ORTHOGONAL FUNCTIONS AND FOURIER SERIES

Orthogonal functions

A function can be considered to be a generalization of a vector. Thus the vector concepts like the inner product and orthogonality of vectors can be extended to functions.

Inner product

Consider the vectors \( \vec{u} = u_1 \hat{i} + u_2 \hat{j} + u_3 \hat{k} \) and \( \vec{v} = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k} \) in \( \mathbb{R}^3 \), then the inner product or dot product of \( \vec{u} \) and \( \vec{v} \) is a real number, a scalar, defined as

\[
(\vec{u}, \vec{v}) = u_1 v_1 + u_2 v_2 + u_3 v_3 = \sum_{k=1}^{3} u_k v_k
\]
The inner product
\[(\vec{u}, \vec{v}) = u_1 v_1 + u_2 v_2 + u_2 v_2 = \sum_{k=1}^{3} u_k v_k\]
possesses the following properties

(i) \[(\vec{u}, \vec{v}) = (\vec{v}, \vec{u})\]
(ii) \[(k\vec{u}, \vec{v}) = k(\vec{u}, \vec{v})\]
(iii) \[(\vec{u}, \vec{u}) = 0 \text{ if } \vec{u} = \vec{0} \text{ and } (\vec{u}, \vec{u}) > 0 \text{ if } \vec{u} \neq \vec{0}\]
(iv) \[(\vec{u} + \vec{v}, \vec{w}) = (\vec{u}, \vec{w}) + (\vec{v}, \vec{w})\]
Suppose that $f_1$ and $f_2$ are piecewise continuous functions defined on an interval $[a, b]$. Since the definite integral on the interval of the product $f_1(x)f_2(x)$ possesses properties (i) - (iv) above.

**Definition: Inner product of functions**

The **inner product** of two functions $f_1$ and $f_2$ on an interval $[a, b]$ is the number

$$ (f_1, f_2) = \int_a^b f_1(x) f_2(x) \, dx $$
Definition: Orthogonal functions

Two functions $f_1$ and $f_2$ are said to be **orthogonal** on an interval $[a, b]$ if

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x) \, dx = 0$$

Example: $f_1(x) = x^2$ and $f_2(x) = x^3$ are orthogonal on the interval $[-1, 1]$ since

$$(f_1, f_2) = \int_{-1}^{1} x^2 \cdot x^3 \, dx = \left[ \frac{1}{6} x^6 \right]_{-1}^{1} = 0$$
Orthogonal sets

We are primarily interested in an infinite sets of orthogonal functions.

Definition: Orthogonal set

A set of real-valued functions \( \{ \phi_0(x), \phi_1(x), \phi_2(x), \ldots \} \) is said to be orthogonal on an interval \([a, b]\) if

\[
(\phi_m, \phi_n) = \int_a^b \phi_m(x) \phi_n(x) \, dx = 0, \quad m \neq n
\]
Orthonormal sets

The norm $||\vec{u}||$ of a vector $\vec{u}$ can be expressed using the inner product:

$$(\vec{u}, \vec{u}) = ||\vec{u}||^2 \Rightarrow ||\vec{u}|| = \sqrt{(\vec{u}, \vec{u})}$$

Similarly the square norm of a function $\phi_n$ is $||\phi_n||^2 = (\phi_n, \phi_n)$, and so the norm is $||\phi_n|| = \sqrt{(\phi_n, \phi_n)}$. In other words, the square norm and the norm of a function $\phi_n$ in an orthogonal set $\{\phi_n(x)\}$ are, respectively,

$$||\phi_n||^2 = \int_a^b \phi_n^2(x) \, dx \quad \text{and} \quad ||\phi_n|| = \sqrt{\int_a^b \phi_n^2(x) \, dx}$$
Example 1: Orthogonal set of functions

