Maynooth University
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MATHEMATICAL PHYSICS

SEMESTER 2, REPEAT

2016–2017

MP204
Electricity and Magnetism

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Time allowed: $1\frac{1}{2}$ hours

Answer **ALL** questions

1. (a) A positive charge q is placed at position $(0, +d/2, 0)$, and a negative charge $-q$ is placed at position $(0, -d/2, 0)$. This is an electric dipole.

   What is the magnitude of the dipole moment $\mathbf{p}$? What is its direction?

   Calculate the electric field at the point $(0, y_0, 0)$ on the y-axis, assuming $y_0 > d$.

   Approximate the expression for the electric field for the case $y_0 \gg d$. Express this large-distance electric field in terms of the vector $\mathbf{p}$.

   [15 marks]

   (b) An electromagnetic system is described by the fields

$$\mathbf{E} = K_0 x \sin(\omega t)\hat{\jmath} \qquad \mathbf{B} = \frac{K_0}{\omega}\cos(\omega t)\hat{k}$$

   Use Maxwell's equations to calculate the charge density ρ and the cuurent density $\mathbf{J}$. Show explicitly that the continuity equation is satisfied.

   [15 marks]

   (c) An infinite wire carrying current I runs along the y axis; the current flows from $y = -\infty$ to $y = +\infty$ through the origin. A square loop lies in the xy plane, with the four corners having coordinates $(2L, 0)$, $(3L, 0)$, $(3L, L)$, and $(2L, L)$. Find the magnetic flux through the square loop.

   Reminder: An infinitely long straight wire produces a magnetic field of strength $\mu_0 I/(2\pi d)$ at a point at distance d from the wire.

   [20 marks]

2. (a) A charged sphere has radius a_0. The charge density is

$$\rho = \begin{cases} \rho_0 (r/a_0) & \text{for } r < a_0 \\ 0 & \text{for } r > a_0 \end{cases}$$

where r is the distance from the center of the sphere.

What is the total charge carried by the sphere?

Using Gauss' law (Gauss' dielectric flux theorem), find the electric field inside and outside the sphere.

[20 marks]

(b) The vector potential in some region is given by

$$\mathbf{A} = \left(-\lambda \frac{z}{2}\right) \hat{j} + \left(\lambda \frac{y}{2}\right) \hat{k}$$

Find the magnetic field $\mathbf{B}$.

Consider adding ∇f to the vector potential, where f is any scalar function. Explain how the magnetic field changes due to this transformation.

Write down or derive a vector potential, different from the one above, which corresponds to the same magnetic field.

[14 marks]

(c) Faraday's law for electromagnetic induction can be written as

$$\oint_C \mathbf{E} \cdot d\mathbf{l} \;=\; -\frac{d\Phi_B}{dt}$$

What do the symbols C and Φ_B refer to?

Starting from this integral relation, derive Maxwell's third equation in differential form. If you use a theorem about vector integrals on the way, name the theorem explicitly.

[16 marks]

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HINTS AND/OR PARTIAL SOLUTIONS

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1. Question 1.

   (a) Question 1(a)

   A positive charge q is placed at position $(0, +d/2, 0)$, and a negative charge $-q$ is placed at position $(0, -d/2, 0)$. This is an electric dipole.

   What is the magnitude of the dipole moment $\mathbf{p}$? What is its direction?

   Calculate the electric field at the point $(0, y_0, 0)$ on the y-axis, assuming $y_0 > d$.

   Approximate the expression for the electric field for the case $y_0 \gg d$. Express this large-distance electric field in terms of the vector $\mathbf{p}$.

   [15 marks]

   [Sample Answser:]

   $p = qd$. The vector $\mathbf{p}$ points from the negative toward the positive charge, i.e., in the positive y direction.

   The electric field is in the positive y direction, with magnitude

$$E = E_+ - E_- = \frac{1}{4\pi\epsilon_0}\frac{q}{(y_0 - d/2)^2} - \frac{1}{4\pi\epsilon_0}\frac{q}{(y_0 + d/2)^2}$$
$$= \frac{q}{4\pi\epsilon_0}\frac{2y_0 d}{(y_0^2 - d^2/4)^2}$$

   In the large-distance limit:

$$E \approx \frac{1}{4\pi\epsilon_0}\frac{2qd}{y_0^3} = \frac{1}{4\pi\epsilon_0}\frac{2p}{y_0^3}$$

   Since the field and the dipole moment point in the same direction, this can be written as a vector equation:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2\mathbf{p}}{y_0^3}$$

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(b) Question 1(b)

An electromagnetic system is described by the fields

$$\mathbf{E} = K_0 x \sin(\omega t)\hat{\jmath} \qquad \mathbf{B} = \frac{K_0}{\omega}\cos(\omega t)\hat{k}$$

Use Maxwell's equations to calculate the charge density ρ and the cuurent density $\mathbf{J}$. Show explicitly that the continuity equation is satisfied.

