Maynooth University
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MATHEMATICAL PHYSICS

SEMESTER 2

2018–2019

MP204
Electricity and Magnetism

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Time allowed: 2 hours

Answer **ALL FOUR** questions

1. A circular ring of radius R is placed in the x-y plane, with its center at the origin $(0,0,0)$. The ring is uniformly charged, with linear charge density λ.

   We will consider the electric field and electric potential due to this charged ring, at the point $(0,0,z)$ on the z axis, for varying z.

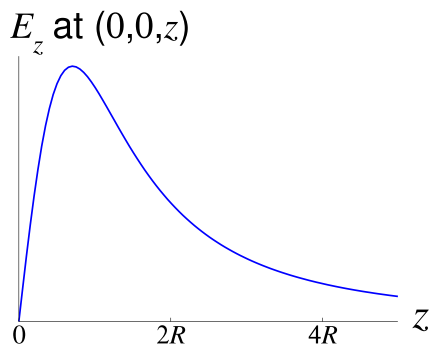
   (a) For $z \gg R$, the z-component of the electric field is well-approximated by $E_z = \dfrac{\alpha}{z^2}$, where α is a constant. Explain physically why this is the case. From your explanation, express α in terms of λ and R.

   [8 marks]

   (b) The z-component of the electric field at the point $(0,0,z)$ is shown in the plot, considering only positive values of z.

   Explain physically why E_z vanishes at $z = 0$.

   Plot E_z against z from $z = -5R$ to $z = +5R$, i.e., extend the curve to the negative part of the z-axis.



   [8 marks]

   (c) Plot E_x, the x-component of the electric field at $(0,0,z)$, as a function of z.

   [3 marks]

   (d) Plot the electric potential at $(0,0,z)$, as a function of z.

   [6 marks]

2. Reminder: An infinitely long straight wire produces a magnetic field of strength $\mu_0 I/(2\pi d)$ at a point at distance d from the wire.

   (a) Steady current I_1 flows through an infinitely long straight wire running parallel to the z axis. The wire lies in the x-z plane and intersects the x axis at the point $(-L, 0, 0)$. The current flows from $z = -\infty$ to $z = +\infty$.

   At the point $(2L, 0, 0)$ on the x-axis, find the magnitude of the magnetic field created by the current. Find also the x-, y-, and z-components of the magnetic field at this point.

   [9 marks]

   (b) An infinite wire carrying current I_2 runs along the y axis; the current flows from $y = -\infty$ to $y = +\infty$ through the origin. A rectangular loop lies in the x-y plane, with the four corners having coordinates $(2L, 0)$, $(4L, 0)$, $(4L, L)$, and $(2L, L)$. Find the magnetic flux through the rectangular loop.

   [16 marks]

3. (a) The magnetic field in some region is uniform: $\mathbf{B} = B_0\hat{k}$. At time $t = 0$, a charged particle (mass m, charge q) is at the origin and has velocity

$$\mathbf{v} = \alpha\hat{\imath} + w\hat{k}$$

At which subsequent times does the particle cross the z axis?

What is the z coordinate of the particle position at these instants?

Hint: The trajectory of the particle is helical.

[12 marks]

(b) A long straight wire carries steady current I_3. The wire cross-section is circular and has radius R. The current density is uniform $\left(= \dfrac{I_3}{\pi R^2}\right)$ inside the wire. Using Ampere's law, calculate the magnetic field *inside* the wire, at distance r from the axis of the wire ($r < R$).

If you use an Amperean loop, explain its position clearly.

[13 marks]

4. (a) In a region of free space, the electromagnetic fields are found to be

$$E_x = 0 \qquad E_y = E_0 \sin(kx + \omega t) \qquad E_z = 0$$
$$B_x = 0 \qquad B_y = 0 \qquad B_z = -B_0 \sin(kx + \omega t)$$

Use Maxwell's equations in free space to find how B_0 is related to E_0, and how ω is related to k.

[12 marks]

(b) An electromagnetic field is described by the scalar potential V and vector potential $\mathbf{A}$, given by

$$V \ = \ 3\alpha c^2 x t \ , \qquad \mathbf{A} \ = \ \alpha \left(c^2 t^2 - y^2\right) \hat{\imath} \ ,$$

where α is a positive constant, and $c = 1/\sqrt{\mu_0 \epsilon_0}$ is the speed of light.
Find the electric and magnetic fields.
Also find the current density.

[13 marks]

Posssibly useful Equations

- Electrostatics: $\mathbf{E} = -\nabla V \quad ; \quad V_{PQ} = -\int_Q^P \mathbf{E} \cdot d\mathbf{l}$

  Electric potential at $\mathbf{r}$ due to a point charge q_1 at $\mathbf{r_1}$: $V = \frac{q_1}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r_1}|}$

  Gauss' law: $\oint_\Sigma \mathbf{E} \cdot d\mathbf{S} = \frac{Q_{\text{enclosed}}}{\epsilon_0}$

- Magnetostatics: Ampere's law: $\int_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enclosed}}$

  Biot-Savart law: $d\mathbf{B} = \left(\frac{\mu_0 I}{4\pi}\right) \frac{d\mathbf{l}' \times \left(\widehat{\mathbf{r} - \mathbf{r}'}\right)}{|\mathbf{r} - \mathbf{r}'|^2}$

- Force on a charge: $\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$

  Magnetic force on a current element: $d\mathbf{F} = I d\mathbf{l} \times \mathbf{B}$

- Fields from potentials: $\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \quad , \quad \mathbf{B} = \nabla \times \mathbf{A}$

- The continuity equation: $\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$

- Faraday's law: $\mathcal{E} = -\frac{d\Phi_B}{dt}$ or $\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\Phi_B = -\frac{d}{dt}\int_\Sigma \mathbf{B} \cdot d\mathbf{S}$

- Maxwell's Equations:

  ① $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$ ② $\nabla \cdot \mathbf{B} = 0$

  ③ $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$

  ④ $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 (\mathbf{J} + \mathbf{J_D})$

- Poynting vector: $\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$ Speed of light: $c = 1/\sqrt{\mu_0 \epsilon_0}$

  Energy density of electromagnetic fields: $u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2$