MP201 – Vector Calculus & Fourier Analysis

Problem Set 6

Due by 5pm on Friday, 10 November 2017

(Please write your name and tutorial day on the front of your assignment.)

1. In lecture, we showed that the line integral of the vector field $\vec{F} = 3xy\hat{\imath} - y^2\hat{\jmath}$ over the curve $\mathcal{C}$ given by the parabola $y = 2x^2$ from the origin to $(1, 2)$ was $-7/6$. We did this two ways, by replacing y by $2x^2$ in both $\vec{F}$ and $\mathrm{d}\vec{r}$, and by parametrising the curve via $x(t) = t$, $y(t) = 2t^2$ for $0 \le t \le 1$.

   Now we do it a third way (suggested by one of you): instead of thinking of y as a function of x, instead think of x as a function of y, namely, $x = \sqrt{y/2}$. Show that when you redo the line integral this way, you once again get $-7/6$.

2. Consider the vector field $\vec{A}(x, y) = (x^2 - y^2)\hat{\imath} - 2xy\hat{\jmath}$. Calculate the line integral of this field over the path that starts from the origin $(0, 0)$, goes straight to the point $(3, 0)$, then goes straight from there to the point $(2, 1)$.

3. One of the most important uses of the line integral in physics is determining the *work* done by a force in moving an object. More precisely, if $\vec{F}(\vec{r})$ is the force acting on an object at $\vec{r}$, then the total work W done by this force field in moving the object along some curve $\mathcal{C}$ is

$$W = \int_{\mathcal{C}} \vec{F} \cdot \mathrm{d}\vec{r}.$$

   (a) An object attached to the origin by a spring of spring constant k feels a force given by Hooke's law: $\vec{F} = -k\vec{r}$. Find the work done by the spring on the object if it moves from the origin to the point $(2, 1, 1)$ along the path

$$\vec{r}(t) = 2t\hat{\imath} + t\hat{\jmath} + \sqrt{t}\hat{k}$$

   from $t = 0$ to $t = 1$.

   (b) Suppose the force field is conservative. Explain why no work is done along any path that begins and ends at the same point.

4. Suppose $\vec{B}$ is the vector field

$$\vec{B}(x, y, z) = y^2 \cos z\hat{\imath} + 2xy \cos z\hat{\jmath} - xy^2 \sin z\hat{k}.$$

   (a) Show that $\vec{B}$ is a conservative field.

   (b) Compute the line integral of $\vec{B}$ along the spiral $x = 1 - \cos z$, $y = \sin z$ from $(0, 0, 0)$ to $(2, 0, \pi)$.

(c) Show that you get the same result when you integrate $\vec{B}$ along the straight line from the origin to $(2, 0, \pi)$.

(Hint for doing the integrals in (b): remenber that $\sin^2 z + \cos^2 z = 1$, $\mathrm{d}(\sin z) = \cos z \, \mathrm{d}z$ and $\mathrm{d}(\cos z) = -\sin z \, \mathrm{d}z$.)

VECTOR CALCULUS

In Cartesian coordinates (x, y, z) with constant unit direction vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$:

- gradient of a scalar field $f(x, y, z)$:

$$\vec{\nabla} f = \hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

- divergence of a vector field $\vec{A}(x, y, z) = A_x(x, y, z)\hat{\imath} + A_y(x, y, z)\hat{\jmath} + A_z(x, y, z)\hat{k}$:

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

- curl of a vector field $\vec{A}(x, y, z) = A_x(x, y, z)\hat{\imath} + A_y(x, y, z)\hat{\jmath} + A_z(x, y, z)\hat{k}$:

$$\begin{aligned} \vec{\nabla} \times \vec{A} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \\ &= \hat{\imath}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \end{aligned}$$

- Laplacian of a scalar field $f(x, y, z)$:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

- Laplacian of a vector field $\vec{A}(x, y, z) = A_x(x, y, z)\hat{\imath} + A_y(x, y, z)\hat{\jmath} + A_z(x, y, z)\hat{k}$:

$$\nabla^2 \vec{A} = \hat{\imath}\left(\nabla^2 A_x\right) + \hat{\jmath}\left(\nabla^2 A_y\right) + \hat{k}\left(\nabla^2 A_z\right)$$

Vector calculus identities:

- Laplacian of a scalar field f:
  $\nabla^2 f = \vec{\nabla} \cdot (\vec{\nabla} f)$
- Curl of the gradient of a scalar field f:
  $\vec{\nabla} \times (\vec{\nabla} f) = \vec{0}$
- Divergence of the curl of a vector field $\vec{A}$:
  $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$
- Curl of the curl of a vector field $\vec{A}$:
  $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$