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THE NATIONAL UNIVERSITY OF IRELAND MAYNOOTH

MATHEMATICAL PHYSICS

EE112

Engineering Mathematics II

Matrices and Basic Opertations

Prof. D. M. Heffernan and Dr. S. Pouryahya

Contents

7 Matrices and Basic Operations

Here we will examine a number of important operations on matrices. These being the *transpose*, the *trace* and the *determinant* of a matrix.

7.1 Operation: Transpose

Definition 7.1 (Transpose)**.**
Given an $m \times n$ matrix $\mathbf{A}$ the transpose of the matrix $\mathbf{A}$ is the matrix, denoted, $\mathbf{A}^\top$, whose consecutive rows are made up of the consecutive columns of $\mathbf{A}$.Thus $\mathbf{A}^\top$ will be a $n \times m$ matrix. We can write this more compactly, if

$$(a_{ij}^T) = (a_{ji}) \quad \text{for all } i, j.$$

Example 7.1.1 (Transpose)**.**
(i)

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & 0 \\ 1 & 2 \end{pmatrix} \quad \text{then} \quad \mathbf{A}^\top = \begin{pmatrix} 2 & 4 & 1 \\ 3 & 0 & 2 \end{pmatrix}$$

(ii)

$$\mathbf{B} = \begin{pmatrix} 2 & 3 & 0 \\ 1 & 2 & 4 \\ 1 & 0 & 6 \end{pmatrix} \quad \text{then} \quad \mathbf{B}^\top = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 0 \\ 0 & 4 & 6 \end{pmatrix}$$

7.1.1 Properties of the transpose

Given the matrices $\mathbf{A}$, $\mathbf{B}$, and a scalar k then the following are true

(i) The transpose of the transpose

$$\left(\mathbf{A}^\top\right)^\top = \mathbf{A}$$

(ii) The transpose of a sum is the sum of the transposes

$$(\mathbf{A} + \mathbf{B})^\top = \mathbf{A}^\top + \mathbf{B}^\top$$

(iii) Factoring scalars

$$(k\mathbf{A})^\top = k\mathbf{A}^\top$$

(iv) The transpose of a product

$$(\mathbf{AB})^\top = \mathbf{B}^\top \mathbf{A}^\top$$

Note 7.1 (Transpose of a product of more than two matrices)**.**
Notice that from our properties of the transpose

$$(\mathbf{AB})^\top = \mathbf{B}^\top\mathbf{A}^\top.$$

More specifically that the order of the product is reversed and that this implies that $(\mathbf{AB})^\top \neq \mathbf{A}^\top\mathbf{B}^\top$. What if we wanted to take the transpose of a product of three matrices? In such a case we have

$$(\mathbf{ABC})^\top = \mathbf{C}^\top\mathbf{B}^\top\mathbf{A}^\top$$

The order is once again reversed. This can be shown without too much difficulty as

$$(\mathbf{ABC})^\top = (\mathbf{A}(\mathbf{BC}))^\top = (\mathbf{BC})^\top\mathbf{A}^\top = \mathbf{C}^\top\mathbf{B}^\top\mathbf{A}^\top$$

7.1.2 Type of matrix: Symmetric

Definition 7.2 (A symmetric matrix)**.**
A matrix $\mathbf{A}$ is said to be symmetric if it satisfies all of the following properties

- The matrix is square.
- The matrix is equal to its transpose.

