- Partial solutions or hints for problem set 6 below.
- I don't always provide pictures, because they are time-intensive to draw eletronically. However, you (the student) should sketch figures whenever relevant. Without sketching the situations, you are unlikely to arrive at correct solutions.
- These hints/solutions have been typed rapidly and have not been carefully proof-read. Quite likely, there are typographical or other errors. Please use with caution and check everything!

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1. Two metallic plates are placed parallel and close to each other. They each have area A. They carry opposite charges $+Q$ and $-Q$. Such an arrangement is known as a *capacitor*.

   (a) The plates are large enough that, if we are considering the region between the two plates (far from the plate edges), we can regard each charged plate as an infinite sheet of charge. Show that the magnitude of the electric field in the region between the plates is $E = Q/(A\epsilon_0)$.

   (Partial) Solution/Hint →

   We learned (in class, in an earlier assignment) that an infinite charged plate with surface charge density σ produces an electric field that is independent of distance, with magnitude $\sigma/(2\epsilon_0)$. The field points perpendicularly away from the plate if the charge is positive and perpendicularly toward the plate if the charge is negative.

   In the region between the plates, we regard each charged plate as infinite, with surface charge density $+Q/A$ on one and $-Q/A$ on the other. A drawing of the situation will show that the two plates cause electric fields which point in the SAME direction. Thus the electric field in the region between the plates is

$$\frac{Q/A}{2\epsilon_0} + \frac{(Q/A)}{2\epsilon_0} = \frac{Q}{A\epsilon_0}$$

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(b) The *capacitance* of a capacitor is the charge on a plate divided by the potential difference between the two plates ($C = Q/V$). If the distance between the plates is d, find the capacitance of the arrangement.

(Partial) Solution/Hint →

What is the potential difference between the two plates? Since we want the difference between two points whose displacement is along the direction of the electric field, the potential difference is $Ed = Qd/(A\epsilon_0)$, ignoring signs. Hence the capacitance is

$$\frac{Q}{V} = \frac{Q}{Qd/(A\epsilon_0)} = \frac{(A\epsilon_0)}{d}$$

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2. An electron (charge$\approx 1.602 \times 10^{-19}$ C; mass$\approx 9.109 \times 10^{-31}$ kg) moving at speed 200 m/s finds itself in a region of constant magnetic field. The magnetic field points perpendicular to the electron velocity and has strength 10^{-3} Tesla.

(a) Find the magnitude of the magnetic force experienced by the electron.

(Partial) Solution/Hint →

As the velocity (of magnitude v = 200m/s) is perpendicular to the magnetic field (of magnitude $B = 10^{-3}$T), their cross product has magnitude vB. Hence the force has magnitude

$$qvB \approx 1.602 \times 10^{-19} \times 200 \times 10^{-3}\text{Newtons} = 3.204 \times 10^{-20}\text{N}$$

Since the input quantities all have SI units, so does the result; we don't have to show explicitly that Coul. × T × m/s = Newtons.

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(b) The magnetic force causes cyclotron motion, so that the electron moves around a circular trajectory. Find the radius of the circular trajectory.

(Partial) Solution/Hint →

We showed in class that the cyclotron radius is found by equating the Lorentz (magnetic) force to the centripetal force:

$$qvB = \frac{mv^2}{R} \qquad \Longrightarrow \qquad R = \frac{mv}{qB}$$

With the values given, the radius is thus

$$R \approx \frac{9.109 \times 10^{-31} \times 200}{1.602 \times 10^{-19} \times 10^{-3}} \mathrm{m} \approx 1.137 \times 10^{-6} \mathrm{m}$$

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(c) How long does it take for the electron to complete one cycle? (i.e., what is the period of the cyclotron motion?)

