# Solutions to EE 112 Exam, Spring 2016-'17



P.1

(a) (i) $\vec{a}\cdot\vec{b} = (\hat{i}-\hat{k})\cdot(2\hat{i}+\hat{j}+\hat{k})$
$= (1)(2) + (0)(1) + (-1)(1)$
$= \boxed{1}$ [3 marks]

(ii) $\vec{a}\times\vec{b} = (\hat{i}-\hat{k})\times(2\hat{i}+\hat{j}+\hat{k})$
$= \hat{i}\times\hat{j} + \hat{i}\times\hat{k} - 2\hat{k}\times\hat{i} - \hat{k}\times\hat{j}$
$= \hat{k} - \hat{j} - 2\hat{j} + \hat{i} = \boxed{\hat{i} - 3\hat{j} + \hat{k}}$ [3 marks]

(iii) $\hat{j}\times(\vec{b}\times\hat{k}) = (\hat{j}\cdot\hat{k})\vec{b} - (\hat{j}\cdot\vec{b})\hat{k}$
$= -[\hat{j}\cdot(2\hat{i}+\hat{j}+\hat{k})]\hat{k}$
$= -\hat{k}$

so

$[\hat{j}\times(\vec{b}\times\hat{k})]\cdot\vec{a} = -\hat{k}\cdot(\hat{i}-\hat{k}) = \boxed{1}$ [3 marks]

(iv) $2(\vec{c}\cdot\vec{a})\hat{j} = 2[(4\hat{j}-2\hat{k})\cdot(\hat{i}-\hat{k})]\hat{j}$
$= 2(2)\hat{j} = 4\hat{j}$

so

$2(\vec{c}\cdot\vec{a})\hat{j} + 3\vec{a} = 4\hat{j} + (\hat{i}-\hat{k}) = \boxed{\hat{i}+4\hat{j}-\hat{k}}$ [3 marks]

(b) $x(t) = 3+2t$, $y(t) = -2t$, $z(t) = t$, so $2x - y + z = 2(3+2t) - (-2t) + t = 7t+6$, so the line crosses the plane at $2x-y+z = 7t+6 = 5$, or $t = -1/7$. Thus, $x(-1/7) = 3 - 2/7 = 19/7$, $y(-1/7) = 2/7$, $z(-1/7) = -1/7$, giving $\boxed{\vec{r}_{int} = 19/7\,\hat{i} + 2/7\,\hat{j} - 1/7\,\hat{k}}$ as the intersection point. [3 marks]

(c) (i) We use the shifting thm that $L[e^{at}f(t)] = F(s-a)$.
$L[t^2] = \frac{2}{s^3}$, so $L[e^t t^2] = \frac{2}{(s-1)^3}$, and
$L[3] = 3/s$, so $L[3e^t] = \frac{3}{s-1}$, giving
$L[e^t(t^2-3)] = \boxed{\frac{2}{(s-1)^3} - \frac{3}{s-1}}$ [4 marks]

(ii) $\frac{s}{s^2-4}$ is the LT of $\cosh(2t)$ and $\frac{1}{s^2-4}$ is the LT of $\frac{1}{2}\sinh(2t)$, so
$L^{-1}\left[\frac{s+1}{s^2-4}\right] = \boxed{\cosh(2t) + \frac{1}{2}\sinh(2t)}$ [4 marks]

(d) The tangent vector

$$\vec{v} = \frac{d\vec{r}}{dt} = 6\cos(3t)\hat{\imath} + 8\hat{\jmath} + 6\sin(3t)\hat{k}$$

which has magnitude

$$|\vec{v}| = \sqrt{36\cos^2(3t) + 64 + 36\sin^2(3t)} = 10$$

so

$$\hat{u} = \frac{3}{5}\cos(3t)\hat{\imath} + \frac{4}{5}\hat{\jmath} + \frac{3}{5}\sin(3t)\hat{k},$$

$$\frac{d\hat{u}}{dt} = -\frac{9}{5}\sin(3t)\hat{\imath} + \frac{9}{5}\cos(3t)\hat{\jmath}, \text{ so the curvature is}$$

$$K = \frac{|d\hat{u}/dt|}{|\vec{v}|} = \frac{\sqrt{81/25\,(\sin^2(3t) + \cos^2(3t))}}{10} = \boxed{\frac{9}{50}}$$ [3 marks]

and the principal unit normal vector is

$$\hat{N} = \frac{d\hat{u}/dt}{|d\hat{u}/dt|} = \frac{-\frac{9}{5}\sin(3t)\hat{\imath} + \frac{9}{5}\cos(3t)\hat{\jmath}}{9/5} = \boxed{-\sin(3t)\hat{\jmath} + \cos(3t)\hat{k}}$$ [3 marks]

