Due on Monday, May 3rd.

If pictures are needed/relevant, please provide them with your solutions.

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1. An electromagnetic system in vacuum (no charges or currents are present!) is described by the fields

$$\mathbf{E} = E_0 \exp\left[-\frac{(x-wt)^2}{2L^2}\right]\hat{j}$$
$$\mathbf{B} = (E_0/w) \exp\left[-\frac{(x-wt)^2}{2L^2}\right]\hat{k}$$

L and w are positive constants.

$\exp[u]$ is common notation for the exponential function e^u.

(a) [**4 pts**] Show that these fields obey Maxwell's third equation (which concerns the curl of the electric field).

(b) [**5 pts**] The fields should also obey Maxwell's fourth equation in vacuum. Find the value of w for which this works.

(c) [**not marked**] Look up the properties of the gaussian function (e.g., wikipedia). You should be able to tell the center and the width of a gaussian by looking at its form.

(d) [**3 pts**] Sketch plots of the magnitude of the electric field as a function of the position, at the time instants $t = 0$, $t = 2L/w$, and $t = 4L/w$. (You can think of these as 'snapshots' at different points of time.)

Please submit a hand-drawn sketch, not a computer printout.

In which direction is our electromagnetic pulse traveling?

(e) [**3 pts**] Calculate the Poynting vector $\mathbf{S}$. In case it has not been treated in class yet: $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$.

(f) [**2 pts**] The Poynting vector gives the 'energy current density' for the flow of electromagnetic energy. Explain whether the calculated direction of $\mathbf{S}$ makes sense.

(g) [**4 pts**] Calculate the energy density for electromagnetic fields,

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)$$

for the given fields.

(h) [**4 pts**] Conservation of electromagnetic energy in vacuum is encoded in the relation (analogous to the continuity equation for charge)

$$\nabla \cdot \mathbf{S} + \frac{\partial u}{\partial t} \;=\; 0\,.$$

Find out whether this is satisfied for our fields.

2. [**6 pts**] Using Maxwell's equations in vacuum and the definitions of $\mathbf{S}$ and u, derive the relation $\nabla \cdot \mathbf{S} = -\partial u/\partial t$. You may need the vector identity

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) \;=\; \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

3. Consider an electromagnetic wave travelling through empty space described by the electric and magnetic fields

$$\mathbf{E} \;=\; 3\alpha \cos\left(\frac{1}{L}(y - ct)\right)\hat{\imath}\,, \qquad \mathbf{B} \;=\; \mathbf{G} \cos\left(\frac{1}{L}(y - ct)\right)$$

where α and L are positive constants and $\mathbf{G}$ is a constant vector.

(a) [**1 pt**] In which direction is this wave traveling?

(b) [**6 pts**] Find the magnitude (in terms of α) and the direction of the constant vector $\mathbf{G}$. You might need to use one of Maxwell's equations.

(c) [**3 pts**] What is the wavelength and the frequency of this wave?

4. An electromagnetic wave has an electric field given by

$$\mathbf{E} = E_0 \cos\left(\frac{2\pi c}{\lambda} t\right) \sin\left(\frac{2\pi z}{\lambda}\right)\hat{\imath}$$

where λ is a positive constant.

(a) [**5 pts**] Use Maxwell's third equation (Faraday's law in differential form) to calculate the associated magnetic field $\mathbf{B}$. You can assume the time-independent additive term (constant of integration) to be zero.

(b) [**1 pts**] Is this wave traveling? In which direction?

(c) [**2+2 pts**] Calculate the Poynting vector and explain the direction of energy flow using your result.