(Partial) Solution/Hint →

Since the cyclotron radius is $R = \frac{qB}{mv}$, the circumference of the trajectory is $S = 2\pi R = 2\pi mv/qB$.
The time it takes for an electron with speed v to cover distance S is

$$T = \frac{S}{v} = \frac{2\pi R}{v} = \frac{2\pi m}{qB}$$

Plugging in the known values into either of these equations gives

$$T \approx 3.57 \times 10^{-8} \mathrm{s}$$

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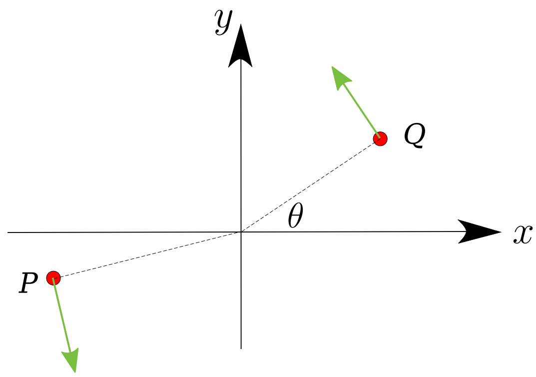
3. Steady current I flows through an infinitely long straight wire placed along the z azis, from $z = -\infty$ to $z = +\infty$ through the origin.

   (a) Sketch a top view of the x-y plane, and show the point P with coordinates $(-4L, -L, 0)$ and the point Q with coordinates $(3L, 2L, 0)$. For each of these points, draw the direction of the magnetic field produced by the steady current I.

   (Partial) Solution/Hint →



   The magnetic field at P should be perpendicular to the line joining the origin to the point P. The direction is shown with a green arrow. Similarly for point Q. (No attempt has been made to make the arrow lengths proportional to the magnitudes of the magnetic field; they indicate only the directions.)

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   (b) Calculate the magnitudes of the magnetic field created at those two points (P and Q).

   (Partial) Solution/Hint →

   The point P has distance $\sqrt{(-4L)^2 + (-L)^2} = \sqrt{17}L$ from the origin, i.e., from the current-carrying wire. Hence, the magnetic field magnitude at this point is

$$\frac{\mu_0 I}{2\pi\sqrt{17}L}$$

The point Q has distance $\sqrt{(3L)^2 + (2L)^2} = \sqrt{13}L$ from the origin, i.e., from the current-carrying wire. Hence, the magnetic field magnitude at this point is

$$\frac{\mu_0 I}{2\pi\sqrt{13}L}$$

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(c) At the point $Q(3L, 2L, 0)$, find the x-, y-, and z- components of the magnetic field. (Sketching a top view of the x-y plane might help.)

(Partial) Solution/Hint →

Let θ be the angle made by the line joining the origin with Q, as also shown in the figure. Consideration of the figure should show that components of the magnetic field at Q are

$$B_x = -B\sin\theta \qquad B_y = B\cos\theta \qquad B_z = 0$$

If this is not entirely clear, you might want to draw a line through the point Q parallel to the y axis, and identify which two lines meet at Q with angle θ between them.
Now from the geometry

$$\sin\theta = \frac{2L}{\sqrt{13}L} = \frac{2}{\sqrt{13}} \qquad \text{and} \qquad \cos\theta = \frac{3L}{\sqrt{13}L} = \frac{3}{\sqrt{13}}$$

so that, using the expression for B derived previously,

$$\begin{aligned} B_x &= -\frac{\mu_0 I}{2\pi\sqrt{13}L}\frac{2}{\sqrt{13}} = -\frac{\mu_0 I}{13\pi L} \\ B_y &= +\frac{\mu_0 I}{2\pi\sqrt{13}L}\frac{3}{\sqrt{13}} = +\frac{3\mu_0 I}{26\pi L} \\ B_z &= 0 \end{aligned}$$

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(d) Now consider a point $(3L, 2L, z_0)$, not necessarily on the x-y plane. What is the magnitude of the magnetic field at this point? What are the x-, y-, and z- components of the magnetic field at this point?

(Partial) Solution/Hint →

This point is at the same distance from the infinite wire and in the same direction, just vertically displaced. Hence the magnetic field at this point (and the components) will be exactly the same as at Q.