More compactly a matrix is symmetric if

$$\mathbf{A} = \mathbf{A}^\top$$

Example 7.1.2 (Symmetric matrices)**.**

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} \quad \text{has transpose} \quad \mathbf{A}^\top = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} \quad \text{hence } \mathbf{A} \text{ is symmetric.}$$

$$\mathbf{B} = \begin{pmatrix} -15 & 2 & 3 \\ 2 & 8 & -17 \\ 3 & -17 & 2 \end{pmatrix} \quad \text{has transpose} \quad \mathbf{B}^\top = \begin{pmatrix} -15 & 2 & 3 \\ 2 & 8 & -17 \\ 3 & -17 & 2 \end{pmatrix} \quad \text{hence } \mathbf{B} \text{ is symmetric.}$$

7.1.3 Type of matrix: Skew-symmetric

Definition 7.3 (A Skew-symmetric matrix)**.**
A matrix **A** is said to be Skew-symmetric or antisymmetric if it satisfies all of the following properties

- The matrix is square.
- The matrix is equal to -1 times its transpose.

More compactly a matrix is Skew-symmetric if

$$\mathbf{A} = -\mathbf{A}^\top$$

Example 7.1.3 (Skew-symmetric matrices)**.**

$$\mathbf{A} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \quad \text{has transpose} \quad \mathbf{A}^\top = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \quad \text{hence } \mathbf{A} \text{ is Skew-symmetric.}$$

$$\mathbf{B} = \begin{pmatrix} 0 & 22 & -9 \\ -22 & 0 & -12 \\ 9 & 12 & 0 \end{pmatrix} \quad \text{has transpose} \quad \mathbf{B}^\top = \begin{pmatrix} 0 & -22 & 9 \\ 22 & 0 & 12 \\ -9 & -12 & 0 \end{pmatrix} \quad \text{hence } \mathbf{B} \text{ is Skew-symmetric.}$$

Notice that a skew-symmetric matrix always has a zeros along its main diagonal.

7.1.4 Type of matrices: Complex, Hermitian and Skew-Hermitian

Definition 7.4 (Complex matrix)**.**
A complex matrix **A** is a matrix which can have complex number elements.

Example 7.1.4 (Complex matrices)**.**

$$\mathbf{Z} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

,

$$\mathbf{W} = \begin{pmatrix} 1-2i & 3 & 4-\pi i & 0.5 \\ -12 & 7+2i & 2i & \sqrt{2} \\ i & 2 & -8 & 10 \end{pmatrix}$$

Definition 7.5 (Hermitian matrix)**.**
A complex matrix $\mathbf{A}$ is a Hermitian matrix if it satisfies all of the following properties

- The matrix is square.
- The transpose of $\mathbf{A}$ is made up of the complex conjugate of $\mathbf{A}$.

More compactly a matrix is Hermitian if

$$\mathbf{A} = \overline{\mathbf{A}}$$

where $\bar{\cdot}$ denotes complex conjugate.

Example 7.1.5 (Hermitian matrices)**.**

$$\mathbf{Z} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{thus} \quad \mathbf{Z}^\top = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{Z}^\top = \overline{\mathbf{Z}} \quad \therefore \mathbf{Z} \quad \text{is Hermitian.}$$

$$\mathbf{Y} = \begin{pmatrix} 12 & 2-i & 4 \\ 2+i & -3 & 1+\sqrt{2}i \\ 4 & 1-\sqrt{2}i & 1 \end{pmatrix} \quad \text{thus} \quad \mathbf{Y}^\top = \begin{pmatrix} 12 & 2+i & 4 \\ 2-i & -3 & 1-\sqrt{2}i \\ 4 & 1+\sqrt{2}i & 1 \end{pmatrix}$$

Since $\mathbf{Y}^\top = \overline{\mathbf{Y}}$ we can say $\mathbf{Y}$ is Hermitian.

Note 7.2.
Notice that a matrix can only be Hermitian if it has real number elements on its main diagonal. The Hermitian property requires elements on the main diagonal to be their own complex conjugate, which can only occur with the elements on the main diagonal are real numbers.

Definition 7.6 (Skew-Hermitian matrix)**.**
A complex matrix $\mathbf{A}$ is a Hermitian matrix if it satisfies all of the following properties

- The matrix is square.
- The transpose of $\mathbf{A}$ is made up of -1 times the complex conjugate of $\mathbf{A}$.