Show that the set \{1, \cos x, \cos 2x, \ldots\} is orthogonal on the interval \([-\pi, \pi]\): 
\(\phi_0 = 1, \ \phi_n = \cos nx\)

\[
(\phi_0, \phi_n) = \int_{-\pi}^{\pi} \phi_0(x) \phi_n(x) \, dx = \int_{-\pi}^{\pi} \cos nx \, dx \\
= \left[ \frac{1}{n} \sin nx \right]_{-\pi}^{\pi} = \frac{1}{n} [\sin n\pi - \sin (-n\pi)] = 0
\]

and for \(m \neq n\), using the triangle identity

\[
(\phi_m, \phi_n) = \int_{-\pi}^{\pi} \phi_m(x) \phi_n(x) \, dx = \int_{-\pi}^{\pi} \cos mx \cos nx \, dx \\
= \frac{1}{2} \int_{-\pi}^{\pi} [\cos(mx + nx) + \cos(mx - nx)] \, dx \\
= \frac{1}{2} \left[ \frac{\sin(mx + nx)}{m + n} + \frac{\sin(mx - nx)}{m - n} \right]_{-\pi}^{\pi} = 0
\]
Example 2: Norms

Find the norms of the functions given in the Example 1 above.

\[
\|\phi_0\|^2 = \int_{-\pi}^{\pi} dx = 2\pi \\
\|\phi_0\| = \sqrt{2\pi} \\
\|\phi_n\|^2 = \int_{-\pi}^{\pi} \cos^2 nx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \left[1 + \cos 2nx\right] \, dx = \pi \\
\|\phi_n\| = \sqrt{\pi}
\]

Any orthogonal set of nonzero functions \(\{\phi_n(x)\}, \, n = 0, 1, 2...\) can be normalized, i.e. made into an orthonormal set.

Example: An orthonormal set on the interval \([-\pi, \pi]\):

\[
\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, ... \right\}
\]
Vector analogy

Suppose \( \vec{v}_1, \vec{v}_2, \) and \( \vec{v}_3 \) are three mutually orthogonal nonzero vectors in \( \mathbb{R}^3 \). Such an orthogonal set can be used as a basis for \( \mathbb{R}^3 \), that is, any three-dimensional vector can be written as a linear combination

\[
\vec{u} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3
\]

where \( c_i, i = 1, 2, 3 \) are scalars called the components of the vector. Each component can be expressed in terms of \( \vec{u} \) and the corresponding vector \( \vec{v}_i \):

\[
\begin{align*}
(\vec{u}, \vec{v}_1) &= c_1 (\vec{v}_1, \vec{v}_1) + c_2 (\vec{v}_2, \vec{v}_1) + c_3 (\vec{v}_3, \vec{v}_1) = c_1 ||\vec{v}_1||^2 + c_2 0 + c_3 0 \\
(\vec{u}, \vec{v}_2) &= c_2 ||\vec{v}_2||^2 \\
(\vec{u}, \vec{v}_3) &= c_3 ||\vec{v}_3||^2
\end{align*}
\]
Hence

\[ c_1 = \frac{(\vec{u}, \vec{v}_1)}{||\vec{v}_1||^2} \quad c_2 = \frac{(\vec{u}, \vec{v}_2)}{||\vec{v}_2||^2} \quad c_3 = \frac{(\vec{u}, \vec{v}_3)}{||\vec{v}_3||^2} \]

and

\[ \vec{u} = \frac{(\vec{u}, \vec{v}_1)}{||\vec{v}_1||^2} \vec{v}_1 + \frac{(\vec{u}, \vec{v}_2)}{||\vec{v}_2||^2} \vec{v}_2 + \frac{(\vec{u}, \vec{v}_3)}{||\vec{v}_3||^2} \vec{v}_3 = \sum_{n=1}^{3} \frac{(\vec{u}, \vec{v}_n)}{||\vec{v}_n||^2} \vec{v}_n \]
Orthogonal series expansion

Suppose \( \{ \phi_n(x) \} \) is an infinite orthogonal set of functions on an interval \([a, b]\). If \( y = f(x) \) is a function defined on the interval \([a, b]\), we can determine a set of coefficients \( c_n, n = 0, 1, 2, \ldots \) for which

\[
f(x) = c_0\phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \ldots + c_n\phi_n(x) + \ldots
\]