This problem requires some 3D thinking. If the solution is not perfectly clear, I suggest drawing the situation from different angles until you have a mental picture of the location of the points and the directions of magnetic fields.

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4. A particle having mass m and charge q is subjected to uniform electric and magnetic fields, both pointing in the z direction: $\mathbf{E} = E_0\hat{k}$, $\mathbf{B} = B_0\hat{k}$.

   At time $t = 0$, the particle is at the origin and has velocity $\mathbf{v} = v\hat{\imath}$.

   (a) At which times does the particle pass through the z axis?

   (Partial) Solution/Hint →

   The magnetic field causes the particle to perform cyclotron motion in the direction perpendicular to the z axis, i.e., parallel to the x-y plane. Meanwhile, the electric field causes the particle to be accelerated in the z direction. The physical insight necessary here is that motion in the x-y plane is decoupled to motion in the z direction.

   The trajectory of the particle is spiral-like. It is not a regular helix, because the velocity in the z direction increases and is not constant. In other words, the turns of the spiral are not equally spaced.

   The particle will hit the z-axis once every cyclotron motion. Since this is perpendicular to the z direction, we can ignore the electric field for calculating the time it takes to complete one cyclotron cycle, and focus only on the x-y part of the motion.

   For the cyclotron (x-y) part of the motion, the cyclotron radius is $R = mv/qB_0$, so that the period of the cyclotron motion is $T = 2\pi R/v = 2\pi m/qB_0$.

   Thus the particle returns to the z axis at times

$$T = \frac{2\pi m}{qB_0}, \quad 2T = \frac{2\pi m}{qB_0}, \quad 3T = \frac{2\pi m}{qB_0}, \quad ....$$

   i.e., at times $nT = n\frac{2\pi m}{qB_0}$, where n is any positive integer.

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(b) What are the z values where the trajectory meets the z axis?

(Partial) Solution/Hint →

We now turn to the motion in the z direction. The particle has constant acceleration in this direction:

$$F = qE_0 \qquad \Longrightarrow \qquad a = \frac{qE_0}{m}$$

So, the z-coordinate of the particle at time t is

$$z = \frac{1}{2}at^2 = \frac{qE_0}{2m}t^2$$

as there is no initial velocity in this direction.
Therefore, the z values where the trajectory meets the z axis are

$$0, \quad \frac{qE_0}{2m}T^2 = \frac{qE_0}{2m}\left(\frac{2\pi m}{qB_0}\right)^2, \quad \frac{qE_0}{2m}(2T)^2 = \frac{qE_0}{2m}\left(\frac{4\pi m}{qB_0}\right)^2, \quad \frac{qE_0}{2m}(3T)^2 = \frac{qE_0}{2m}\left(\frac{6\pi m}{qB_0}\right)^2, \quad \frac{qE_0}{2m}(4T)^2 = \frac{qE_0}{2m}\left(\frac{8\pi m}{qB_0}\right)^2, \ldots$$

i.e.,

$$\frac{qE_0}{2m}(nT)^2 \;=\; \frac{qE_0}{2m}\left(\frac{2n\pi m}{qB_0}\right)^2 \;=\; \frac{2n^2\pi^2 mE_0}{qB_0^2}$$

where n is zero or a positive integer.

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5. A particle with charge Q and mass m is placed in a region with uniform electric field in the y-direction, $\mathbf{E} = E\hat{\jmath}$, and uniform magnetic field in the z direction, $\mathbf{B} = B\hat{k}$. The particle position coordinates are x, y, z, and its velocity components are denoted as v_x, v_y, v_z.

   (a) Calculate the force $\mathbf{F}$. Write out separately the expressions for its components, F_x, F_y, F_z. (These expressions will involve the quantities Q, B, E, v_x, and v_y.)