More compactly a matrix is Hermitian if

$$\mathbf{A} = -\overline{\mathbf{A}}$$

where $\bar{\cdot}$ denotes complex conjugate.

Example 7.1.6 (Hermitian matrices).

$$\mathbf{Z} = \begin{pmatrix} 0 & 1-i \\ -1-i & i \end{pmatrix} \quad \text{now} \quad \mathbf{Z}^\top = \begin{pmatrix} 0 & -1-i \\ 1-i & i \end{pmatrix} \quad \text{and}$$
$$\overline{\mathbf{Z}} = \begin{pmatrix} 0 & 1+i \\ -1+i & -i \end{pmatrix} \quad \text{hence} \quad \mathbf{Z}^\top = -\overline{\mathbf{Z}}$$

$\therefore \mathbf{Z}$ is Skew-Hermitian.

Note 7.3.
Notice that a matrix can only be skew-Hermitian if the elements on its main diagonal are complex numbers or zero.

7.2 Operation: trace

Definition 7.7 (The trace).
The trace of a square matrix $\mathbf{A}$ of order n is the sum of the elements on the main diagonal. More compactly,

$$\mathrm{Tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}$$

where a_{ii} is the i^{th}-row i^{th}-column element of $\mathbf{A}$.

The trace takes a matrix and returns a scalar result.

Example 7.2.1 (trace).

Given the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 5 & 12 \\ -1 & -8 & 1 \\ 0.5 & 7 & 4 \end{pmatrix}$$

The trace is found by summing the elements on the main diagonal.

$$\begin{pmatrix} \boxed{1} & 5 & 12 \\ -1 & \boxed{-8} & 1 \\ 0.5 & 7 & \boxed{4} \end{pmatrix}$$

thus,

$$\mathrm{Tr}(\mathbf{A}) = 1 + (-8) + 4 = -3.$$

7.2.1 Properties of the trace

(i) The trace of a sum

$$\mathrm{Tr}(\mathbf{A}+\mathbf{B}) = \mathrm{Tr}(\mathbf{A}) + \mathrm{Tr}(\mathbf{B})$$

(ii) The trace of a scalar times a matrix

$$\mathrm{Tr}(k\mathbf{A}) = k\,\mathrm{Tr}(\mathbf{A})$$

(iii) The trace of the transpose

$$\mathrm{Tr}(\mathbf{A}) = \mathrm{Tr}(\mathbf{A}^\top)$$

(iv) The trace of a product. Given an $n \times m$ matrix $\mathbf{X}$ and a $m \times n$ matrix $\mathbf{Y}$

$$\mathrm{Tr}(\mathbf{X}\mathbf{Y}) = \sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij}y_{ji}$$

This property can be used to show that

$$\mathrm{Tr}(\mathbf{X}\mathbf{Y}) = \mathrm{Tr}(\mathbf{Y}\mathbf{X})$$

7.3 Type of matrices: upper triangular, lower triangular and diagonal matrices.

Definition 7.8 (A lower triangular matrix)**.**
A *square* matrix is said to be a *lower triangular matrix* if all the elements above the main diagonal of the square matrix are all zero.

$$\mathbf{L} = \begin{pmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & & 0 \\ l_{31} & l_{32} & l_{33} & & 0 \\ \vdots & & & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{pmatrix}$$

Example 7.3.1. Examples of lower triangular matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 4 & 0 \\ -1 & 8 & 10 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 9 & 0 & 0 \\ 12 & 0 & 10 & 21 \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$$

Definition 7.9 (An upper triangular matrix)**.**
A *square* matrix is said to be an *upper triangular matrix* if all the elements below the main diagonal of the square matrix are all zero.

$$\mathbf{U} = \begin{pmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ 0 & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{pmatrix}$$