(1)

using the inner product. Multiplying the expression above by \( \phi_m(x) \) and integrating over the interval \([a, b]\) gives

\[
\int_a^b f(x) \phi_m(x) \, dx =
\]

\[
= c_0 \int_a^b \phi_0(x) \phi_m(x) \, dx + c_1 \int_a^b \phi_1(x) \phi_m(x) \, dx + \ldots + c_n \int_a^b \phi_n(x) \phi_m(x) \, dx + \ldots
\]

\[
= c_0(\phi_0, \phi_m) + c_1(\phi_1, \phi_m) + \ldots + c_n(\phi_n, \phi_m) + \ldots
\]
By orthogonality, each term on r.h.s. is zero except when $m = n$, in which case we have

$$
\int_a^b f(x) \phi_n(x) \, dx = c_n \int_a^b \phi_n^2(x) \, dx
$$

The required coefficients are then

$$
c_n = \frac{\int_a^b f(x) \phi_n(x) \, dx}{\int_a^b \phi_n^2(x) \, dx}, \quad n = 0, 1, 2, \ldots
$$

In other words

$$
f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x) = \sum_{n=0}^{\infty} \frac{\int_a^b f(x) \phi_n(x) \, dx}{\|\phi_n(x)\|^2} \phi_n(x) = \sum_{n=0}^{\infty} \frac{(f, \phi_n)}{\|\phi_n(x)\|^2} \phi_n(x)
$$
Definition: Orthogonal set / weight function

A set of real-valued functions \( \{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\} \) is said to be **orthogonal with respect to a weight function** \( w(x) \) on an interval \([a, b]\) if

\[
\int_a^b w(x) \phi_m(x) \phi_n(x) \, dx = 0, \quad m \neq n
\]

The usual assumption is that \( w(x) > 0 \) on the interval of orthogonality \([a, b]\).

For example, the set \( \{1, \cos x, \cos 2x, \ldots\} \) is orthogonal w.r.t. the weight function \( w(x) = 1 \) on the interval \([-\pi, \pi]\).
If \( \{\phi_n(x)\} \) is orthogonal w.r.t. a weight function \( w(x) \) on the interval \([a, b]\), then multiplying the expansion (1),

\[
f(x) = c_0\phi_0(x) + c_1\phi_1(x) + \ldots,
\]

by \( w(x) \) and integrating by parts yields

\[
c_n = \frac{\int_a^b f(x) \, w(x) \, \phi_n(x) \, dx}{\|\phi_n(x)\|^2}
\]

where

\[
\|\phi_n(x)\|^2 = \int_a^b w(x) \, \phi_n^2(x) \, dx
\]
The series

\[ f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x) \]  

(2)

with the coefficients given either by

\[ c_n = \frac{\int_a^b f(x) \phi_n(x) \, dx}{\|\phi_n(x)\|^2} \quad \text{or} \quad c_n = \frac{\int_a^b f(x) w(x) \phi_n(x) \, dx}{\|\phi_n(x)\|^2} \]  

(3)

is said to be an orthogonal series expansion of \( f \) or a generalized Fourier series.

**Complete sets**

We shall assume that an orthogonal set \( \{\phi_n(x)\} \) is complete. Under this assumption \( f \) can not be orthogonal to each \( \phi_n \) of the orthogonal set.
Fourier series

Trigonometric series

The set of functions

\[ \left\{ 1, \cos \frac{\pi}{p} x, \cos \frac{2\pi}{p} x, \cos \frac{3\pi}{p} x, \ldots, \sin \frac{\pi}{p} x, \sin \frac{2\pi}{p} x, \sin \frac{3\pi}{p} x, \ldots \right\} \]

is orthogonal on the interval \([-p, \ p]\).

We can expand a function \( f \) defined on \([-p, \ p]\) into the trigonometric series

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right) \quad (4)
\]
Determining the coefficients $a_0, a_1, a_2, ..., b_1, b_2, ...$:

We multiply by 1 (the first function in our orthogonal set) and integrate both sides of the expansion (4) from $-p$ to $p$

$$\int_{-p}^{p} f(x) \, dx = \frac{a_0}{2} \int_{-p}^{p} \, dx + \sum_{n=1}^{\infty} \left( a_n \int_{-p}^{p} \cos \frac{n\pi}{p} x \, dx + b_n \int_{-p}^{p} \sin \frac{n\pi}{p} x \, dx \right)$$

