ORTHOGONAL FUNCTIONS AND FOURIER SERIES

Orthogonal functions

A function can be considered to be a generalization of a vector. Thus the vector concepts like the inner product and orthogonality of vectors can be extended to functions.

Inner product

Consider the vectors $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ and $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ in $\mathbb{R}^3$, then the inner product or dot product of $\vec{u}$ and $\vec{v}$ is a real number, a **scalar**, defined as

$$(\vec{u}, \vec{v}) = u_1v_1 + u_2v_2 + u_2v_2 = \sum_{k=1}^{3} u_kv_k$$

The inner product

$$(\vec{u}, \vec{v}) = u_1 v_1 + u_2 v_2 + u_2 v_2 = \sum_{k=1}^{3} u_k v_k$$

possesses the following properties

(i) $(\vec{u}, \vec{v}) = (\vec{v}, \vec{u})$

(ii) $(k\vec{u}, \vec{v}) = k(\vec{u}, \vec{v})$

(iii) $(\vec{u}, \vec{u}) = 0$ if $\vec{u} = \vec{0}$ and $(\vec{u}, \vec{u}) > 0$ if $\vec{u} \neq \vec{0}$

(iv) $(\vec{u} + \vec{v}, \vec{w}) = (\vec{u}, \vec{w}) + (\vec{v}, \vec{w})$

Suppose that f_1 and f_2 are piecewise continuous functions defined on an interval $[a, b]$. Since the definite integral on the interval of the product $f_1(x)f_2(x)$ possesses properties (i) - (iv) above.

Definition: Inner product of functions

The **inner product** of two functions f_1 and f_2 on an interval $[a, b]$ is the number

$$(f_1, f_2) = \int_a^b f_1(x)\, f_2(x)\, dx$$

Definition: Orthogonal functions

Two functions f_1 and f_2 are said to be **orthogonal** on an interval $[a, b]$ if

$$(f_1, f_2) = \int_a^b f_1(x)\ f_2(x)\ dx = 0$$

<u>Example:</u> $f_1(x) = x^2$ and $f_2(x) = x^3$ are orthogonal on the interval $[-1, 1]$ since

$$(f_1, f_2) = \int_{-1}^{1} x^2\ .\ x^3\ dx = \left[\frac{1}{6}\ x^6\right]_{-1}^{1} = 0$$

Orthogonal sets

We are primarily interested in an infinite sets of orthogonal functions.

Definition: Orthogonal set

A set of real-valued functions $\{\phi_0(x), \phi_1(x), \phi_2(x), ...\}$ is said to be **orthogonal** on an interval $[a, b]$ if

$$(\phi_m, \phi_n) = \int_a^b \phi_m(x)\, \phi_n(x)\, dx = 0, \qquad m \neq n$$

Orthonormal sets

The norm $||\vec{u}||$ of a vector $\vec{u}$ can be expressed using the inner product:

$$(\vec{u}, \vec{u}) \quad = \quad ||\vec{u}||^2 \qquad \Rightarrow \qquad ||\vec{u}|| = \sqrt{(\vec{u}, \vec{u})}$$

Similarly the square norm of a function ϕ_n is $||\phi_n||^2 = (\phi_n, \phi_n)$, and so the **norm** is $||\phi_n|| = \sqrt{(\phi_n, \phi_n)}$. In other words, the square norm and the norm of a function ϕ_n in an orthogonal set $\{\phi_n(x)\}$ are, respectively,

$$||\phi_n||^2 = \int_a^b \phi_n^2(x)\, dx \quad \text{and} \quad ||\phi_n|| = \sqrt{\int_a^b \phi_n^2(x)\, dx}$$

Example 1: Orthogonal set of functions

Show that the set $\{1, \cos x, \cos 2x, ...\}$ is orthogonal on the interval $[-\pi, \pi]$:

$\phi_0 = 1$, $\phi_n = \cos nx$

$$
\begin{aligned}
(\phi_0, \phi_n) &= \int_{-\pi}^{\pi} \phi_0(x)\, \phi_n(x)\, dx = \int_{-\pi}^{\pi} \cos nx\, dx \\
&= \left[\frac{1}{n} \sin nx\right]_{-\pi}^{\pi} = \frac{1}{n}[\sin n\pi - \sin(-n\pi)] = 0
\end{aligned}
$$

and for $m \neq n$, using the triangle identity

$$
\begin{aligned}
(\phi_m, \phi_n) &= \int_{-\pi}^{\pi} \phi_m(x)\, \phi_n(x)\, dx = \int_{-\pi}^{\pi} \cos mx \cos nx\, dx \\
&= \frac{1}{2} \int_{-\pi}^{\pi} [\cos(m+n)x + \cos(m-n)x]\ dx \\
&= \frac{1}{2} \left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi} = 0
\end{aligned}
$$

Example 2: Norms

Find the norms of the functions given in the Example 1 above.

$$\begin{aligned}
||\phi_0||^2 &= \int_{-\pi}^{\pi} dx = 2\pi \\
||\phi_0|| &= \sqrt{2\pi} \\
||\phi_n||^2 &= \int_{-\pi}^{\pi} \cos^2 nx\, dx = \frac{1}{2}\int_{-\pi}^{\pi} [1 + \cos 2nx]\, dx = \pi \\
||\phi_n|| &= \sqrt{\pi}
\end{aligned}$$

Any orthogonal set of nonzero functions $\{\phi_n(x)\}$, $n = 0, 1, 2...$ can be *normalized*, i.e. made into an orthonormal set.

Example: An orthonormal set on the interval $[-\pi, \pi]$:

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, ...\right\}$$

Vector analogy

Suppose $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are three mutually orthogonal nonzero vectors in $\mathbb{R}^3$. Such an orthogonal set can be used as a basis for $\mathbb{R}^3$, that is, any three-dimensional vector can be written as a linear combination

$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$$

where c_i, $i = 1, 2, 3$ are scalars called the components of the vector. Each component can be expressed in terms of $\vec{u}$ and the corresponding vector $\vec{v}_i$:

$$\begin{aligned}
(\vec{u}, \vec{v}_1) &= c_1(\vec{v}_1, \vec{v}_1) + c_2(\vec{v}_2, \vec{v}_1) + c_3(\vec{v}_3, \vec{v}_1) = c_1||\vec{v}_1||^2 + c_2.0 + c_3.0 \\
(\vec{u}, \vec{v}_2) &= c_2||\vec{v}_2||^2 \\
(\vec{u}, \vec{v}_3) &= c_3||\vec{v}_3||^2
\end{aligned}$$

Hence

$$c_1 = \frac{(\vec{u}, \vec{v}_1)}{\|\vec{v}_1\|^2} \qquad c_2 = \frac{(\vec{u}, \vec{v}_2)}{\|\vec{v}_2\|^2} \qquad c_3 = \frac{(\vec{u}, \vec{v}_3)}{\|\vec{v}_3\|^2}$$

and

$$\vec{u} = \frac{(\vec{u}, \vec{v}_1)}{\|\vec{v}_1\|^2}\,\vec{v}_1 + \frac{(\vec{u}, \vec{v}_2)}{\|\vec{v}_2\|^2}\,\vec{v}_2 + \frac{(\vec{u}, \vec{v}_3)}{\|\vec{v}_3\|^2}\,\vec{v}_3 = \sum_{n=1}^{3} \frac{(\vec{u}, \vec{v}_n)}{\|\vec{v}_n\|^2}\,\vec{v}_n$$

Orthogonal series expansion

Suppose $\{\phi_n(x)\}$ is an infinite orthogonal set of functions on an interval $[a, b]$. If $y = f(x)$ is a function defined on the interval $[a, b]$, we can determine a set of coefficients c_n, $n = 0, 1, 2, ...$ for which

$$f(x) = c_0\phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + ... + c_n\phi_n(x) + ... \tag{1}$$

using the inner product. Multiplying the expression above by $\phi_m(x)$ and integrating over the interval $[a, b]$ gives

$$\begin{aligned}
\int_a^b & f(x)\,\phi_m(x)\,dx = \\
= \quad & c_0 \int_a^b \phi_0(x)\,\phi_m(x)\,dx + c_1 \int_a^b \phi_1(x)\,\phi_m(x)\,dx + ... + c_n \int_a^b \phi_n(x)\,\phi_m(x)\,dx + ... \\
= \quad & c_0(\phi_0, \phi_m) + c_1(\phi_1, \phi_m) + ... + c_n(\phi_n, \phi_m) + ...
\end{aligned}$$