[15 marks]

[**Sample Answser:**]

Charge density: use Maxwell's first eq:

$$\rho \ = \ \epsilon_0 \nabla \cdot \mathbf{E} \ = \ \epsilon_0 \frac{\partial E_y}{\partial y} \ = \ 0$$

Current density: use Maxwell's fourth eq:

$$\begin{aligned} \mu_0 \mathbf{J} \ &= \ \nabla \times \mathbf{B} \ - \ \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \ = \ 0 - \mu_0 \epsilon_0 K_0 x \omega \cos(\omega t)\hat{\jmath} \\ \mathbf{J} \ &= \ - \epsilon_0 K_0 \omega x \cos(\omega t)\hat{\jmath} \end{aligned}$$

Is the continuity equation satisfied? The results above indicate that

$$\frac{\partial \rho}{\partial t} \ = \ 0 \qquad \text{and} \qquad \nabla \cdot \mathbf{J} \ = \ \frac{\partial J_y}{\partial y} \ = \ 0$$

Hence the continuity equation ($\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} \ = \ 0$) is rather trivially satisfied.

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(c) Question 1(c)

An infinite wire carrying current I runs along the y axis; the current flows from $y = -\infty$ to $y = +\infty$ through the origin. A square loop lies in the xy plane, with the four corners having coordinates $(2L, 0)$, $(3L, 0)$, $(3L, L)$, and $(2L, L)$. Find the magnetic flux through the square loop.

Reminder: An infinitely long straight wire produces a magnetic field of strength $\mu_0 I/(2\pi d)$ at a point at distance d from the wire.

[20 marks]

[**Sample Answser:**]

A reasonable figure, top view of the xy plane, is expected. (It's unlikely that an examinee could get this right without a sketch.)

As long as we are considering points on the xy plane, the magnetic field due to the wire is $\frac{\mu_0 I}{2\pi x}$, because x is the perpendicular distance to the wire. The field is perpendicular to the xy plane, hence perpendicular to the square loop surface; we thus don't have to worry about angles in calculating the flux, $\mathbf{B} \cdot d\mathbf{S} = BdS$, where $d\mathbf{S}$ is the area element. The flux is therefore

$$\Phi = \int BdS = \int_0^L dy \int_{2L}^{3L} dx \frac{\mu_0 I}{2\pi x} = \frac{\mu_0 I}{2\pi} L \int_{2L}^{3L} dx \frac{1}{x}$$

$$= \frac{\mu_0 IL}{2\pi} \Big[\ln(3L) - \ln(2L) \Big] = \frac{\mu_0 IL}{2\pi} \ln\left(\frac{3}{2}\right)$$

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2. Question 2.

   (a) Question 2(a)
   A charged sphere has radius a_0. The charge density is

$$\rho = \begin{cases} \rho_0(r/a_0) & \text{for } r < a_0 \\ 0 & \text{for } r > a_0 \end{cases}$$

   where r is the distance from the center of the sphere.
   What is the total charge carried by the sphere?
   Using Gauss' law (Gauss' dielectric flux theorem), find the electric field inside and outside the sphere.

[20 marks]

[**Sample Answswer:**]
Total charge:

$$Q = \int_0^{a_0} 4\pi r^2 \rho(r) dr = 4\pi \frac{\rho_0}{a_0} \int_0^{a_0} r^3 dr = \frac{4\pi\rho_0}{a_0} \frac{a_0^4}{4} = \pi\rho_0 a_0^3$$

To find the electric field inside, we construct a spherical gaussian surface Σ of radius r concentric to the charged sphere. (A sketch is expected from the examinees.) The charge enclosed within this surface is

$$q_{\text{encl.}} = 4\pi \frac{\rho_0}{a_0} \int_0^r r^3 dr = \frac{\pi \rho_0 r^4}{a_0}$$

and the electric flux through the surface is

$$\oint_\Sigma \mathbf{E} \cdot d\mathbf{S} = E \times \text{area} = E(4\pi r^2)$$

because the electric field by symmetry has the same magnitude at all points of the surface and points radially outwards, hence is perpendicular to the surface (parallel to the surface-vector $d\mathbf{S}$) at all points of the surface. Thus Gauss's law gives

$$E(4\pi r^2) = \frac{q_{\text{encl.}}}{\epsilon_0} = \frac{\pi\rho_0 r^4}{\epsilon_0 a_0} \implies E = \left(\frac{\rho_0}{4\epsilon_0 a_0}\right) r^2$$