Since $\cos(n\pi x/p)$ and $\sin(n\pi x/p)$, $n \geq 1$, are orthogonal to 1 on the interval, the r.h.s. reduces as follows

$$\int_{-p}^{p} f(x) \, dx = \frac{a_0}{2} \int_{-p}^{p} \, dx = \left[ \frac{a_0}{2} x \right]_{-p}^{p} = pa_0$$

Solving for $a_0$ yields

$$a_0 = \frac{1}{p} \int_{-p}^{p} f(x) \, dx$$ (5)
Now, we multiply (4) by \( \cos(m\pi x/p) \) and integrate

\[
\int_{-p}^{p} f(x) \cos \frac{m\pi}{p} x \, dx = \frac{a_0}{2} \int_{-p}^{p} \cos \frac{m\pi}{p} x \, dx 
+ \sum_{n=1}^{\infty} \left( a_n \int_{-p}^{p} \cos \frac{m\pi}{p} x \cos \frac{n\pi}{p} x \, dx + b_n \int_{-p}^{p} \cos \frac{m\pi}{p} x \sin \frac{n\pi}{p} x \, dx \right)
\]

By orthogonality, we have

\[
\int_{-p}^{p} \cos \frac{m\pi}{p} x \, dx = 0, \quad m > 0 \\
\int_{-p}^{p} \cos \frac{m\pi}{p} x \sin \frac{n\pi}{p} x \, dx = 0 \\
\int_{-p}^{p} \cos \frac{m\pi}{p} x \cos \frac{n\pi}{p} x \, dx = p \delta_{mn}
\]

where the Kronecker delta \( \delta_{mn} = 0 \) if \( m \neq n \), and \( \delta_{mn} = 1 \) if \( m = n \).
Thus the equation (6) above reduces to

\[ \int_{-p}^{p} f(x) \cos \frac{n\pi}{p} x \, dx = a_n p \]

and so

\[ a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi}{p} x \, dx \]  \hspace{1cm} (6)
Finally, multiplying (4) by \( \sin(m\pi x/p) \), integrating and using the orthogonality relations

\[
\int_{-p}^{p} \sin \left( \frac{m\pi}{p} x \right) dx = 0, \quad m > 0 \\
\int_{-p}^{p} \sin \left( \frac{m\pi}{p} x \right) \cos \left( \frac{n\pi}{p} x \right) dx = 0 \\
\int_{-p}^{p} \sin \left( \frac{m\pi}{p} x \right) \sin \left( \frac{n\pi}{p} x \right) dx = p \delta_{mn}
\]

we find that

\[
b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin \left( \frac{n\pi}{p} x \right) dx \quad (7)
\]

The trigonometric series (4) with coefficients \( a_0, a_n, \) and \( b_n \) defined by (5), (6) and (7), respectively are said to be the \textbf{Fourier series} of the function \( f \). The coefficients obtained from (5), (6) and (7) are referred as \textbf{Fourier coefficients} of \( f \).
Definition: Fourier series

The **Fourier series** of a function $f$ defined on the interval $(-p, p)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$$

where

$$a_0 = \frac{1}{p} \int_{-p}^{p} f(x) \, dx$$

$$a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi}{p} x \, dx$$

$$b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin \frac{n\pi}{p} x \, dx$$
Example 1: Expansion in a Fourier series

\[ f(x) = \begin{cases} 
0, & -\pi < x < 0 \\
\pi - x, & 0 \leq x < \pi
\end{cases} \]

With \( p = \pi \) we have from (9) and (10) that

\[
\begin{align*}
a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{\pi} \left[ \int_{-\pi}^{0} 0 \, dx + \int_{0}^{\pi} (\pi - x) \, dx \right] = \frac{1}{\pi} \left[ \pi x - \frac{x^2}{2} \right]_{0}^{\pi} = \frac{\pi}{2} \\
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \left[ \int_{-\pi}^{0} 0 \, dx + \int_{0}^{\pi} (\pi - x) \cos nx \, dx \right] \\
&= \frac{1}{\pi} \left\{ \left[ (\pi - x) \frac{\sin nx}{n} \right]_{0}^{\pi} + \frac{1}{n} \int_{0}^{\pi} \sin nx \, dx \right\} \\
&= -\frac{1}{n\pi} \left[ \frac{\cos nx}{n} \right]_{0}^{\pi} \\
&= -\frac{\cos n\pi + 1}{n^2\pi} = \frac{1 - (-1)^n}{n^2\pi}
\end{align*}
\]
Similarly, we find from (11)

\[ b_n = \frac{1}{n} \int_0^\pi (\pi - x) \sin nx \, dx = \frac{1}{n} \]