(e) (i)

$$AB = \begin{pmatrix} 5 & 0 \\ 0 & -1 \\ -8 & 3 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 10+0 & 10+0 \\ 0+1 & 0-1 \\ -16-3 & -16+3 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 10 & 10 \\ 1 & -1 \\ -19 & -13 \end{pmatrix}}$$ [2 marks]

(ii) $$A^T = \begin{pmatrix} 5 & 0 \\ 0 & -1 \\ -8 & 3 \end{pmatrix}^T = \boxed{\begin{pmatrix} 5 & 0 & -8 \\ 0 & -1 & 3 \end{pmatrix}}$$ [2 marks]

(iii) $$B^T = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}^T = \boxed{\begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}}$$ [2 marks]

(iv) $$(AB)^T = \begin{pmatrix} 10 & 10 \\ 1 & -1 \\ -19 & -13 \end{pmatrix}^T = \boxed{\begin{pmatrix} 10 & 1 & -19 \\ 10 & -1 & -13 \end{pmatrix}}$$ [2 marks]

(f) The matrix is lower-triangular, so its determinant is the product of its diagonal elements:

$$\det\begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 7 & -3 \end{pmatrix} = (5)(1)(-3) = \boxed{-15}$$ [3 marks]

$$\mathrm{tr}\begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 7 & -3 \end{pmatrix} = 5 + 1 + (-3) = \boxed{3}$$ [3 marks]

(3)

(5) Reduce the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 0 \\ 1 & -3 & -1 & 0 \end{array}\right)$$

Multiply 1st row by $-1$ and add to 2nd and 3rd row:

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 1-1 & 1-2 & 1-2 & 0-2 \\ 1-1 & -3-2 & -1-2 & 0-2 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & -1 & -1 & -2 \\ 0 & -5 & -3 & -2 \end{array}\right)$$

Multiply 2nd row by $-5$ and add to 3rd row:

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & -1 & -1 & -2 \\ 0 & -5+5 & -3+5 & -2+10 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & 2 & 8 \end{array}\right)$$

Multiply 2nd row by 2 and add to 1st:

$$\left(\begin{array}{ccc|c} 1+0 & 2-2 & 2-2 & 2-4 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & 2 & 8 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & 2 & 8 \end{array}\right)$$

Multiply 3rd row by $\frac{1}{2}$ and add to 2nd:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & -1+0 & -1+1 & -2+4 \\ 0 & 0 & 2 & 8 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 2 & 8 \end{array}\right)$$

so $x=-2,\ -y=2,\ 2z=8 \Rightarrow \boxed{x=-2,\ y=-2,\ z=4}$ [7 marks]

P.2

(a) $AK_i = \lambda_i K_i$ give the eigenvalues $\lambda_i$, so

$$\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2+4+3 \\ 2+2+6 \\ -1-4+0 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ -5 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

so $AK_1 = \lambda_1 K_1 \Rightarrow \boxed{\lambda_1 = 5.}$ [3 marks]

$$\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4+2+0 \\ -4+1+0 \\ 2-2+0 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 0 \end{pmatrix} \Rightarrow -3\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$ [3 marks]

so $AK_2 = \lambda_2 K_2 \Rightarrow \boxed{\lambda_2 = -3}$

$$\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -6+0-3 \\ 6+0-6 \\ -3+0+0 \end{pmatrix} = \begin{pmatrix} -9 \\ 0 \\ -3 \end{pmatrix} = -3\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

so $AK_3 = \lambda_3 K_3 \Rightarrow \boxed{\lambda_3 = -3}$ [3 marks]

(b) $L\left[\frac{d^2y}{dt^2}\right] = s^2L[y] - sy(0) - y'(0) = s^2L[y] - 5s - 5$

