MP201 – Vector Calculus & Fourier Analysis

Problem Set 5

Due by 5pm on Friday, 27 October 2017

(Please write your name and tutorial day on the front of your assignment.)

1. Consider the vector field

$$\vec{V}(x,y,z) \quad = \quad \hat{\jmath} e^{x^2+2y-z}.$$

   (a) Compute $\vec{\nabla}(\vec{\nabla}\cdot\vec{V})$.

   (b) Confirm the identity

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{V}\right) \quad = \quad \vec{\nabla}\left(\vec{\nabla}\cdot\vec{V}\right) - \nabla^2\vec{V}$$

   by explicitly computing the left-hand side (i.e. take two curls of $\vec{V}$) and showing it is equal to the right hand side. Remember that the Laplacian of a vector field $\vec{V} = V_x\hat{\imath} + V_y\hat{\jmath} + V_z\hat{k}$ is defined as

$$\nabla^2\vec{V} \quad = \quad \hat{\imath}\left(\nabla^2 V_x\right) + \hat{\jmath}\left(\nabla^2 V_y\right) + \hat{k}\left(\nabla^2 V_z\right).$$

2. Two extremely important vector fields that occur in nature are the electric field $\vec{E}(t,x,y,z)$ and the magnetic field $\vec{B}(t,x,y,z)$. As you'll see in MP204 next semester, these fields must obey four famous equations first formulated by James Clerk Maxwell in 1861-62 (called, predictably enough, "Maxwell's equations"). Two of these equations are

$$\vec{\nabla}\cdot\vec{B} \quad = \quad 0, \tag{1}$$

$$\vec{\nabla}\times\vec{E} \quad = \quad -\frac{\partial\vec{B}}{\partial t}. \tag{2}$$

   (a) Another field that appears in electromagnetism is the "vector potential" $\vec{A}$ which is related to the magnetic field via $\vec{B} = \vec{\nabla}\times\vec{A}$. Explain why this is consistent with Equation (1).

   (b) Use Equation (2) to explain why the vector field

$$\vec{E} + \frac{\partial\vec{A}}{\partial t}$$

   is a conservative vector field.

3. In empty space, the other two of Maxwell's equations are

$$\vec{\nabla}\cdot\vec{E} \quad = \quad 0, \tag{3}$$

$$\vec{\nabla}\times\vec{B} \quad = \quad \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}. \tag{4}$$

where $c = 3 \times 10^8\,\mathrm{m} \cdot \mathrm{s}^{-1}$ is the speed of light. Using all four of these equations, show that the electric and magnetic fields both satisfy the wave equations

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}, \qquad \nabla^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}.$$

Hint: take the curl of both sides of Equation (2) and use whatever vector calculus identities you think are relevant, and then do the same for Equation (4).

(This result was one of the great triumphs of Maxwell's equations, as it confirmed what many had suspected, that light was, in fact, an electromagnetic wave.)

VECTOR CALCULUS

In Cartesian coordinates (x, y, z) with constant unit direction vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$:

- gradient of a scalar field $f(x, y, z)$:

$$\vec{\nabla} f = \hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

- divergence of a vector field $\vec{A}(x, y, z) = A_x(x, y, z)\hat{\imath} + A_y(x, y, z)\hat{\jmath} + A_z(x, y, z)\hat{k}$:

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

- curl of a vector field $\vec{A}(x, y, z) = A_x(x, y, z)\hat{\imath} + A_y(x, y, z)\hat{\jmath} + A_z(x, y, z)\hat{k}$:

$$\begin{aligned} \vec{\nabla} \times \vec{A} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \\ &= \hat{\imath}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \end{aligned}$$

- Laplacian of a vector field $\vec{A}(x, y, z) = A_x(x, y, z)\hat{\imath} + A_y(x, y, z)\hat{\jmath} + A_z(x, y, z)\hat{k}$:

$$\nabla^2 \vec{A} = \hat{\imath}\left(\nabla^2 A_x\right) + \hat{\jmath}\left(\nabla^2 A_y\right) + \hat{k}\left(\nabla^2 A_z\right)$$

Vector calculus identities:

- Laplacian of a scalar field f:
  $\nabla^2 f = \vec{\nabla} \cdot (\vec{\nabla} f)$
- Curl of the gradient of a scalar field f:
  $\vec{\nabla} \times (\vec{\nabla} f) = \vec{0}$
- Divergence of the curl of a vector field $\vec{A}$:
  $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$
- Curl of the curl of a vector field $\vec{A}$:
  $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$