MATHEMATICAL PHYSICS

SEMESTER 2

2017–2018

MP204
Electricity and Magnetism

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Time allowed: 2 hours

Answer **ALL** questions

1. (a) A positive charge $2q$ is placed at the origin, and two negative charges, $-q$ each, are placed at positions $(0, +d, 0)$ and $(0, -d, 0)$. Calculate the electric field due to this collection of three point charges at the point $(x_0, 0, 0)$ on the x-axis.

[12 marks]

(b) Consider the static electric field $\mathbf{E} = 4\alpha(x - L)\hat{\imath} + 2\alpha y\hat{\jmath}$.

Here α and L are positive constants.

Find the electric potential difference V_{QP} between the point Q at $(L, 2L, 0)$ and the point P at $(0, L, 0)$.

The planar surface Σ_1 lies in the y-z plane (perpendicular to the x axis) and has area A_1. Find the electric flux through this surface.

[23 marks]

2. Reminder: An infinitely long straight wire produces a magnetic field of strength $\mu_0 I/(2\pi d)$ at a point at distance d from the wire.

   (a) An infinite wire carrying current I runs along the x axis; the current flows from $x = -\infty$ to $x = +\infty$ through the origin. A rectangular loop lies in the xy plane, with the four corners having coordinates $(0, 2L)$, $(0, 4L)$, $(L, 4L)$, and $(L, 2L)$. Find the magnetic flux through the rectangular loop.

   [16 marks]

   (b) Steady current I flows through an infinitely long straight wire placed parallel to the z axis, from $z = -\infty$ to $z = +\infty$. The wire runs through the point $(2L, 0, 0)$.

   At the point $M(0, 2L, 0)$ in the x-y plane, find the magnitude of the magnetic field created by the current. Find also the x-, y-, and z-components of the magnetic field at this point.

   Now consider the point $N(0, 2L, -5L)$, not in the x-y plane. Find the x-, y-, and z- components of the magnetic field at this point.

   [14 marks]

3. (a) The current density in some region is

$$\mathbf{J} = \gamma \left(\frac{4y}{x^2 + y^2} \hat{\imath} - \frac{4x}{x^2 + y^2} \hat{\jmath} \right)$$

where γ is a positive constant.
How is the charge density ρ in this region changing?
Which physical principle is encoded by the continuity equation?

[7 marks]

(b) An electromagnetic field is described by the scalar potential V and vector potential $\mathbf{A}$, given by

$$V = \alpha c^2 z t \,, \qquad \mathbf{A} = \alpha \left(c^2 t^2 - x^2 \right) \hat{k} \,,$$

where α is a positive constant, and $c = 1/\sqrt{\mu_0 \epsilon_0}$ is the speed of light.
Find the electric and magnetic fields.
Also find the current density and the displacement current density.

[16 marks]

(c) Show how Ampere's law in differential form is inconsistent with the continuity equation.
Show how the introduction of Maxwell's correction term resolves this inconsistency.

[12 marks]

Posssibly useful Equations

- Electrostatics: $\mathbf{E} = -\nabla V$; $V_{PQ} = -\int_Q^P \mathbf{E} \cdot d\mathbf{l}$

  Electric potential at $\mathbf{r}$ due to a point charge q_1 at $\mathbf{r_1}$: $V = \dfrac{q_1}{4\pi\epsilon_0} \dfrac{1}{|\mathbf{r} - \mathbf{r_1}|}$

  Gauss' law: $\oint_\Sigma \mathbf{E} \cdot d\mathbf{S} = \dfrac{Q_{\text{enclosed}}}{\epsilon_0}$

- Magnetostatics: Ampere's law: $\int_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enclosed}}$

  Biot-Savart law: $d\mathbf{B} = \left(\dfrac{\mu_0 I}{4\pi}\right) \dfrac{d\mathbf{l}' \times \left(\widehat{\mathbf{r} - \mathbf{r}'}\right)}{|\mathbf{r} - \mathbf{r}'|^2}$

