# Exponential and Weibull Distributions

# Exponential Random Variables

1. Poisson random variables are concerned with the number of ‘arrivals’, ‘flaws’ or ‘successes’ in a given interval of time or length.
2. The mean number of successes in a unit interval is $\lambda$.
3. The waiting time or distance between ‘successes’ is an exponential random variable with density

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

# The Exponential Distribution

1. The expected value and variance of an exponential distribution with parameter $\lambda$ are

$$E(X) = \frac{1}{\lambda}, \quad V(X) = \frac{1}{\lambda^2}.$$

2. It can be easily verified that for an exponential random variable with parameter $\lambda$,

$$\begin{aligned} P(X > x) &= e^{-\lambda x} \\ P(X \leq x) &= 1 - e^{-\lambda x}. \end{aligned}$$

for $x \geq 0$.

# Exponential Distribution - Example

**Example**

Log ons to a departmental computer network follow a Poisson distribution with a mean of 2 log ons per minute.

(i) What is the probability that there are no log ons in the next 3 minutes?

(ii) What is the probability that the time to the next log on is between 1 and 2.5 minutes?

(iii) What is the mean time between successive log ons to this network?

Here the unit of time is the minute.

# Exponential Distribution - Example

## Example

(i) Let $X$ denote the waiting time in minutes to the next log on. Then $X$ has an exponential distribution with $\lambda = 2$. Hence

$$P(X > 3) = e^{-2(3)} = 0.0025.$$

(ii)

$$\begin{aligned} P(1 \leq X \leq 2.5) &= P(X \leq 2.5) - P(X < 1) \\ &= e^{-2(1)} - e^{-2(2.5)} \\ &= 0.129 \end{aligned}$$

(iii) The mean of an exponential distribution with parameter $\lambda$ is $\frac{1}{\lambda}$ so the mean time between successive log ons is 0.5 minutes.

# Exponential Distribution - Example

## Example

The time between calls to a help desk is exponentially distributed with a mean time between calls of 5 minutes.

(i) What is the probability that there are no calls in an interval of 8 minutes?

(ii) What is the probability that there is at least 1 call in a 6 minute interval?

(iii) Determine the length of a time interval such that the probability of at least one call in the interval is 0.8.

(iv) If 5 minutes have passed without a call, what is the probability of a call in the next 6 minutes?

# Exponential Distribution - Example

## Example

(i) Let $X$ denote the time in minutes between calls. Then if the mean time between calls is 5 minutes, the parameter $\lambda = 0.2$. Hence

$$P(X > 8) = e^{-0.2(8)} = 0.202.$$

(ii) We are looking for the probability that $X \leq 6$.

$$P(X \leq 6) = 1 - e^{-0.2(6)} = 0.699.$$

(iii) We want to find $x$ such that $P(X \leq x) = 0.8$. This means

$$\begin{aligned} 1 - e^{-0.2x} &= 0.8 \\ e^{-0.2x} &= 0.2 \\ 0.2x &= log(5) \end{aligned}$$

So $x = 8.05$ minutes is the desired length of time.

**Example**

(iv) We want to find $P(X \leq 11 | X > 5)$. By the definition of conditional probability this is

$$\frac{P(5 < X \leq 11)}{P(X > 5)} = \frac{e^{-0.2(5)} - e^{-0.2(11)}}{e^{-0.2(5)}} = 0.699$$

But this is the same as $P(X \leq 6)$.

# Exponential Distribution - Lack of Memory

The last part of the previous exercise illustrates a key property of the exponential distribution - lack of memory.

Having waited for a call for 5 minutes, the probability that there are no calls in the next 8 minutes is the same as the probability of no calls in the 8 minutes from when we started waiting.

**Theorem**

*For an exponential random variable $X$, and $t_1 > 0, t_2 > 0$*

$$P(X \leq t_1 + t_2 | X > t_1) = P(X \leq t_2).$$

# The Weibull Distribution

1. If a device fails due to sudden shocks rather than due to slow wear and tear, the exponential distribution can be used to model its time to failure.
2. In situations where failure is due to slow deterioration over time, the Weibull distribution is a more appropriate model.

The probability density function of the Weibull distribution is

$$f(x) = \frac{\beta}{\delta}\left(\frac{x}{\delta}\right)^{\beta-1} e^{-(x/\delta)^{\beta}}$$

for $x > 0$.