The function \( f(x) \) is thus expanded as

\[ f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left\{ \frac{1 - (-1)^n}{n^2\pi} \cos nx + \frac{1}{n} \sin nx \right\} \]

(12)

We also note that

\[ 1 - (-1)^n = \begin{cases} 
0, & n \text{ even} \\
2, & n \text{ odd}.
\end{cases} \]
Convergence of a Fourier series

Theorem: Conditions for convergence

Let $f$ and $f'$ be piecewise continuous on the interval $(-p, p)$; that is, let $f$ and $f'$ be continuous except at a finite number of points in the interval and have only finite discontinuities at these points. Then the Fourier series of $f$ on the interval converges to $f(x)$ at a point of continuity. At a point of discontinuity, the Fourier series converges to the average

$$\frac{f(x+) + f(x-)}{2},$$

where $f(x+)$ and $f(x-)$ denote the limit of $f$ at $x$ from the right and from the left, respectively.
Example 2: Convergence of a point of discontinuity

The expansion (12) of the function (Example 1)

\[ f(x) = \begin{cases} 
0, & -\pi < x < 0 \\
\pi - x, & 0 \leq x < \pi 
\end{cases} \]

will converge to \( f(x) \) for every \( x \) from the interval \((-\pi, \pi)\) except at \( x = 0 \) where it will converge to

\[
\frac{f(0^+) + f(0^-)}{2} = \frac{\pi + 0}{2} = \frac{\pi}{2}.
\]
Periodic extension

Observe that each of the functions in the basis set

\[ \left\{ 1, \cos \frac{\pi}{p} x, \cos \frac{2\pi}{p} x, \cos \frac{3\pi}{p} x, \ldots, \sin \frac{\pi}{p} x, \sin \frac{2\pi}{p} x, \sin \frac{3\pi}{p} x, \ldots \right\} \]

has a different fundamental period \(2p/n, \ n \geq 1\), but since a positive integer multiple of a period is also a period, we see that all the functions have in common the period \(2p\). Thus the r.h.s. of

\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right) \]

is \(2p\)-periodic; indeed \(2p\) is the fundamental period of the sum.

We conclude that a Fourier series not only represents the function on the interval \((-p, p)\) but also gives the periodic extension of \(f\) outside this interval.
We can now apply the Theorem on conditions for convergence to the periodic extension or simply assume the function is periodic, \( f(x + T) = f(x) \), with period \( T = 2p \) from the outset. When \( f \) is piecewise continuous and the right- and left-hand derivatives exist at \( x = -p \) and \( x = p \), respectively, then the Fourier series converges to the average \( \left[ f(p-) + f(p+) \right] / 2 \) at these points and also to this value extended periodically to \( \pm 3p, \pm 5p, \pm 7p \), and so on.

Example: The Fourier series of the function \( f(x) \) in the Example 1 converges to the periodic extension of the function on the entire \( x \)-axis. At \( 0, \pm 2\pi, \pm 4\pi, \ldots \), and at \( \pm \pi, \pm 3\pi, \pm 5\pi, \ldots \), the series converges to the values

\[
\frac{f(0+) + f(0-)}{2} = \frac{\pi}{2} \quad \text{and} \quad \frac{f(\pi+) + f(\pi-)}{2} = 0
\]
Sequence of partial sums

It is interesting to see how the sequence of partial sums \( \{S_N(x)\} \) of a Fourier series approximates a function. For example

\[
S_1(x) = \frac{\pi}{4}, \quad S_2(x) = \frac{\pi}{4} + \frac{2}{\pi} \cos x + \sin x, \quad S_3(x) = \frac{\pi}{4} + \frac{2}{\pi} \cos x + \sin x + \frac{1}{2} \sin 2x
\]