- Force on a charge: $\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$

  Magnitic force on a current element: $d\mathbf{F} = I d\mathbf{l} \times \mathbf{B}$

- Fields from potentials: $\mathbf{E} = -\nabla V - \dfrac{\partial \mathbf{A}}{\partial t}$, $\mathbf{B} = \nabla \times \mathbf{A}$

- The continuity equation: $\nabla \cdot \mathbf{J} + \dfrac{\partial \rho}{\partial t} = 0$

- Maxwell's Equations:

$$① \quad \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad ② \quad \nabla \cdot \mathbf{B} = 0$$

$$③ \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$④ \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 (\mathbf{J} + \mathbf{J_D})$$

- Poynting vector: $\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$ Speed of light: $c = 1/\sqrt{\mu_0 \epsilon_0}$

  Energy density of electromagnetic fields: $u = \dfrac{1}{2}\epsilon_0 E^2 + \dfrac{1}{2\mu_0} B^2$

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HINTS AND/OR PARTIAL SOLUTIONS

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1. Question 1.

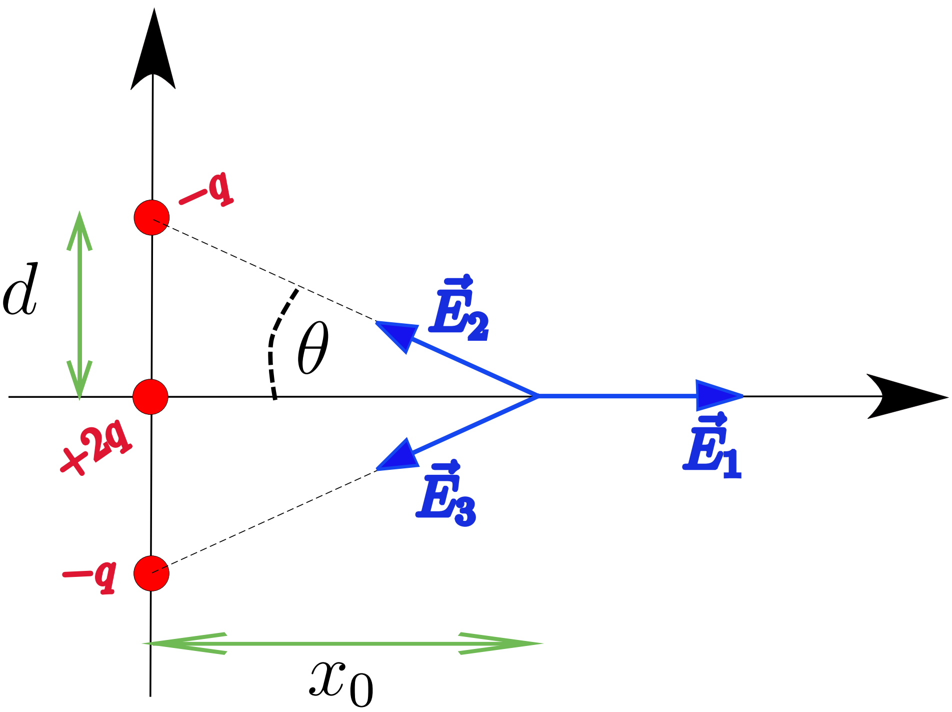
   (a) Question 1(a)

   A positive charge $2q$ is placed at the origin, and two negative charges, $-q$ each, are placed at positions $(0, +d, 0)$ and $(0, -d, 0)$. Calculate the electric field due to this collection of three point charges at the point $(x_0, 0, 0)$ on the x-axis.

   [12 marks]

   [Sample Answser:]

   A clear sketch or sketches of the geometry is essential.



   The point $(x_0, 0, 0)$ is at distance x_0 from the charge $2q$ and at distance $\sqrt{x_0^2 + d^2}$ from each of the two charges $-q$.