By orthogonality, each term on r.h.s. is zero *except* when $m = n$, in which case we have

$$\int_a^b f(x)\,\phi_n(x)\,dx = c_n \int_a^b \phi_n^2(x)\,dx$$

The required coefficients are then

$$c_n = \frac{\int_a^b f(x)\,\phi_n(x)\,dx}{\int_a^b \phi_n^2(x)\,dx}, \quad n = 0,\ 1,\ 2...$$

In other words

$$f(x) = \sum_{n=0}^{\infty} c_n\,\phi_n(x) = \sum_{n=0}^{\infty} \frac{\int_a^b f(x)\,\phi_n(x)\,dx}{\|\phi_n(x)\|^2}\,\phi_n(x) = \sum_{n=0}^{\infty} \frac{(f, \phi_n)}{\|\phi_n(x)\|^2}\,\phi_n(x)$$

Definition: Orthogonal set / weight function

A set of real-valued functions $\{\phi_0(x), \phi_1(x), \phi_2(x), ...\}$ is said to be **orthogonal with respect to a weight function** $w(x)$ on an interval $[a, b]$ if

$$\int_a^b w(x)\ \phi_m(x)\ \phi_n(x)\ dx = 0, \quad m \neq n$$

The usual assumption is that $w(x) > 0$ on the interval of orthogonality $[a, b]$.

For example, the set $\{1, \cos x, \cos 2x, ...\}$ is orthogonal w.r.t. the weight function $w(x) = 1$ on the interval $[-\pi, \pi]$.

If $\{\phi_n(x)\}$ is orthogonal w.r.t. a weight function $w(x)$ on the interval $[a, b]$, them multiplying the expansion (1),$f(x) = c_0\phi_0(x) + c_1\phi_1(x)...$, by $w(x)$ and integrating by parts yields

$$c_n = \frac{\int_a^b \ f(x)\ w(x)\ \phi_n(x)\ dx}{||\phi_n(x)||^2}$$

where

$$||\phi_n(x)||^2 = \int_a^b \ w(x)\ \phi_n^2(x)\ dx$$

The series

$$f(x) = \sum_{n=0}^{\infty} c_n \, \phi_n(x) \tag{2}$$

with the coefficients given either by

$$c_n = \frac{\int_a^b f(x) \, \phi_n(x) \, dx}{||\phi_n(x)||^2} \quad \text{or} \quad c_n = \frac{\int_a^b f(x) \, w(x) \, \phi_n(x) \, dx}{||\phi_n(x)||^2} \tag{3}$$

is said to be an **orthogonal series expansion** of f or a **generalized Fourier series**.

Complete sets

We shall assume that an orthogonal set $\{\phi_n(x)\}$ is **complete**. Under this assumption f can not be orthogonal to each ϕ_n of the orthogonal set.

Fourier series

Trigonometric series

The set of functions

$$\left\{1,\ \cos\frac{\pi}{p}x,\ \cos\frac{2\pi}{p}x,\ \cos\frac{3\pi}{p}x, ...,\ \sin\frac{\pi}{p}x,\ \sin\frac{2\pi}{p}x,\ \sin\frac{3\pi}{p}x, ...\right\}$$

is orthogonal on the interval $[-p,\ p]$.

We can expand a function f defined on $[-p,\ p]$ into the trigonometric series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\ \cos\frac{n\pi}{p}x + b_n\ \sin\frac{n\pi}{p}x\right) \tag{4}$$

Determining the ocefficients $a_0,\ a_1,\ a_2, ...,\ b_1,\ b_2, ...$:
We multiply by 1 (the first function in our orthogonal set) and integrate both sides of the expansion (4) from $-p$ to p

$$\int_{-p}^{p} f(x)\,dx = \frac{a_0}{2}\int_{-p}^{p} dx + \sum_{n=1}^{\infty}\left(a_n \int_{-p}^{p} \cos\frac{n\pi}{p}x\,dx + b_n \int_{-p}^{p} \sin\frac{n\pi}{p}x\,dx\right)$$

