Recurrence relations = Difference equations

Sometimes adjacent terms of a sequence are related to each other. For example, the terms of the sequence $\{x_k\} = \{2^k\}$ are such that $x_{k+1} = 2^{k+1} = 2 \times 2^k = 2x_k$. That is

$$x_{k+1} = 2x_k$$

The equation holds for all adjacent terms of the sequence - we say it *recurs* for all values of k.

The equation is called a **linear, first order, constant coefficient recurrence relation**.

Example: a recurrence relation of the second order

$$x_{k+2} - x_{k+1} - x_k = 1$$

Initial terms

A recurrence relation can be used to generate the terms of a sequence provided initial terms are given - equal in number to the order of the equation.

Example 1:
Given the sequence $\{x_k\}$ where $x_{k+1} = 3x_k$ with the initial term $x_0 = 2$ generates the sequence

$$\{x_k\} = \{2, 6, 18, 54, \ldots\}$$

Since $x_{k+1} = 3x_k$, where $x_0 = 3$ gives

$$\begin{aligned} x_1 &= 3x_0 = 3 \times 2 = 6 \\ x_2 &= 3x_1 = 3 \times 6 = 18 \\ x_3 &= 3x_2 = 3 \times 18 = 54 \end{aligned}$$

Example 2:

Similarly , if another sequence has terms that satisfy the second-order recurrence relation

$$x_{k+2} - 3x_{k+1} + 2x_k = 1$$

where $x_0 = 0$ and $x_1 = 1$ then the first five terms of the sequence are

$$\{x_k\} = \{0, 1, 4, 11, 26, \ldots\}$$

Because

$$\begin{aligned}
x_2 - 3x_1 + 2x_0 &= x_2 - 3 \times 1 + 2 \times 0 = 1 && \Rightarrow && x_2 = 4 \\
x_3 - 3x_2 + 2x_1 &= x_3 - 3 \times 4 + 2 \times 1 = 1 && \Rightarrow && x_3 = 11 \\
x_4 - 3x_3 + 2x_2 &= x_4 - 3 \times 11 + 2 \times 4 = 1 && \Rightarrow && x_4 = 26
\end{aligned}$$

Example 3:

$$x_{k+2} - x_k = 1$$

where $x_0 = 0$ and $x_1 = -1$.

$$\begin{aligned}
x_2 - x_0 &= x_2 - 0 = 1 & \Rightarrow \quad & x_2 = 1 \\
x_3 - x_1 &= x_3 - (-1) = 1 & \Rightarrow \quad & x_3 = 0 \\
x_4 - x_2 &= x_4 - 1 = 1 & \Rightarrow \quad & x_4 = 2 \\
x_5 - x_3 &= x_5 - 0 = 1 & \Rightarrow \quad & x_5 = 1
\end{aligned}$$

Therefore

$$\{x_k\} = \{0, -1, 1, 0, 2, 1, \ldots\}$$

Solving the recurrence relation

If a sequence $\{x_k\}$ satisfies a recurrence relation with given initial conditions then the general term of the sequence can be found by using the Z transform where $\mathcal{Z}\{x_k\}$ = F(z).

Example 4:
Solve the recurrence relation

$$x_{k+2} - 3x_{k+1} + 2x_k = 1$$

where $x_0 = 0$ and $x_1 = 1$.

Since this recurrence relation is true for all values of k it can itself be used to form a sequence $\{y_k\}$, namely

$$\{y_k\} = \{x_{k+2} - 3x_{k+1} + 2x_k\} = \{1\}$$

Now taking the Z transform of both sides of this equation gives

$$\mathcal{Z}\{y_k\} = \mathcal{Z}\{x_{k+2} - 3x_{k+1} + 2x_k\} = \mathcal{Z}\{1\}$$
$$\mathcal{Z}\{x_{k+2}\} - 3\mathcal{Z}\{x_{k+1}\} + 2\mathcal{Z}\{x_k\} = \mathcal{Z}\{1\}$$

