MP201 – Vector Calculus & Fourier Analysis

Problem Set 4

Due by 5pm on Friday, 20 October 2017

(Please write your name and tutorial day on the front of your assignment.)

1. Compute the Laplacians of the following scalar fields and determine their values at the point $(2, -1, 1)$:

$$\begin{aligned} P(x,y,z) &= y^2(x+z)^2, \\ Q(x,y,z) &= z - 9\sqrt{x^2+y^2}, \\ R(x,y,z) &= e^{-xyz}. \end{aligned}$$

2. (a) Compute the divergences of the following vector fields:

$$\begin{aligned} \vec{u}(x,y,z) &= \left(x^3+y^3\right)\hat{\imath} + 3xy^2\hat{\jmath} + 3zy^2\hat{k}, \\ \vec{v}(x,y,z) &= \frac{x\hat{\imath}+y\hat{\jmath}+z\hat{k}}{\left(x^2+y^2+z^2\right)^{3/2}}, \\ \vec{w}(x,y,z) &= \hat{\imath}\sin y + \hat{\jmath}\cos z - \hat{k}\tan x. \end{aligned}$$

   (b) Compute the curls of the vector fields given in (a).

3. Determine which (if any) of the following three vector fields are conservative:

$$\begin{aligned} \vec{A}(x,y,z) &= -z^2\hat{\jmath}, \\ \vec{B}(x,y,z) &= -y^2\hat{\imath} - 2xy\hat{\jmath} + z\hat{k}, \\ \vec{C}(x,y,z) &= x\hat{\imath} - y\hat{\jmath} + z\hat{k}. \end{aligned}$$

4. (a) Let $f(x,y,z)$ be any scalar field. Prove that

$$\vec{\nabla} \times \left(\vec{\nabla} f\right) = \vec{0},$$

   (b) Let $\vec{A}(x,y,z)$ be any scalar field. Prove that

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) = 0.$$

   Note: by "any" I mean to show that the above are true no matter what f or $\vec{A}$ might be. *Don't* assume that either one has any particular functional form.

VECTOR CALCULUS

In Cartesian coordinates (x, y, z) with constant unit direction vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$:

- gradient of a scalar field $f(x, y, z)$:

$$\vec{\nabla} f = \hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

- divergence of a vector field $\vec{A}(x, y, z) = A_x(x, y, z)\hat{\imath} + A_y(x, y, z)\hat{\jmath} + A_z(x, y, z)\hat{k}$:

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

- curl of a vector field $\vec{A}(x, y, z) = A_x(x, y, z)\hat{\imath} + A_y(x, y, z)\hat{\jmath} + A_z(x, y, z)\hat{k}$:

$$\begin{aligned} \vec{\nabla} \times \vec{A} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \\ &= \hat{\imath}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \end{aligned}$$

Vector calculus identities:

- Laplacian of a scalar field f:
  $\nabla^2 f = \vec{\nabla} \cdot (\vec{\nabla} f)$

- Curl of the gradient of a scalar field f:
  $\vec{\nabla} \times (\vec{\nabla} f) = \vec{0}$

- Divergence of the curl of a vector field $\vec{A}$:
  $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$

- Curl of the curl of a vector field $\vec{A}$:
  $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$