Since $\cos(n\pi x/p)$ and $\sin(n\pi x/p)$, $n \geq 1$, are orthogonal to 1 on the interval, the r.h.s. reduces as follows

$$\int_{-p}^{p} f(x)\,dx = \frac{a_0}{2}\int_{-p}^{p} dx = \left[\frac{a_0}{2}x\right]_{-p}^{p} = pa_0$$

Solving for a_0 yields

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx \tag{5}$$

Now, we multiply (4) by $\cos(m\pi x/p)$ and integrate

$$\begin{aligned}\int_{-p}^{p} f(x)\ \cos\frac{m\pi}{p}x\,dx &= \frac{a_0}{2}\int_{-p}^{p}\cos\frac{m\pi}{p}x\,dx \\ &+ \sum_{n=1}^{\infty}\left(a_n\int_{-p}^{p}\cos\frac{m\pi}{p}x\ \cos\frac{n\pi}{p}x\,dx + b_n\int_{-p}^{p}\cos\frac{m\pi}{p}x\ \sin\frac{n\pi}{p}x\,dx\right)\end{aligned}$$

By orthogonality, we have

$$\begin{aligned}&\int_{-p}^{p}\cos\frac{m\pi}{p}x\,dx = 0, \qquad m > 0 \\ &\int_{-p}^{p}\cos\frac{m\pi}{p}x\ \sin\frac{n\pi}{p}x\,dx = 0 \\ &\int_{-p}^{p}\cos\frac{m\pi}{p}x\ \cos\frac{n\pi}{p}x\,dx = p\,\delta_{mn}\end{aligned}$$

where the Kronecker delta $\delta_{mn} = 0$ if $m \neq n$, and $\delta_{mn} = 1$ if $m = n$.

Thus the equation (6) above reduces to

$$\int_{-p}^{p} f(x)\ \cos\frac{n\pi}{p}x\,dx = a_n p$$

and so

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\ \cos\frac{n\pi}{p}x\,dx \tag{6}$$

Finally, multiplying (4) by $\sin(m\pi x/p)$, integrating and using the orthogonality relations

$$\int_{-p}^{p} \sin \frac{m\pi}{p}x \, dx = 0, \qquad m > 0$$
$$\int_{-p}^{p} \sin \frac{m\pi}{p}x \; \cos \frac{n\pi}{p}x \, dx = 0$$
$$\int_{-p}^{p} \sin \frac{m\pi}{p}x \; \sin \frac{n\pi}{p}x \, dx = p \, \delta_{mn}$$

we find that

$$b_n = \frac{1}{p} \int_{-p}^{p} f(x) \; \sin \frac{n\pi}{p}x \, dx \tag{7}$$

The trigonometric series (4) with coefficients a_0, a_n, and b_n defined by (5), (6) and (7), respectively are said to be the **Fourier series** of the function f. The coefficients obtained from (5), (6) and (7) are referred as **Fourier coefficients** of f.

Definition: Fourier series

The **Fourier series** of a function f defined on the interval $(-p, p)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right) \tag{8}$$

where

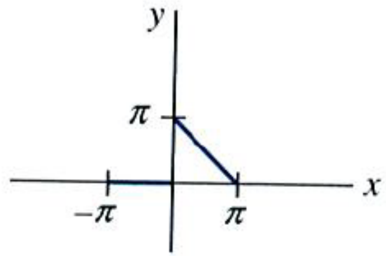
$$a_0 = \frac{1}{p} \int_{-p}^{p} f(x)\, dx \tag{9}$$

$$a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi}{p} x\, dx \tag{10}$$

$$b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin \frac{n\pi}{p} x\, dx \tag{11}$$

Example 1: Expansion in a Fourier series

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$$



With $p = \pi$ we have from (9) and (10) that

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi}(\pi - x)\,dx\right] = \frac{1}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi} = \frac{\pi}{2} \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi}(\pi - x)\,\cos nx\,dx\right] \\
&= \frac{1}{\pi}\left\{\left[(\pi - x)\frac{\sin nx}{n}\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\sin nx\,dx\right\} \\
&= -\frac{1}{n\pi}\left[\frac{\cos nx}{n}\right]_0^{\pi} \\
&= \frac{-\cos n\pi + 1}{n^2\pi} = \frac{1 - (-1)^n}{n^2\pi}
\end{aligned}$$