Using the first shift theorem and $\mathcal{Z}\{x_k\} = F(z)$ and the initial conditions $x_0 = 0$ and $x_1 = 1$, this becomes

$$\left(z^2F(z) - z^2x_0 - zx_1\right) - 3\left(zF(z) - zx_0\right) + 2F(z) = \frac{z}{z-1}$$
$$\left(z^2F(z) - z\right) - 3\left(zF(z)\right) + 2F(z) = \frac{z}{z-1}$$
$$\left(z^2 - 3z + 2\right)F(z) - z = \frac{z}{z-1}$$
$$\Rightarrow \qquad F(z) = \frac{z^2}{(z-1)^2(z-2)} \quad \text{and so} \quad \frac{F(z)}{z} = \frac{z}{(z-1)^2(z-2)}$$

Now we perform the partial fraction decomposition of $\frac{F(z)}{z}$:

$$\begin{aligned} \frac{F(z)}{z} &= \frac{z}{(z-1)^2(z-2)} = \frac{A}{(z-1)^2} + \frac{B}{z-1} + \frac{C}{z-2} \\ &= \frac{A(z-2) + B(z-1)(z-2) + C(z-1)^2}{(z-1)^2(z-2)} \end{aligned}$$

and so $z = A(z-2) + B(z-1)(z-2) + C(z-1)^2$ giving

$$\begin{aligned} [z^2]: &\quad B + C = 0 \\ [z^1]: &\quad A - 3B - 2C = 1 \\ [z^0]: &\quad -2A + 2B + C = 0 \end{aligned}$$

with solution $A = -1$, $B = -2$ and $C = 2$. Therefore

$$\frac{F(z)}{z} = -\frac{1}{(z-1)^2} - \frac{2}{z-1} + \frac{2}{z-2}$$

Now we take the inverse Z transform:

$$\begin{aligned}
\frac{F(z)}{z} &= -\frac{1}{(z-1)^2} - \frac{2}{z-1} + \frac{2}{z-2} \\
F(z) &= -\frac{z}{(z-1)^2} - \frac{2z}{z-1} + \frac{2z}{z-2} \\
\mathcal{Z}^{-1}F(z) &= -\mathcal{Z}^{-1}\left(\frac{z}{(z-1)^2}\right) - 2\mathcal{Z}^{-1}\left(\frac{z}{z-1}\right) + 2\mathcal{Z}^{-1}\left(\frac{z}{z-2}\right) \\
&= \left\{-k - 2(1^k) + 2(2^k)\right\} \\
&= \left\{-k - 2 + 2^{k+1}\right\}
\end{aligned}$$

Indeed $\{x_k\} = \left\{-k - 2 + 2^{k+1}\right\}$ is the solution to the recurrence relation.

Verifying the solution:

$$
\begin{aligned}
&x_{k+2} - 3x_{k+1} + 2x_k \\
&= \left(-[k+2] - 2 + 2^{[k+2]+1}\right) - 3\left(-[k+1] - 2 + 2^{[k+1]+1}\right) + 2\left(-k - 2 + 2^{k+1}\right) \\
&= \left(-k - 4 + 8 \times 2^k\right) - 3\left(-k - 3 + 4 \times 2^k\right) + 2\left(-k - 2 + 2 \times 2^k\right) \\
&= -k - 4 + 8 \times 2^k + 3k + 9 - 12 \times 2^k - 2k - 4 + 4 \times 2^k \\
&= 1
\end{aligned}
$$

Example 5:

Solve the second-order recurrence relation

$$x_{k+2} - x_k = 1$$

where $x_0 = 0$ and $x_1 = -1$.

Taking the Z transform of the equation gives

$$\mathcal{Z}\{x_{k+2} - x_k\} = \mathcal{Z}\{1\}$$
$$\mathcal{Z}\{x_{k+2}\} - \mathcal{Z}\{x_k\} = \mathcal{Z}\{1\}$$
$$\left(z^2 F(z) - z^2 x_0 - z x_1\right) - F(z) = \frac{z}{z-1}$$