The calculation outside is very similar. The answer should be

$$E = \frac{1}{4\pi\epsilon_0 a_0} \frac{Q}{r^2} = \left(\frac{\rho_0 a_0^2}{4\epsilon_0}\right) \frac{1}{r^2}$$

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(b) Question 2(b)
The vector potential in some region is given by

$$\mathbf{A} = \left(-\lambda\frac{z}{2}\right)\hat{\jmath} + \left(\lambda\frac{y}{2}\right)\hat{k}$$

Find the magnetic field **B**.
Consider adding ∇f to the vector potential, where f is any scalar function. Explain how the magnetic field changes due to this transformation.
Write down or derive a vector potential, different from the one above, which corresponds to the same magnetic field.

[14 marks]

[**Sample Answser:**]

$$\mathbf{B} = \nabla\times\mathbf{A} \; = \; \left(\frac{\lambda}{2} - \left(-\frac{\lambda}{2}\right)\right)\hat{\imath} \; + \; 0\hat{\jmath} \; + \; 0\hat{k} \; = \; \lambda\hat{\imath}$$

What happens if ∇f is added? The magnetic field does not change, because the curl of a gradient is zero:

$$\nabla\times(\mathbf{A}+\nabla f) \;=\; \nabla\times\mathbf{A} \;+\; \nabla\times(\nabla f) \;=\; \nabla\times\mathbf{A}$$

i.e., the magnetic field corresponding to the vector potential $\mathbf{A}+\nabla f$ is the same as that due to the vector potential **A**.
Constructing another vector potential that gives the same magnetic field: can be done in several ways.
By inspection of the curl calculation above, we see that each component of the given vector potential contributes $(\lambda/2)\hat{\imath}$ to the magnetic field. We could instead choose all of it to come from one of the two components:

$$\mathbf{A} = (-\lambda z)\,\hat{\jmath} \qquad \text{or} \qquad \mathbf{A} = (\lambda y)\,\hat{k}\;.$$

More generally, any combination of the form

$$\mathbf{A} = (-\lambda z\alpha)\,\hat{\jmath} + (\lambda y(1-\alpha))\,\hat{k}$$

would have the same curl.

More straightforwardly, if one chooses any scalar field f and adds its gradient to the given vector potential, the resulting vector field would have the same curl:

$$\mathbf{A} = \left(-\lambda\frac{z}{2}\right)\hat{j} + \left(\lambda\frac{y}{2}\right)\hat{k} \;+\; \nabla f$$

One has now to simply choose a scalar function that has nonzero gradient. The number of possible examples are infinite.
How about $f = xyz$, which gives $\nabla f = yz\hat{i} + zx\hat{j} + xy\hat{k}$; then a different vector potential giving the same magnetic field is

$$\mathbf{A} \;=\; \left(-\lambda\frac{z}{2}\right)\hat{j} \;+\; \left(\lambda\frac{y}{2}\right)\hat{k} \;+\; yz\hat{i} \;+\; zx\hat{j} \;+\; xy\hat{k}$$

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(c) Question 2(c)
Faraday's law for electromagnetic induction can be written as

$$\oint_C \mathbf{E}\cdot d\mathbf{l} \;=\; -\frac{d\Phi_B}{dt}$$

What do the symbols C and Φ_B refer to?
Starting from this integral relation, derive Maxwell's third equation in differential form. If you use a theorem about vector integrals on the way, name the theorem explicitly.

[16 marks]

[**Sample Answser:**]
C is a closed curve — there may or may not be a wire loop on this curve.
Φ_B is the total flux through the surface enclosed within the closed curve C:

$$\Phi_B \;=\; \int_\Sigma \mathbf{B}\cdot d\mathbf{S}$$

where Σ is a surface having C as boundary. Actually, there are many such surfaces; any one of them will do.
To derive Maxwell's third equation, we use Stokes' theorem, which turns the line integral into a surface integral:

$$\oint_C \mathbf{E}\cdot d\mathbf{l} \;=\; \int_\Sigma (\nabla\times\mathbf{E})\cdot d\mathbf{S}$$

where Σ is a surface enclosed by the curve C.
Thus we have

$$\int_\Sigma (\nabla\times\mathbf{E})\cdot d\mathbf{S} \;=\; -\frac{d}{dt}\int_\Sigma \mathbf{B}\cdot d\mathbf{S} \;=\; -\int_\Sigma \frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{S}$$

Since this equality of integrals holds for ANY surface Σ, the integrands must be equal at all points of space. Thus

$$\nabla\times\mathbf{E} \;=\; -\frac{\partial\mathbf{B}}{\partial t}$$

which is Maxwell's third equation.

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