$L[1] = 1/s$, $L[e^{3t}] = \frac{1}{s-3}$

so if $Y(s) = L[y]$,

$$-2(s^2Y - 5s - 5) + 2Y = -\frac{1}{s} - \frac{1}{s-3}$$

or

$$-2(s^2-1)Y + 10(s+1) = -\frac{1}{s} - \frac{1}{s-3}.$$

Thus,

$$Y(s) = \frac{5}{s-1} + \frac{1}{2s(s^2-1)} + \frac{1}{2(s-3)(s^2-1)}$$

$L^{-1}\left(\frac{1}{s-1}\right) = e^t$, so the 1st term is done. $L[\sinh(t)] = \frac{1}{s^2-1}$, so

$$L^{-1}\left[\frac{1}{s(s^2-1)}\right] = \int_0^t \sinh(\tau)d\tau = \cosh(\tau)\Big|_0^t = \cosh(t) - 1.$$

Using partial fractions,

$$\frac{1}{(s-3)(s^2-1)} = \frac{A}{s-3} + \frac{B}{s+1} + \frac{C}{s-1}$$

$$\Rightarrow 1 = A(s^2-1) + B(s^2-4s+3) + C(s^2-2s-3)$$
$$= (A+B+C)s^2 + (-4B-2C)s + (-A+3B-3C)$$

$\Rightarrow A+B+C=0$, $-4B-2C=0$, $-A+3B-3C=1$.

This can be solved any number of ways, like Gauss-Jordan Elimination:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -4 & -2 & 0 \\ -1 & 3 & -3 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -4 & -2 & 0 \\ 0 & 4 & -2 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -4 & -2 & 0 \\ 0 & 0 & -4 & 1 \end{array}\right)$$

so $-4C = 1 \Rightarrow C = -1/4$, $-4B-2C=0 \Rightarrow B = 1/8$,

$A+B+C=0 \Rightarrow A = 1/8$, so

$$\frac{1}{(s-3)(s^2-1)} = \frac{1}{8(s-3)} + \frac{1}{8(s+1)} - \frac{1}{4(s-1)}$$

so

$$L^{-1}\left[\frac{1}{(s-3)(s^2-1)}\right] = L^{-1}\left[\frac{1}{8(s-3)} + \frac{1}{8(s+1)} - \frac{1}{4(s-1)}\right] = \frac{1}{8}e^{3t} + \frac{1}{8}e^{-t} - \frac{1}{4}e^t$$

and so

$$L^{-1}[Y(s)] = y(t) = 5e^t + \frac{1}{2}(\cosh(t)-1) + \frac{1}{2}\left(\frac{1}{8}e^{3t} + \frac{1}{8}e^{-t} - \frac{1}{4}e^t\right)$$

$$= \boxed{\frac{1}{16}e^{3t} - \frac{1}{2} + \frac{41}{8}e^t + \frac{5}{16}e^{-t}}$$ [16 marks]

P.3

(a) $x+y+z=1=\hat{n}_1\cdot\vec{r}$, so $\hat{n}_1=\hat{i}+\hat{j}+\hat{k}$ is perpendicular to the plane, and $x-y+2z=0=\hat{n}_2\cdot\vec{r}$, so $\hat{n}_2=\hat{i}-\hat{j}+2\hat{k}$ is perpendicular to the plane. $\hat{n}_1\times\hat{n}_2$ is the direction vector of the line of intersection of the planes, which is

$$\vec{a}=\hat{n}_1\times\hat{n}_2=(\hat{i}+\hat{j}+\hat{k})\times(\hat{i}-\hat{j}+2\hat{k})=(-\hat{i}\times\hat{j}+2\hat{i}\times\hat{k}+\hat{j}\times\hat{i}+2\hat{j}\times\hat{k}+\hat{k}\times\hat{i}-\hat{k}\times\hat{j})$$
$$=-\hat{k}-2\hat{j}-\hat{k}+2\hat{i}+\hat{j}+\hat{i}=3\hat{i}-\hat{j}-2\hat{k}.$$

Now we need a point $\vec{r}_0$ which is shared by both planes. The shared point that lies in the xy plane has $z=0$, i.e. $x+y=1$ and $x-y=0$, i.e. $x=y=\frac{1}{2}$.
Thus, $\vec{r}_0=\frac{1}{2}\hat{i}+\frac{1}{2}\hat{j}$ lies in the line of intersection, so

$$\boxed{\vec{r}(t)=\left(\tfrac{1}{2}\hat{i}+\tfrac{1}{2}\hat{j}\right)+(3\hat{i}-\hat{j}-2\hat{k})t}$$ [10 marks]

gives the vector form of the line of intersection of the two planes.

