnetic flux through the loop changes.

By analyzing many different examples and experiments, it becomes clear that an EMF is generated whenever there is a change in the magnetic flux, and the magnitude of the induced EMF is equal to the rate of change of the magnetic flux. This is codified in Faraday's law:

$$\boxed{\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\int \mathbf{B}\cdot d\mathbf{S}} \tag{2}$$

- The law was reported by Faraday in 1831.
- For motional EMF like in Example 1, it gives the same result as one gets by using the Lorentz force law.
- The flux could change due to several reasons: due to motion of the loop through a region where the magnetic field is changing, due to a changing magnetic field, due to a change in the angle between field and loop surface. You should convince yourself that any of these effects can change a flux.
- The flux integral above has been written with a single integral. You should know from the context that this is a surface integral. The integral sign is $\int$ and not $\oint$, because the surface is not a closed surface. (Its boundary is a closed loop.)
- Faraday's law holds not only when there is a physical wire loop present, but also for any imagined or theoretical closed loop.
- What about the minus sign? The sign tells you the direction of the induced EMF and induced current. We will understand this properly later, when we look at **Lenz's law**.

From Faraday's law to Maxwell's third equation

We will now turn Faraday's law in integral form to an equation in differential form. Similar derivations — going from an integral vector equation to a corresponding differential vector equation — has been done a few times before in this semester.

Let us consider a closed curve Γ enclosing a surface Σ.

The EMF along Γ can be written as

$$\mathcal{E} = \oint_\Gamma \mathbf{E}\cdot d\mathbf{l} = \int_\Sigma (\nabla\times\mathbf{E})\cdot d\mathbf{S}$$

using Stokes' theorem.

Also, the right side in Eq. (2) can be written as

$$-\frac{d\Phi_B}{dt} = -\frac{d}{dt}\int_\Sigma \mathbf{B}\cdot d\mathbf{S} = \int_\Sigma \left(-\frac{\partial \mathbf{B}}{\partial t}\right)\cdot d\mathbf{S}$$

Comment: The flux derivative is a total derivative because Φ_B depends only on time. The derivative of $\mathbf{B}$ is a partial derivative because $\mathbf{B}$ also depends on space. Question: Why doesn't Φ_B depend on space?

Thus Faraday's law becomes

$$\int_\Sigma (\nabla\times\mathbf{E})\cdot d\mathbf{S} = \int_\Sigma \left(-\frac{\partial \mathbf{B}}{\partial t}\right)\cdot d\mathbf{S}$$

Since this is true for any surface Σ, the integrands on both sides must be equal. Hence we have

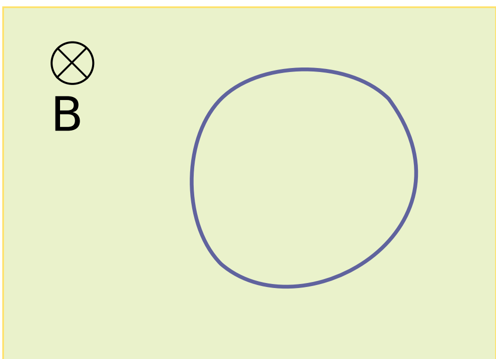
$$\boxed{\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}$$

This is Maxwell's third equation.

At this stage, we have obtained ALL of Maxwell's equations! I strongly urge you to go back and review the derivation of each of Maxwell's equations.

Another example

Here's an example of electromagnetic induction which is not due to motion.



Example 2

$$B = B_0 \exp(-t/T)$$

Now, the wire loop is sitting still in a region of magnetic field, but the magnetic field itself is changing. The magnetic flux through the loop is then changing. EMF will be induced in the loop.

Because there is no motion, we cannot use the Lorentz force law to find the induced EMF. However, we can use Faraday's law.

Let's say the magnetic field is decreasing as $B(t) = B_0 e^{-t/T}$, where T is a positive constant.

If the area of the loop is A, then the flux through the loop is

$$\Phi_B = B \cdot A \cdot \cos\left(\frac{\pi}{2}\right) = B_0 e^{-t/T} \cdot A \cdot 1 = B_0 A e^{-t/T}$$

If you are not 100% comfortable with the exponential function, please review it, and sketch plots of $e^{-t/T}$ and $e^{+t/T}$ as functions of t for various values of T.

The rate of change of the magnetic flux is

$$\frac{d\Phi_B}{dt} = B_0 A (-1/T) e^{-t/T}$$

Therefore, by Faraday's law, the EMF induced in the loop will have magnitude

$$|\mathcal{E}| = \left|\frac{d\Phi_B}{dt}\right| = \frac{B_0 A}{T} e^{-t/T}$$

As the magnetic field and its rate of change dies off with time, the EMF will also die off with time.

Sign? We didn't worry about the sign of the EMF. That depends on convention.

Direction of induced current? The induced EMF will cause current to flow around the loop. Which direction? The direction can be figured out using Lenz's law. (To be discussed in a later lecture.) In the geometry shown above, the induced current will be counter-clockwise.