The charge $2q$ creates electric field of magnitude

$$E_1 = \frac{1}{4\pi\epsilon_0}\frac{2q}{x_0^2}$$

and the negative charges create electric fields of equal magnitude

$$E_2 = E_3 = \frac{1}{4\pi\epsilon_0}\frac{q}{x_0^2 + d^2}$$

From the symmetry of the problem, the y-components of the electric field should cancel and the we only need to worry about the x components. Since $\vec{E}_1$ is already in the x direction, it's x component is equal to its magnitude, $E_{1x} = E_1$.
For $\vec{E}_2$ and $\vec{E}_3$, the x components are:

$$E_{2x} = -E_2\cos\theta = -\frac{1}{4\pi\epsilon_0}\frac{q}{x_0^2 + d^2}\frac{x_0}{\sqrt{x_0^2 + d^2}} = -\frac{q}{4\pi\epsilon_0}\frac{x_0}{(x_0^2 + d^2)^{3/2}}$$

$$E_{3x} = -E_3\cos\theta = E_{2x}$$

Hence the total x component is

$$E_{1x} + E_{2x} + E_{3x} \;=\; E_1 + 2E_{2x} \;=\; \frac{2q}{4\pi\epsilon_0}\frac{1}{x_0^2} - \frac{2q}{4\pi\epsilon_0}\frac{x_0}{(x_0^2 + d^2)^{3/2}}$$

Net electric field is thus

$$\mathbf{E} \;=\; \frac{2q}{4\pi\epsilon_0}\left(\frac{1}{x_0^2} - \frac{x_0}{(x_0^2 + d^2)^{3/2}}\right)\hat{i}$$

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(b) Question 1(b)

Consider the static electric field $\mathbf{E} = 4\alpha(x-L)\hat{\imath} + 2\alpha y\hat{\jmath}$.

Here α and L are positive constants.

Find the electric potential difference V_{QP} between the point Q at $(L, 2L, 0)$ and the point P at $(0, L, 0)$.

The planar surface Σ_1 lies in the y-z plane (perpendicular to the x axis) and has area A_1. Find the electric flux through this surface.

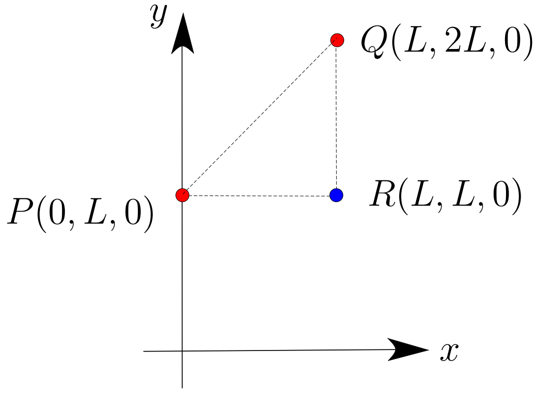
[23 marks]

[**Sample Answser:**]

Potential difference:

$$V_{QP} = -\int_P^Q \mathbf{E}\cdot d\mathbf{l}$$

The line integral may be performed along any path starting at P and ending at Q. Here is one possibility.



Define the intermediate point R at $(L, L, 0)$. Then we can take the path composed of the two straight-line segments PR and RQ:

$$\begin{aligned} V_{QP} = V_{RP} + V_{QR} &= -\int_P^R \mathbf{E}\cdot d\mathbf{l} - \int_R^Q \mathbf{E}\cdot d\mathbf{l} \\ &= -\int_{x=0}^{x=L} \mathbf{E}\cdot\ \hat{\imath}\ dx - \int_{y=L}^{y=2L} \mathbf{E}\cdot\ \hat{\jmath}\ dy \\ &= -\int_{x=0}^{x=L} 4\alpha(x-L)dx - \int_{y=L}^{y=2L} 2\alpha y dy \\ &= -\frac{4\alpha}{2}\left[(x-L)^2\right]_0^L - \frac{2\alpha}{2}\left[y^2\right]_L^{2L} \\ &= -2\alpha(-L^2) - \alpha 3L^2 = -\alpha L^2 \end{aligned}$$