Substituting for $x_0 = 0$ and $x_1 = -1$ gives

$$\left(z^2 F(z) + z\right) - F(z) = \frac{z}{z-1}$$

$$\left(z^2 - 1\right) F(z) + z = \frac{z}{z-1}$$

$$F(z) = \frac{z}{(z^2-1)(z-1)} - \frac{z}{(z^2-1)}$$

$$\begin{aligned}\frac{F(z)}{z} &= \frac{1}{(z^2-1)(z-1)} - \frac{1}{(z^2-1)} \\ &= \frac{1}{(z+1)(z-1)^2} - \frac{1}{(z+1)(z-1)} \\ &= \frac{-z+2}{(z+1)(z-1)^2}\end{aligned}$$

We now perform the partial fraction decomposition

$$\begin{aligned}
\frac{F(z)}{z} &= \frac{-z+2}{(z+1)(z-1)^2} \\
&= \frac{A}{(z+1)} + \frac{B}{(z-1)} + \frac{C}{(z-1)^2} \\
&= \frac{A(z-1)^2 + B(z+1)(z-1) + C(z+1)}{(z+1)(z-1)^2}
\end{aligned}$$

Equating numerators and comparing coefficients of powers of z gives

$$\begin{aligned}
[z^2]: &\quad A + B = 0 \\
[z^1]: &\quad -2A + C = -1 \\
[z^0]: &\quad A - B + C = 2
\end{aligned}$$

with solution $A = 3/4$, $B = -3/4$ and $C = 1/2$.

By inverting the transform

$$F(z) = \frac{3}{4}\frac{z}{(z+1)} - \frac{3}{4}\frac{z}{(z-1)} + \frac{1}{2}\frac{z}{(z-1)^2}$$

we obtain the final solution:

$$\begin{aligned}
\mathcal{Z}^{-1}F(z) &= \frac{3}{4}\mathcal{Z}^{-1}\left\{\frac{z}{z+1}\right\} - \frac{3}{4}\mathcal{Z}^{-1}\left\{\frac{z}{z-1}\right\} + \frac{1}{2}\mathcal{Z}^{-1}\left\{\frac{z}{(z-1)^2}\right\} \\
&= \frac{3}{4}\left\{(-1)^k\right\} - \frac{3}{4}\left\{1^k\right\} + \frac{1}{2}\{k\}
\end{aligned}$$

$$\{x_k\} = \left\{\frac{3}{4}(-1)^k - \frac{3}{4} + \frac{k}{2}\right\}$$

so that

$$x_k = \begin{cases} k/2 & k \text{ even} \\ (k-3)/2 & k \text{ odd} \end{cases}$$

Sampling

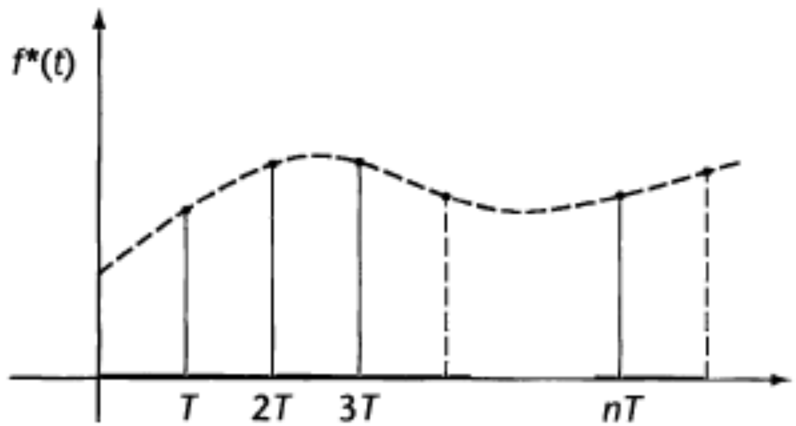
If a continuous function $f(t)$ of time t progresses from $t = 0$ onwards and is measured at every time interval T, then the result is a sequence of values

$$\{f(kT)\} = \{f(0), f(T), f(2T), f(3T), \ldots\}$$

A new piecewise continuous function $f^*(t)$ can be created from the sequence of sampled values such that

$$f^*(t) = \begin{cases} f(kT) & \text{if } t = kT \\ 0 & \text{otherwise} \end{cases}$$

The graph of this new function consists of a series of spikes at regular intervals $t = kT$.



