9 Solving Linear Systems of Equations

An equation of the form $ax + by = c$ where a,b and c are real numbers (e.g. $2x + 5y = 3$) is said to be a linear equation in the variables x and y.

For real numbers a,b,c and d, the equation $ax + by + cz = d$ is a linear equation in the variable x, y, z and is the equation of a plane.

In general, any equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n$ and b are real numbers is a linear equation in the n variables $x_1, x_2, \ldots, x_n$.

We will examine various techniques to solve *systems* of linear equations.

For example, consider the following system of two linear equations

$$\begin{aligned} 2x + 3y &= 2 \\ 3x + 4y &= 4 \end{aligned}$$

Finding a solution to this system of equations means we are looking for values of x and y which simultaneously satisfy both of the above linear equations. The values $x = 4$ and $y = -2$ satisfies the system and together constitute a solution to our system of equations.

9.1 Linear Systems in Matrix Form

Any linear system of equations can be expressed in the form of a matrix equation. For example, consider the matrix equation

$$\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \tag{9.1.1}$$

performing the multiplication on the left-hand side of (9.1.1) yields

$$\begin{pmatrix} 2x + 3y \\ 3x + 4y \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

which we can read as

$$\begin{aligned} 2x + 3y &= 2 \\ 3x + 4y &= 4 \end{aligned} \tag{9.1.2}$$

The linear system (9.1.2) can thus be written in the form (9.1.1), i.e, as a matrix equation

$$\mathbf{Ax} = \mathbf{B}$$

where

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$$ is the variable coefficient matrix

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$ holds the system variables

$$\mathbf{B} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$ holds the constants of the system.

In general, any system of m linear equations in n unknowns, $x_1, x_2, \ldots, x_n$

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{gathered}$$

can be written compactly as a matrix equation

$$\mathbf{Ax} = \mathbf{B}$$

where

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

9.2 Using an inverse matrix to solve a linear system

We have seen that a linear system of n equations in n unknowns

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{gathered}$$

can be written as a matrix equation

$$\mathbf{Ax} = \mathbf{B}$$

with

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

If we have the linear equations then we will know what the matrices **A** and **B** are, our aim is to find the value of the variables that compose **x**. We can find the values of **x** by multiplying both sides of our matrix equation $\mathbf{Ax} = \mathbf{B}$ by $\mathbf{A}^{-1}$ which yields

$$\begin{aligned} \mathbf{A}^{-1}(\mathbf{Ax}) &= \mathbf{A}^{-1}\mathbf{B} \\ (\mathbf{A}^{-1}\mathbf{A})\mathbf{x} &= \mathbf{A}^{-1}\mathbf{B} \qquad \text{and as } \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \\ \mathbf{Ix} &= \mathbf{A}^{-1}\mathbf{B} \end{aligned}$$

which brings us the solution to our linear system of equations

Given a system of linear equations $\mathbf{Ax} = \mathbf{B}$ and provided that $\mathbf{A}^{-1}$ exists them

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{B}$$

Example 9.2.1 (Solution to a linear system with an inverse (2×2))**.**

Find the solution to the following linear system

$$\begin{aligned} 2x - 9y &= 15 \\ 4x + 6y &= 16 \end{aligned}$$

using an inverse matrix.

Solution:
Writing the system in matrix form we have

$$\underbrace{\begin{pmatrix} 2 & -9 \\ 3 & 6 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 15 \\ 16 \end{pmatrix}}_{\mathbf{B}}$$

We require the inverse of the coefficient matrix **A**

$$\mathbf{A}^{-1} = \frac{1}{\det\mathbf{A}}\,\text{adj}\mathbf{A}$$

which we can show (you should do this as an exercise) to be

$$\mathbf{A}^{-1} = \frac{1}{39}\begin{pmatrix} 6 & 9 \\ -3 & 2 \end{pmatrix}$$

thus the solution to our system of equations $\mathbf{x}$ is

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{B} = \frac{1}{39}\begin{pmatrix} 6 & 9 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 15 \\ 16 \end{pmatrix} = \frac{1}{39}\begin{pmatrix} 234 \\ -13 \end{pmatrix} = \begin{pmatrix} 6 \\ -1/3 \end{pmatrix}$$

finally

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ -1/3 \end{pmatrix}$$

which means our solution is

$$x = 6, \qquad y = -\frac{1}{3}$$

Example 9.2.2 (Solution to a linear system with an inverse (3×3))**.**

Using an inverse we can solve the system

$$\begin{aligned} 2x_1 + x_3 &= 2 \\ -2x_1 + 3x_2 + 4x_3 &= 4 \\ -5x_1 + 5x_2 + 6x_3 &= 1 \end{aligned}$$

Solution:
We have

$$\begin{matrix} \begin{pmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{pmatrix} & \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} & = & \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} \\ \mathbf{A} & \mathbf{x} & = & \mathbf{B} \end{matrix}$$