Electric flux:

A perpendicular unit vector for the surface Σ_1 (anywhere on the surface) is in the x direction, i.e., $\hat{\imath}$ or $-\hat{\imath}$. Either would do — the problem description doesn't specify one of these. Let's choose $\hat{\imath}$. A surface element of Σ_1 is thus

$$d\mathbf{S} \;=\; \hat{\imath} dS.$$

Hence the flux is

$$\Phi \;=\; \int_{\Sigma_1} \mathbf{E}\cdot d\mathbf{S} \;=\; \int_{\Sigma_1} \mathbf{E}\cdot\hat{\imath} dS \;=\; \int_{\Sigma_1} 4\alpha(x-L)dS$$

Since the surface Σ_1 lies in the y-z plane, the x coordinate is zero at any point of the plane. Thus

$$\Phi \;=\; \int_{\Sigma_1} 4\alpha(0-L)dS \;=\; -4\alpha L\int_{\Sigma_1} dS \;=\; -4\alpha L A_1$$

The flux is thus $-4\alpha L A_1$ or $4\alpha L A_1$ (sign is not defined).

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2. Question 2.

   Reminder: An infinitely long straight wire produces a magnetic field of strength $\mu_0 I/(2\pi d)$ at a point at distance d from the wire.

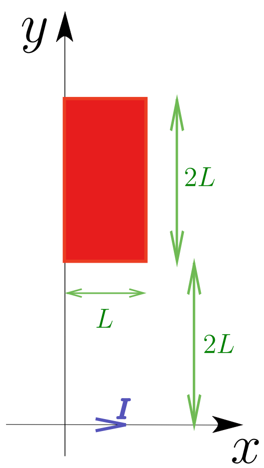
   (a) Question 2(a)

   An infinite wire carrying current I runs along the x axis; the current flows from $x = -\infty$ to $x = +\infty$ through the origin. A rectangular loop lies in the xy plane, with the four corners having coordinates $(0, 2L)$, $(0, 4L)$, $(L, 4L)$, and $(L, 2L)$. Find the magnetic flux through the rectangular loop.

   [16 marks]

   [Sample Answser:]

   A reasonable figure, showing a top view of the xy plane, is expected.
   (It's unlikely that an examinee gets this right without a sketch.)
   The rectangular loop is shown red-shaded in the sketch here.



   As long as we are considering points on the xy plane, the magnetic field due to the wire is $\dfrac{\mu_0 I}{2\pi y}$, because y is the perpendicular distance to the wire. The field (at any point of the xy plane) is perpendicular to the xy plane, hence perpendicular to the rectangular loop surface. We thus don't have to worry about angles in calculating the flux, $\mathbf{B} \cdot d\mathbf{S} = BdS$, where $d\mathbf{S}$ is the area element. The flux is therefore

$$\Phi = \int B dS = \int_0^L dx \int_{2L}^{4L} dy \frac{\mu_0 I}{2\pi y} = \frac{\mu_0 I}{2\pi} L \int_{2L}^{4L} dy \frac{1}{y}$$
$$= \frac{\mu_0 I L}{2\pi} \Big[\ln(4L) - \ln(2L) \Big] = \frac{\mu_0 I L}{2\pi} \ln 2$$

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(b) Question 2(b)

Steady current I flows through an infinitely long straight wire placed parallel to the z axis, from $z = -\infty$ to $z = +\infty$. The wire runs through the point $(2L, 0, 0)$.