   (Partial) Solution/Hint →

   The force is

   $$\mathbf{F} \;=\; Q\mathbf{E} + Q\mathbf{v} \times \mathbf{B} \;=\; QE\hat{\jmath} + QB\mathbf{v} \times \hat{k}$$

   In terms of components:

   $$\begin{aligned} F_x &= QBv_y \\ F_y &= QE - QBv_x \\ F_z &= 0 \end{aligned}$$

   — —

   (b) Using your results for F_x, F_y and Newton's 2nd law, $\mathbf{F} = m\frac{d\mathbf{v}}{dt}$, calculate expressions for $\dfrac{dv_x}{dt}$ and $\dfrac{dv_y}{dt}$. You should thus obtain coupled first-order differential equations for $v_x(t)$ and $v_y(t)$.

   (Partial) Solution/Hint →

   Using $\dfrac{dv_x}{dt} = \frac{1}{m}F_x$ and $\dfrac{dv_y}{dt} = \frac{1}{m}F_y$, and the expressions obtained previously for F_x and F_y,

   $$\begin{aligned} \frac{dv_x}{dt} &= \frac{QB}{m}v_y \\ \frac{dv_y}{dt} &= \frac{QE}{m} - \frac{QB}{m}v_x \end{aligned}$$

   — —

(c) If the particle starts at rest, i.e., if the initial conditions are $v_x(0) = v_y(0) = 0$, then the solutions to the above equations are of the form

$$v_x = \frac{E}{B} - \frac{E}{B}\cos(\omega t) \qquad v_y = \frac{E}{B}\sin(\omega t) \tag{1}$$

Substituting these into your first differential equation, find an expression for ω. Substituting these into your second differential equation, show that you find the same expression for ω.

(Partial) Solution/Hint →

Substituting the solution form into the first differential equation gives

$$0 - \frac{E}{B}(-\omega)\sin(\omega t) = \frac{QB}{m}\frac{E}{B}\sin(\omega t) \qquad \Longrightarrow \qquad \omega = \frac{QB}{m}$$

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(d) The equations (1) are themselves differential equations for $x(t)$ and $y(t)$, since $v_x = \frac{dx}{dt}$ and $v_y = \frac{dy}{dt}$. Assume that the particle starts at the origin, so that the initial conditions are $x(0) = y(0) = 0$. Find the solutions for $x(t)$ and $y(t)$.

(Partial) Solution/Hint →

First $x(t)$:

$$\frac{dx}{dt} = v_x = \frac{E}{B} - \frac{E}{B}\cos(\omega t)$$

$$\Longrightarrow \qquad x(t) = \frac{E}{B}t - \frac{E}{B\omega}\sin(\omega t) + \text{const.}$$

Since $x(0) = 0$, the constant is zero:

$$x(t) = \frac{E}{B}t - \frac{E}{B\omega}\sin(\omega t) = \frac{E}{B}t - \frac{Em}{QB^2}\sin(\omega t)$$

Now for $y(t)$:

$$\frac{dy}{dt} = v_y = \frac{E}{B}\sin(\omega t) \qquad \Longrightarrow \qquad y(t) = -\frac{E}{B\omega}\cos(\omega t) + \text{const.}$$

Since $y(0) = 0$, the constant is found to be $E/(B\omega)$. Thus

$$y(t) = -\frac{E}{B\omega}\cos(\omega t) + \frac{E}{B\omega} = \frac{E}{B\omega}\Big[1 - \cos(\omega t)\Big]$$

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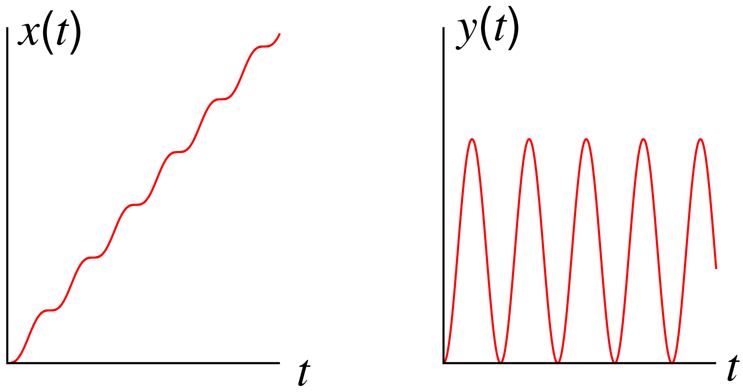
(e) Sketch $x(t)$ as a function of t and $y(t)$ as a function of t. Sketch the trajectiory of the particle in the x-y plane.