Example 7.3.2 (Examples of upper triangular matrices)**.**

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0.25 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Definition 7.10 (An diagonal matrix)**.**
A *square* matrix is said to be a *diagonal* matrix if it is **both** upper and lower triangular

$$\mathbf{D} = \begin{pmatrix} d_{11} & 0 & 0 & \cdots & 0 \\ 0 & d_{22} & 0 & \cdots & 0 \\ 0 & 0 & d_{33} & \cdots & 0 \\ 0 & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_{nn} \end{pmatrix}$$

Example 7.3.3 (Examples of upper triangular matrices)**.**

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}$$

7.4 Operation: Determinant

Earlier in the course, we introduced the determinant of a matrix as a means of finding the cross product between vectors. We have already seen that the determinant of a $2x2$ matrix

$$M_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

is defined as

Formula 7.1.

$$\det(M_2) \equiv \begin{vmatrix} a & b \\ c & d \end{vmatrix} \equiv ad - bc.$$

and for a $3x3$ matrix

$$M_3 = \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}.$$

The determinant is defined as

Formula 7.2.

$$\det(M_3) \equiv \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \equiv A_1 \begin{vmatrix} B_2 & B_3 \\ C_2 & C_3 \end{vmatrix} - A_2 \begin{vmatrix} B_1 & B_3 \\ C_1 & C_3 \end{vmatrix} + A_3 \begin{vmatrix} B_1 & B_2 \\ C_1 & C_2 \end{vmatrix}$$

It will transpire that the determinant of a matrix is a rather powerful function when it comes to solving linear systems of equations. Due to this, we will here spend some time examining the process of finding the determinant of a matrix more formally.

7.4.1 Cofactor expansion of the determinant

In order to find the determinant of a matrix that is of order larger than three we can employ what is called a *cofactor* expansion across any row or any column of a given matrix.

Definition 7.11 (The Minor of a matrix entry)**.** Given the square matrix **A**, the *minor* of the entry in the ith-row and jth column denoted

$$M_{ij}$$

is the determinant of the submatrix formed by deleting the ith-row and jth column from A.

Example 7.4.1. Take for example the following matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 4 \\ 0.5 & -1 & 2 \\ 2 & 4 & 0 \end{pmatrix}$$

To find the Minor of 2nd-row 1st-column entry, we require

$$M_{21} = \det \begin{pmatrix} \blacksquare & 3 & 4 \\ \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & 4 & 0 \end{pmatrix} = \det \begin{pmatrix} 3 & 4 \\ 4 & 0 \end{pmatrix} = (3)(0) - (4)(4) = -16.$$

To find the Minor M_{32} we have

$$M_{32} = \det \begin{pmatrix} 1 & \blacksquare & 4 \\ 0.5 & \blacksquare & 2 \\ \blacksquare & \blacksquare & \blacksquare \end{pmatrix} = \det \begin{pmatrix} 1 & 4 \\ 0.5 & 2 \end{pmatrix} = (1)(2) - (4)(0.5) = 0.$$

Each of the entries of the matrix **A** has a minor associated with it.

Definition 7.12 (The Cofactor of a matrix entry)**.** Given the square matrix $\mathbf{A}$, the *Cofactor* of the entry in the ith-row and jth column denoted

$$C_{ij}$$

is given by

$$C_{ij} = (-1)^{i+j} M_{ij}$$

where M_{ij} is the minor of the ith-row, jth column element of the matrix $\mathbf{A}$.

Lets revisit our previous example and find some of its cofactors.

