

MP201 – Vector Calculus & Fourier Series

Problem Set 11

Due by 5pm on Friday, 15 December 2017

1. For both of the following vector fields, closed surfaces and volumes, verify Gauss' theorem, namely, show that

$$\oiint_{\Sigma} \vec{F} \cdot d\vec{\sigma} = \iiint_{\mathcal{V}} (\vec{\nabla} \cdot \vec{F}) dV$$

where \mathcal{V} is the region of space enclosed by Σ .

- (a) $\vec{F} = y\hat{i} + 3y\hat{j} - \hat{k}$, Σ is the closed parabolic surface $z = 1 - x^2 - y^2$ where $0 \leq z \leq 1$.
- (b) $\vec{F} = 4y\hat{i} - 4x\hat{j} - 2z^2\hat{k}$, \mathcal{V} is the cylindrical region defined by $\sqrt{x^2 + y^2} \leq 2$, $0 \leq z \leq 1$.

2. Let \vec{A} be a vector field. Using Gauss' theorem, explain why the integral of $\nabla \times \vec{A}$ over any closed surface is zero.

3. Find the Fourier series of the following two functions:

(a) $f(x) = -|x|$ on $-1 \leq x \leq 1$, where $f(x+2) = f(x)$

(b) $g(x) = \begin{cases} 2 & \text{on } -2 \leq x < -1 \\ 0 & \text{on } -1 \leq x < 1 \\ 1 & \text{on } 1 \leq x \leq 2 \end{cases}$ where $g(x+4) = g(x)$.

4. Let $f(x)$ be the function that has period 2π and is equal to $-2x$ between $-\pi$ and π .

- (a) Find the Fourier series of $f(x)$.
- (b) Use your answer to (a) to prove the very famous result

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots \\ &= \frac{\pi}{4} \end{aligned}$$

(Hint: think of what happens when $x = \pi/2$ or $x = -\pi/2$.)

VECTOR CALCULUS

In Cartesian coordinates (x, y, z) with constant unit direction vectors $\hat{i}, \hat{j}, \hat{k}$:

- gradient of a scalar field $f(x, y, z)$:

$$\vec{\nabla} f = \hat{i} \frac{\partial f}{\partial x} + \hat{j} \frac{\partial f}{\partial y} + \hat{k} \frac{\partial f}{\partial z}$$

- divergence of a vector field $\vec{A}(x, y, z) = A_x(x, y, z)\hat{i} + A_y(x, y, z)\hat{j} + A_z(x, y, z)\hat{k}$:

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

- curl of a vector field $\vec{A}(x, y, z) = A_x(x, y, z)\hat{i} + A_y(x, y, z)\hat{j} + A_z(x, y, z)\hat{k}$:

$$\begin{aligned} \vec{\nabla} \times \vec{A} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \\ &= \hat{i} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{j} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{k} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \end{aligned}$$

- Laplacian of a scalar field $f(x, y, z)$:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

- Laplacian of a vector field $\vec{A}(x, y, z) = A_x(x, y, z)\hat{i} + A_y(x, y, z)\hat{j} + A_z(x, y, z)\hat{k}$:

$$\nabla^2 \vec{A} = \hat{i} (\nabla^2 A_x) + \hat{j} (\nabla^2 A_y) + \hat{k} (\nabla^2 A_z)$$

Vector calculus identities:

- Divergence of the gradient of a scalar field f :

$$\vec{\nabla} \cdot (\vec{\nabla} f) = \nabla^2 f$$

- Curl of the gradient of a scalar field f :

$$\vec{\nabla} \times (\vec{\nabla} f) = \vec{0}$$

- Divergence of the curl of a vector field \vec{A} :

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$$

- Curl of the curl of a vector field \vec{A} :

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$

- Stokes' theorem: if Σ is a surface that has boundary \mathcal{C} , then for any vector field \vec{F} , $\oint_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \iint_{\Sigma} (\vec{\nabla} \times \vec{F}) \cdot d\vec{\sigma}$.

- Gauss' theorem: if \mathcal{V} is a three-dimensional region of space bounded by a surface Σ , then for any vector field \vec{F} , $\oiint_{\Sigma} \vec{F} \cdot d\vec{\sigma} = \iiint_{\mathcal{V}} (\vec{\nabla} \cdot \vec{F}) dV$.

COORDINATE TRANSFORMATIONS

1. Two-dimensional coordinate systems:

Cartesian (x, y) to plane polar (r, ϕ) :

$$r = \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1} \left(\frac{y}{x} \right).$$

Plane polar (r, ϕ) to Cartesian (x, y) :

$$x = r \cos \phi, \quad y = r \sin \phi.$$

Area element:

$$dS = dx dy = r dr d\phi.$$

2. Three-dimensional coordinate systems:

- Cartesian (x, y, z) to cylindrical (r, ϕ, z) :

$$r = \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1} \left(\frac{y}{x} \right), \quad z = z.$$

Cylindrical (r, ϕ, z) to Cartesian (x, y, z) :

$$x = r \cos \phi, \quad y = r \sin \phi, \quad z = z$$

Volume element:

$$dV = dx dy dz = r dr d\phi dz.$$

- Cartesian (x, y, z) to spherical (r, θ, ϕ) :

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \tan^{-1} \left(\frac{\sqrt{x^2 + y^2}}{z} \right), \quad \phi = \tan^{-1} \left(\frac{y}{x} \right).$$

Spherical (r, θ, ϕ) to Cartesian (x, y, z) :

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta.$$

Volume element:

$$dV = dx dy dz = r^2 \sin \theta dr d\theta d\phi.$$

FOURIER SERIES

If $f(x)$ is a function with period $2L$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left(\frac{n\pi x}{L} \right) + \sum_{n=1}^{\infty} b_n \sin \left(\frac{n\pi x}{L} \right)$$

where

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \left(\frac{n\pi x}{L} \right) dx, \quad n = 0, 1, 2, \dots,$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \left(\frac{n\pi x}{L} \right) dx, \quad n = 1, 2, \dots$$