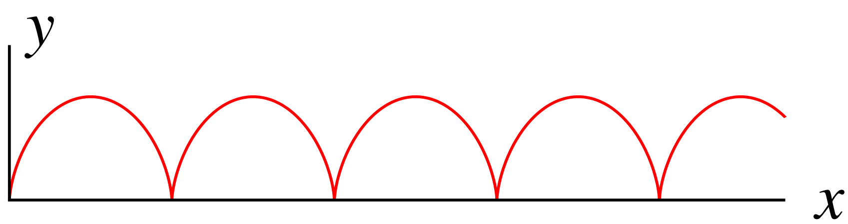
(Partial) Solution/Hint →

The x coordinate has dependence $\frac{E}{B}t - \frac{E}{B\omega}\sin(\omega t)$ on time, which meanst it is linearly increasing on average with oscillations at frequency *omega* on top of the linear increase. This is shown in the figure below.



The y coordinate has dependence $\frac{E}{B\omega}\left[1 - \cos(\omega t)\right]$ on time, which is simply a trigonometric oscillation shifted from zero. The function $1 - \cos(u)$ oscillates is between 0 and 2, starting at 2. Hence the function $y(t)$ oscillates between 0 and $\frac{E}{B\omega}$. This is also shown in the figure above.

As for the trajectory, one can see physically that the particle will move first in the y direction due the the E-field and be forced to turn around due to the B-field, but before it continues the full cyclotron path (due to B-field) it is stopped by the E-field, and then the whole motion starts again. The result is a series of semicircles, as shown below.



Of course, the trajectory is consistent with the $x(t)$ and $y(t)$ functions described/ derived above: $x(t)$ increases with some oscillations around its linear increase while $y(t)$ oscillates.

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6. Two infinite straight wires are placed parallel to each other, at a distance of 5 meters from each other. Each carries a current of 0.2 Amperes.

(a) Look up and report the value of μ_0 (permeability of free space) in SI units. Also calculate μ_0/π.

(Partial) Solution/Hint →

$$\mu_0 = 4\pi \times 10^{-7} N/A^2 \qquad \text{so that} \quad \mu_0/\pi = 4 \times 10^{-7} N/A^2$$

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(b) If the currents in the two wires are in the same direction, calculate the magnitude of the magnetic field at a point halfway between the two wires.

(Partial) Solution/Hint →

A carefully drawn figure will show that, midway between the wires, the magnetic fields due to the two wires cancel each other out. $B = 0$.

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(c) If the currents in the two wires are in opposite directions, calculate the magnitude of the magnetic field at a point halfway between the two wires.

(Partial) Solution/Hint →

A carefully drawn figure will show that, midway between the wires, the magnetic fields due to the two wires are equal and point in the same direction. Each wire creates field of strength

$$\frac{\mu_0 I}{2\pi d} = \frac{\mu_0}{\pi}\frac{I}{2d} = 4 \times 10^{-7} \times \frac{0.2}{2(5/2)}\text{Tesla} = 1.6 \times 10^{-8}\text{T}$$

Thus the total magnetic field has magnitude

$$B = 3.2 \times 10^{-8}\text{T}$$

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7. The magnetic field in some region is given by

$$\mathbf{B} = \frac{\mu_0}{\pi}\left(\frac{Cy}{x^2+y^2}\hat{\imath} + \frac{2x}{x^2+y^2}\hat{\jmath}\right) .$$

Use one of Maxwell's equations to determine the constant C.