The function can alternatively be described in terms of the delta function $\delta(t)$ as

$$\begin{aligned} f^*(t) &= f(0)\delta(t) + f(T)\delta(t-T) + f(2T)\delta(t-2T) + f(3T)\delta(t-3T) + \ldots \\ &= \sum_{k=0}^{\infty} f(kT)\delta(t-kT) \end{aligned}$$

The Laplace transform is then given as

$$\begin{aligned} F^*(s) &= \mathcal{L}\{f^*(t)\} \\ &= \int_0^{\infty} \{f(0)\,\delta(t) + f(T)\,\delta(t-T) + f(2T)\,\delta(t-2T) + \ldots\}\; e^{-st}\,dt \\ &= f(0) + f(T)e^{-sT} + f(2T)e^{-2sT} + f(3T)e^{-3sT} + \ldots \\ &= \sum_{k=0}^{\infty} f(kT)e^{-ksT} \end{aligned}$$

Define a new variable $z = e^{sT}$ and we see that

$$\mathcal{L}\{f^*(t)\} = \sum_{k=0}^{\infty} f(kT)z^{-k} = \sum_{k=0}^{\infty} \frac{f(kT)}{z^k}$$

which is the Z transform of the sequence $\{f(kT)\}$.

Example 1:

The function $f(t) = e^{-at}$ is sampled every interval T. Calculate the Z-transform of the sampled function.

Defining $f^*(t) = \sum_{k=0}^{\infty} f(kT)\ \delta(t - kT) = \sum_{k=0}^{\infty} e^{-akT}\ \delta(t - kT)$, then the Laplace transform of $f^*(t)$ is given as

$$F^*(s) = \sum_{k=0}^{\infty} e^{-akT}\ e^{-ksT}$$

and thus the Z transform of $\{f(kT)\}$ is

$$F(z) = \sum_{k=0}^{\infty} \frac{e^{-kaT}}{z^k} = \frac{1}{1 - \frac{e^{-aT}}{z}} = \frac{z}{z - e^{-aT}}$$

Notice that this agrees with the Z transform of the sequence $\{b^k\}$, which is $\frac{z}{z-b}$ when b is replaced by e^{-aT}.

Example 2:

The function $f(t) = t$ is sampled every interval T. The Z transform of the sampled function $\{f(kT)\} = \{kT\}$ is

$$
\begin{aligned}
F(z) &= \sum_{k=0}^{\infty} \frac{f(kT)}{z^k} = \sum_{k=0}^{\infty} \frac{kT}{z^k} \\
&= T\left(\frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \ldots\right) \\
&= \frac{T}{z}\left(1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + \ldots\right) \\
&= -Tz\frac{\mathrm{d}}{\mathrm{d}z}\left(1 + z^{-1} + z^{-2} + z^{-3} + \ldots\right) \\
&= -Tz\frac{\mathrm{d}}{\mathrm{d}z}\left(1 - \frac{1}{z}\right)^{-1} = \frac{T}{z}\left(1 - \frac{1}{z}\right)^{-2} = \frac{Tz}{(z-1)^2}
\end{aligned}
$$

Example 3:

The function $f(t) = \cos t$ is sampled every interval of T. We first rewrite

$$f(t) = \cos t = \frac{e^{it} + e^{-it}}{2}$$
$$f(kT) = \frac{e^{ikT} + e^{-ikT}}{2}$$

The Z transform of $\{e^{-kaT}\}$ is $F(z) = \frac{z}{z-e^{-aT}}$. Therefore the Z transform of the sampled function $\{\cos kT\}$ is

$$F(z) = \frac{1}{2}\left(\frac{z}{z - e^{-iT}} + \frac{z}{z - e^{iT}}\right) = \frac{1}{2}\left[\frac{z\left(z - e^{iT}\right) + z\left(z - e^{-iT}\right)}{\left(z - e^{-iT}\right)\left(z - e^{iT}\right)}\right]$$
$$= \frac{1}{2}\left[\frac{2z^2 - z\left(e^{iT} + e^{-iT}\right)}{z^2 - \left(e^{iT} + e^{-iT}\right)z + 1}\right] = \frac{z(z - \cos T)}{z^2 - 2z\cos T + 1}$$