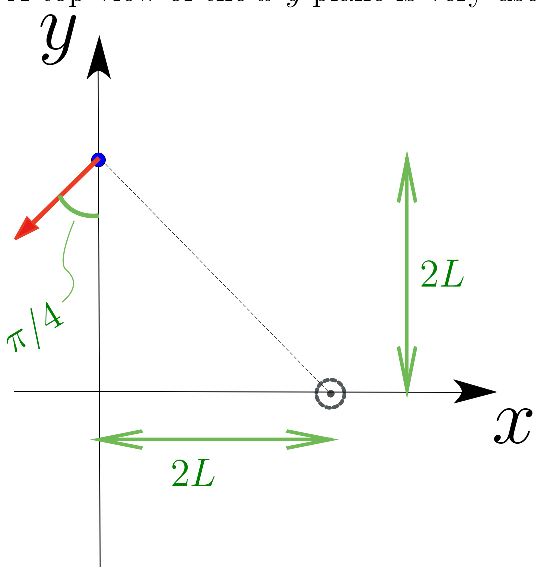
At the point $M(0, 2L, 0)$ in the x-y plane, find the magnitude of the magnetic field created by the current. Find also the x-, y-, and z-components of the magnetic field at this point.

Now consider the point $N(0, 2L, -5L)$, not in the x-y plane. Find the x-, y-, and z- components of the magnetic field at this point.

[14 marks]

[Sample Answser:]

A top view of the x-y plane is very useful.



The magnetic field at $M(0, 2L, 0)$ points perpendicular to the line joining M to the nearest point on the wire, as shown by the arrow in the figure above. The perpendicular distance to the wire is

$$\sqrt{(2L)^2 + (2L)^2} = 2\sqrt{2}L \quad \text{or} \quad \sqrt{8}L$$

Hence the magnitude of the magnetic field is

$$B = \frac{\mu_0 I}{2\pi(\text{distance})} = \frac{\mu_0 I}{4\pi\sqrt{2}L} \quad \text{or} \quad \frac{\mu_0 I}{2\pi\sqrt{8}L}$$

The field direction is in the x-y plane, hence $B_z = 0$. The x and y coordinates are

$$B_x = -B\sin(\pi/4) = -\frac{\mu_0 I}{4\pi\sqrt{2}L}\frac{1}{\sqrt{2}} = -\frac{\mu_0 I}{8\pi L}$$

and

$$B_y = -B\cos(\pi/4) = -\frac{\mu_0 I}{8\pi L}$$

Thus the magnetic field created at point M has the components

$$B_x = -\frac{\mu_0 I}{8\pi L}, \quad B_y = -\frac{\mu_0 I}{8\pi L}, \quad B_z = 0.$$

The point $N(0, 2L, -5L)$ lies directly under the point M, at the same distance and direction from the infinite wire. Therefore by symmetry the magnetic field created at N is exactly the same as that created at M, and hence has components

$$B_x = -\frac{\mu_0 I}{8\pi L}, \quad B_y = -\frac{\mu_0 I}{8\pi L}, \quad B_z = 0.$$

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3. Question 3.

   (a) Question 3(a)
   The current density in some region is

$$\mathbf{J} = \gamma \left(\frac{4y}{x^2+y^2}\hat{i} - \frac{4x}{x^2+y^2}\hat{j} \right)$$

   where γ is a positive constant.
   How is the charge density ρ in this region changing?
   Which physical principle is encoded by the continuity equation?

   [7 marks]

   [**Sample Answser:**]
   Using the continuity equation, we find

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= -\nabla \cdot \mathbf{J} = -4\gamma y \frac{\partial}{\partial x}\frac{1}{x^2+y^2} + 4\gamma x \frac{\partial}{\partial y}\frac{1}{x^2+y^2} \\ &= -4\gamma y \left(-\frac{2x}{(x^2+y^2)^2} \right) + 4\gamma x \left(-\frac{2y}{(x^2+y^2)^2} \right) \\ &= \frac{8\gamma xy - 8\gamma xy}{(x^2+y^2)^2} = 0 \end{aligned}$$

   The temporal derivative of the charge density is zero. This means the charge density in this region does not change at all.
   The continuity equation embodies the conservation of charge. When the charge in a region is reduced, it can only because of current flows out of that region.