Similarly, we find from (11)

$$b_n = \frac{1}{n} \int_0^{\pi} (\pi - x) \sin nx \, dx = \frac{1}{n}$$

The function $f(x)$ is thus expanded as

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left\{ \frac{1 - (-1)^n}{n^2 \pi} \cos nx + \frac{1}{n} \sin nx \right\} \tag{12}$$

We also note that

$$1 - (-1)^n = \begin{cases} 0, & n \text{ even} \\ 2, & n \text{ odd.} \end{cases}$$

Convergence of a Fourier series

Theorem: Conditions for convergence

Let f and f' be piecewise continuous on the interval $(-p, p)$; that is, let f and f' be continuous except at a finite number of points in the interval and have only finite discontinuities at these points. Then the Fourier series of f on the interval converges to $f(x)$ at a point of continuity. At a point of discontinuity, the Fourier series converges to the average

$$\frac{f(x+) + f(x-)}{2},$$

where $f(x+)$ and $f(x-)$ denote the limit of f at x from the right and from the left, respectively.

Example 2: Convergence of a point of discontinuity

The expansion (12) of the function (Example 1)

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$$

will converge to $f(x)$ for every x from the interval $(-\pi, \pi)$ except at $x = 0$ where it will converge to

$$\frac{f(0+) + f(0-)}{2} = \frac{\pi + 0}{2} = \frac{\pi}{2}.$$

Periodic extension

Observe that each of the functions in the basis set

$$\left\{1,\ \cos\frac{\pi}{p}x,\ \cos\frac{2\pi}{p}x,\ \cos\frac{3\pi}{p}x, ...,\ \sin\frac{\pi}{p}x,\ \sin\frac{2\pi}{p}x,\ \sin\frac{3\pi}{p}x, ...\right\}$$

has a different fundamental period $2p/n$, $n \geq 1$, but since a positive integer multiple of a period is also a period, we see that all the functions have in common the period $2p$. Thus the r.h.s. of

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\ \cos\frac{n\pi}{p}x + b_n\ \sin\frac{n\pi}{p}x\right)$$

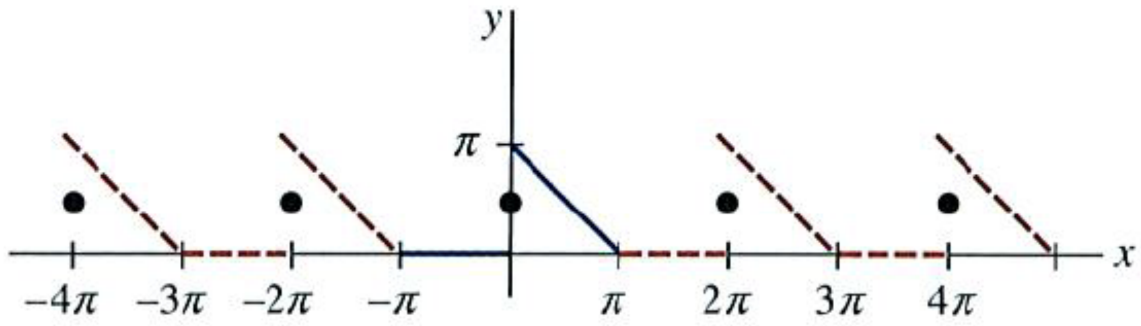
is $2p$-periodic; indeed $2p$ is the fundamental period of the sum.

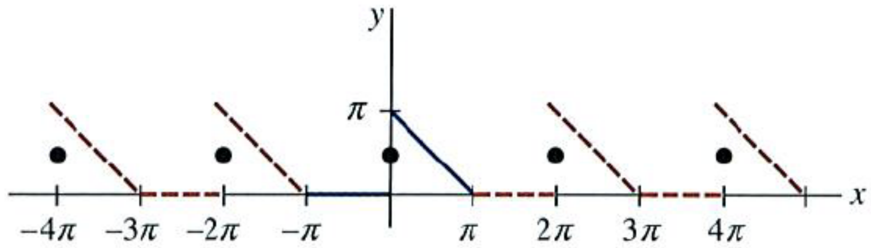
We conclude that a Fourier series not only represents the function on the interval $(-p, p)$ but also gives the **periodic extension** of f outside this interval.