(b) Let's find the inverse using the cofactor method, i.e. $A^{-1}=\frac{1}{\det A}\operatorname{adj}A$, where $\operatorname{adj}A$ is the adjugate of $A$. First, we find $C$, the cofactor of $A$, given by

$$C=\begin{pmatrix}\begin{vmatrix}-1&2\\0&1\end{vmatrix} & -\begin{vmatrix}-4&2\\-2&1\end{vmatrix} & \begin{vmatrix}-4&-1\\-2&0\end{vmatrix}\\ -\begin{vmatrix}2&-9\\0&1\end{vmatrix} & \begin{vmatrix}19&-9\\-2&1\end{vmatrix} & -\begin{vmatrix}19&2\\-2&0\end{vmatrix}\\ \begin{vmatrix}2&-9\\-1&2\end{vmatrix} & -\begin{vmatrix}19&-9\\-4&2\end{vmatrix} & \begin{vmatrix}19&2\\-4&-1\end{vmatrix}\end{pmatrix}=\begin{pmatrix}-1&0&-2\\-2&1&-4\\-5&-2&-11\end{pmatrix}$$

so $$\operatorname{adj}A=C^T=\begin{pmatrix}-1&-2&-5\\0&1&-2\\-2&-4&-11\end{pmatrix}$$

Now, $$\det A=\begin{vmatrix}19&2&-9\\-4&-1&2\\-2&0&1\end{vmatrix}=-2\begin{vmatrix}2&-9\\-1&2\end{vmatrix}+\begin{vmatrix}19&2\\-4&-1\end{vmatrix}=-2(-5)+(-11)$$
$$=-1$$

so $$\boxed{A^{-1}=-1\operatorname{adj}A=\begin{pmatrix}1&2&5\\0&-1&2\\2&4&11\end{pmatrix}}$$ [15 marks]

P.4

(a) Let's solve this using $A\cdot X=K$, where

$$A=\begin{pmatrix}0.5&-1&0\\1&1&1\\0.5&0&-3\end{pmatrix},\quad X=\begin{pmatrix}I_1\\I_2\\I_3\end{pmatrix}\text{ and }K=\begin{pmatrix}2\\0\\4\end{pmatrix}.$$

This will give $X = A^{-1} \cdot K$, and

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A = -\frac{2}{10} \begin{pmatrix} -3 & 3.5 & -0.5 \\ -3 & -1.5 & -0.5 \\ -1 & -0.5 & 1.5 \end{pmatrix}^T \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

so $\boxed{I_1 = 2A,\ I_2 = I_3 = -1A}$ [4 marks each]

(b) The characteristic polynomial is $p(\lambda) = \det(M - \lambda I)$, so

$$p(\lambda) = \begin{vmatrix} 2-\lambda & -4 \\ 1 & -3-\lambda \end{vmatrix} = (2-\lambda)(-3-\lambda) - (-4)(1)$$

$$= \lambda^2 + \lambda - 2$$

so the characteristic eq'n is $p(M) = 0$, ie

$\boxed{M^2 + M - 2I = 0}$ [5 marks]

Thus,

$$M^2 = -M + 2I = -\begin{pmatrix} 2 & -4 \\ 1 & -3 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 0 & 4 \\ -1 & 5 \end{pmatrix}}$$ [4 marks]

and, since $M^{-1}(M^2 + M - 2I) = M + I - 2M^{-1} = 0$,

$$M^{-1} = \frac{1}{2}(M + I) = \frac{1}{2}\left[\begin{pmatrix} 2 & -4 \\ 1 & -3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \boxed{\begin{pmatrix} 3/2 & -2 \\ 1/2 & -1 \end{pmatrix}}$$ [4 marks]