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(b) Question 3(b)

An electromagnetic field is described by the scalar potential V and vector potential $\mathbf{A}$, given by

$$V = \alpha c^2 zt\,, \qquad \mathbf{A} = \alpha\left(c^2t^2 - x^2\right)\hat{k}$$

where α is a positive constant, and $c = 1/\sqrt{\mu_0\epsilon_0}$ is the speed of light.
Find the electric and magnetic fields.
Also find the current density and the displacement current density.

[16 marks]

[Sample Answser:]
Electric and magnetic fields:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -\alpha c^2 t\hat{k} - 2\alpha c^2 t\hat{k} = -3\alpha c^2 t\hat{k}$$

$$\begin{aligned} \mathbf{B} &= \nabla\times\mathbf{A} \\ &= \left(-\partial_x A_z\right)\hat{j} \\ &= 2\alpha x\hat{j} \end{aligned}$$

A has only a z-component, which depends on x but has no y-dependence. Hence $\nabla\times\mathbf{A} = \left(-\partial_x A_z\right)\hat{j}$

Displacement current density:

$$\mathbf{J_D} = \epsilon_0\frac{\partial \mathbf{E}}{\partial t} = -3\alpha c^2\epsilon_0\hat{k} = -\frac{3\alpha}{\mu_0}\hat{k}$$

We can now use Maxwell's 4th equation to calculate the current density:

$$\begin{aligned} \mu_0\mathbf{J} &= \nabla\times\mathbf{B} - \mu_0\mathbf{J_D} \\ &= \left(\partial_x B_y\right)\hat{k} - \mu_0\mathbf{J_D} \\ &= 2\alpha\hat{k} + 3\alpha\hat{k} = 5\alpha\hat{k} \end{aligned}$$

B has only a y-component, which has an x-dependence. Hence $\nabla\times\mathbf{B} = \left(\partial_x B_y\right)\hat{k}$.

so that

$$\mathbf{J} = \frac{5\alpha}{\mu_0}\hat{k} = 5\alpha c^2\epsilon_0\hat{k}$$

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(c) Question 3(c)
Show how Ampere's law in differential form is inconsistent with the continuity equation.
Show how the introduction of Maxwell's correction term resolves this inconsistency.

[12 marks]

[Sample Answser:]
Ampere's law in differential form is

$$\nabla \times \mathbf{B} \;=\; \mu_0 \mathbf{J}$$

Taking the divergence of both sides, we get

$$\nabla \cdot (\nabla \times \mathbf{B}) \;=\; \mu_0 \nabla \cdot \mathbf{J} \tag{1}$$

Since the divergence of the curl of any vector is zero, the left side is zero. However, according to the continuity equation, the right side is

$$\mu_0 \nabla \cdot \mathbf{J} \;=\; -\,\mu_0 \frac{\partial \rho}{\partial t}$$

which is not necessarily zero. Thus Ampere's law in differential form is inconsistent with the continuity equation.
To correct this inconsistency, the term

$$\mu_0 \mathbf{J_D} \;=\; \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

is added to the right side of Ampere's law. This is Maxwell's correction. Then the right side of Eq. (1) becomes

$$\begin{aligned} \mu_0 \nabla \cdot \mathbf{J} + \mu_0 \nabla \cdot \mathbf{J_D} \;&=\; -\,\mu_0 \frac{\partial \rho}{\partial t} \;+\; \mu_0 \epsilon_0 \nabla \cdot \left(\frac{\partial \mathbf{E}}{\partial t} \right) \\ &=\; -\,\mu_0 \frac{\partial \rho}{\partial t} \;+\; \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(\nabla \cdot \mathbf{E} \right) \\ &=\; -\,\mu_0 \frac{\partial \rho}{\partial t} \;+\; \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(\frac{\rho}{\epsilon_0} \right) \\ &=\; -\,\mu_0 \frac{\partial \rho}{\partial t} \;+\; \mu_0 \frac{\partial \rho}{\partial t} \;=\; 0 \end{aligned}$$

This is consistent with the left hand side of Eq. (1). Thus including Maxwell's correction makes Ampere's law consistent with the continuity equation.

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