(Partial) Solution/Hint →

The divergence of $\mathbf{B}$ is

$$\begin{aligned}\nabla\cdot\mathbf{B} &= \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \\ &= \frac{\partial}{\partial x}\left(\frac{\mu_0}{\pi}\frac{Cy}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{\mu_0}{\pi}\frac{2x}{x^2+y^2}\right)\end{aligned}$$

$$= \frac{\mu_0}{\pi}\left[Cy\times\frac{-2x}{(x^2+y^2)^2} + 2x\times\frac{-2y}{(x^2+y^2)^2}\right] = -\frac{\mu_0}{\pi}\frac{2xy}{(x^2+y^2)^2}(C+2)$$

Maxwell's second equation demands $\nabla\cdot\mathbf{B}=0$, so that

$$C+2=0 \quad\Longrightarrow\quad C=-2$$

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8. (a) A long straight wire carrying steady current 0.4 Amperes is placed on the x-y plane, passes through the origin, and makes an angle of $30° = \pi/6$ with the x-axis. The region is subjected to a uniform magnetic field of $\mathbf{B} = (10^{-2}T)\hat{\jmath}$, pointing in the y-direction. What is the force experienced by a segment of the wire of length 0.8m? Give both the magnitude and the direction of the force.

(Partial) Solution/Hint →

Please do make a drawing of the situation before attempting this problem. A sketch is very essential for figuring out the directions.

The force on an element $d\mathbf{l}$ of the wire is

$$d\mathbf{F} \;=\; Id\mathbf{l} \times \mathbf{B}$$

Since the wire and hence $d\mathbf{l}$ makes angle $30° = \pi/6$ with the x-axis, the angle with the magnetic field (in the y direction) is $60° = \pi/3$. Hence the cross product has magnitude

$$d\mathbf{l}B\sin(\pi/3) = \frac{\sqrt{3}}{2}Bd\mathbf{l}$$

and points in the positive z or the negative z direction, depending on the direction of the current.

We see that $dF = \frac{1}{2}IBd\mathbf{l}$ does not depend on the location of the element on the wire. Hence the magnitude of total force on a segment of length L of the wire is

$$F \;=\; \frac{\sqrt{3}}{2}IBL$$

For the particular values given,

$$F \;=\; \frac{\sqrt{3}}{2} \times 0.4 \times 10^{-2} \times 0.8\text{Newtons} \;\approx\; 2.77 \times 10^{-3}\text{N}$$

The direction of the force is in the positive z or the negative z direction, depending on the direction of the current.

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(b) A long straight wire carrying steady current I_1 runs along the y axis: The current flows from $y = -\infty$ to $y = +\infty$. The region is subjected to a magnetic field of $\mathbf{B} = \lambda y^2 \hat{k}$, pointing in the z-direction. Find the force experienced by the segment of the wire between $y = L$ and $y = 2L$.

(Note that the magnetic field is not uniform. Here λ is a positive constant.)

(Partial) Solution/Hint →

A sketch, e.g., of the x-y plane or the y-z plane, should help understand the directions mentioned below. (If you do not draw sketches, there is little chance of solving such a problem correctly.)

The force on an element $d\mathbf{l}$ of the wire is

$$d\mathbf{F} \;=\; I_1 d\mathbf{l} \times \mathbf{B}$$

Since the current and hence $d\mathbf{l}$ is in the positive y direction and the field is in the positive z direction, the force will be in the positive x direction ($\hat{\jmath} \times \hat{k} = \hat{\imath}$). Also

$$d\mathbf{l} \times \mathbf{B} = dy\hat{\jmath} \times \lambda y^2 \hat{k} = \lambda y^2 dy \hat{\imath}$$

Thus the magnitude of the total force on the segment is

$$F \;=\; I_1 \int_L^{2L} \lambda y^2 dy \;=\; I_1 \lambda \int_L^{2L} y^2 dy \;=\; I_1 \lambda \left(\frac{(2L)^3}{3} - \frac{(L)^3}{3} \right) \;=\; \frac{7}{3} I_1 \lambda L^3$$

and this force points in the positive positive x direction:

$$\mathbf{F} \;=\; \frac{7}{3} I_1 \lambda L^3 \hat{\imath}$$

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