We can now apply the Theorem on conditions for convergence to the periodic extension or simply assume the function is periodic, $f(x+T) = f(x)$, with period $T = 2p$ from the outset. When f is piecewise continuous and the right- and left-hand derivatives exist at $x = -p$ and $x = p$, respectively, then the Fourier series converges to the average $[f(p-)+f(p+)]/2$ at these points and also to this value extended periodically to $\pm 3p, \pm 5p, \pm 7p$, and so on.

Example: The Fourier series of the function $f(x)$ in the Example 1 converges to the periodic extension of the function on the entire x-axis. At $0, \pm 2\pi, \pm 4\pi$, ..., and at $\pm\pi, \pm 3\pi, \pm 5\pi$, ... , the series converges to the values

$$\frac{f(0+)+f(0-)}{2} = \frac{\pi}{2} \quad \text{and} \quad \frac{f(\pi+)+f(\pi-)}{2} = 0$$

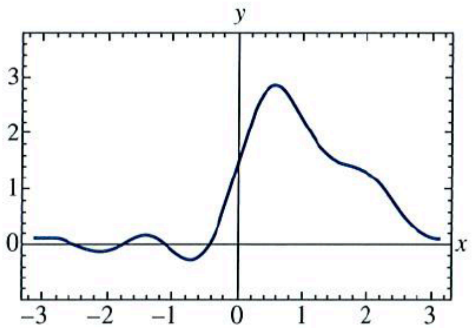




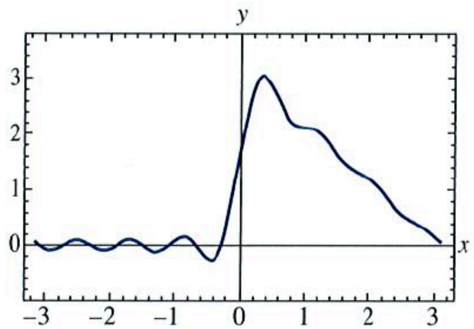
Sequence of partial sums

It is interesting to see how the sequence of partial sums $\{S_N(x)\}$ of a Fourier series approximates a function. For example

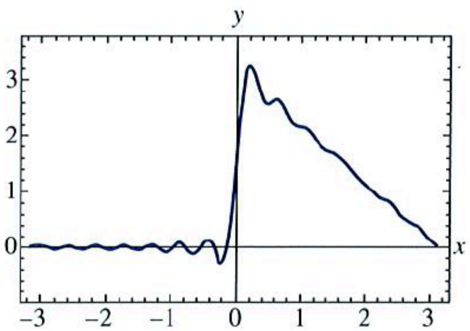
$$S_1(x) = \frac{\pi}{4}, \qquad S_2(x) = \frac{\pi}{4} + \frac{2}{\pi}\cos x + \sin x, \qquad S_3(x) = \frac{\pi}{4} + \frac{2}{\pi}\cos x + \sin x + \frac{1}{2}\sin 2x$$



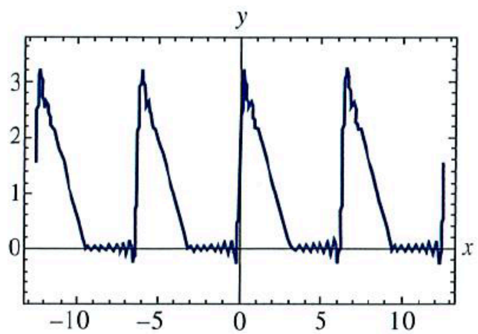
(a) $S_5(x)$ on $(-\pi, \pi)$



(b) $S_8(x)$ on $(-\pi, \pi)$



(c) $S_{15}(x)$ on $(-\pi, \pi)$



(d) $S_{15}(x)$ on $(-4\pi, 4\pi)$