Example 7.4.2. Starting with the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 4 \\ 0.5 & -1 & 2 \\ 2 & 4 & 0 \end{pmatrix}$$

To find the cofactor of its 2nd-row 3rd-column element, C_{23}, for example, we first require the elements minor,

$$M_{23} = \det \begin{pmatrix} 1 & 3 & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \\ 2 & 4 & \blacksquare \end{pmatrix} = \det \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = (1)(4) - (3)(2) = -2.$$

Now we can find the cofactor for this element, from our definition of the cofactor we have

$$C_{ij} = (-1)^{i+j} M_{ij}$$

for the 2nd-row 3rd-column element we have

$$C_{23} = (-1)^{2+3} M_{23} = (-1)^5(-2) = 2.$$

To find another cofactor we repeat this same process, take for example

$$C_{22} = (-1)^{2+2} M_{22}$$

earlier we had found the $M_{22} = 0$, thus,

$$C_{22} = (-1)^4(0) = 0.$$

We are now in a position to find the determinant of a general $n \times n$ matrix.

The determinant of a matrix $\mathbf{A}$ can be found by performing a cofactor expansion about any row or column of the matrix $\mathbf{A}$. That is

$$\det(\mathbf{A}) = \sum_{j=1}^{n} a_{ij}C_{ij} \quad \text{expansion about the i}^{th}\text{-row}$$

$$\det(\mathbf{A}) = \sum_{i=1}^{n} a_{ij}C_{ij} \quad \text{expansion about the j}^{th}\text{-column}$$

where a_{ij} is the ith-row jth-column element of $\mathbf{A}$ and C_{ij} is the cofactor associated with a_{ij}.

Example 7.4.3. Given

$$\mathbf{A} = \begin{pmatrix} 2 & 4 & 7 \\ 6 & 0 & 3 \\ 1 & 5 & 3 \end{pmatrix}$$

We can evaluate the determinant of $\mathbf{A}$ by performing a cofactor expansion about the first row.

$$\begin{aligned} \det(\mathbf{A}) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= 2C_{11} + 4C_{12} + 7C_{13} \end{aligned}$$

We need the cofactors for each of the elements on the first row, which involve the minors of the matrix elements

$$M_{11} = \det\begin{pmatrix} \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & 0 & 3 \\ \blacksquare & 5 & 3 \end{pmatrix} = \det\begin{pmatrix} 0 & 3 \\ 5 & 3 \end{pmatrix} = -15$$

$$C_{11} = (-1)^{1+1}M_{11} = -15.$$

our next cofactor

$$M_{12} = \det\begin{pmatrix} \blacksquare & \blacksquare & \blacksquare \\ 6 & \blacksquare & 3 \\ 1 & \blacksquare & 3 \end{pmatrix} = \det\begin{pmatrix} 6 & 3 \\ 1 & 3 \end{pmatrix} = 15$$

$$C_{12} = (-1)^{1+2}M_{12} = -15.$$

the final cofactor we need is C_{13}

$$M_{13} = \det\begin{pmatrix} \blacksquare & \blacksquare & \blacksquare \\ 6 & 0 & \blacksquare \\ 1 & 5 & \blacksquare \end{pmatrix} = \det\begin{pmatrix} 6 & 0 \\ 1 & 5 \end{pmatrix} = 30$$

$$C_{13} = (-1)^{1+3}M_{13} = 30.$$

returning to our cofactor expansion we have

$$\begin{aligned}\det(\mathbf{A}) &= 2C_{11} + 4C_{12} + 7C_{13} \\ &= 2(-15) + 4(15) + 7(30) = 120.\end{aligned}$$

We could have found this determinant had we been a little more clever with our cofactor expansion. Since we are free to expand about any row or column it would take less work to expand about a row or column that has *zero* entries. As an illustration look at a cofactor expansion about the second column

$$\begin{aligned}\det(\mathbf{A}) &= a_{12}C_{12} + a_{22}C_{22} + a_{13}C_{32} \\ &= (4)C_{12} + (0)C_{12} + (5)C_{32} \\ &= 4C_{12} + 5C_{32}\end{aligned}$$

We only need to work out 2 cofactors using this expansion!

$$M_{12} = \det\begin{pmatrix} \blacksquare & \blacksquare & \blacksquare \\ 6 & \blacksquare & 3 \\ 1 & \blacksquare & 3 \end{pmatrix} = \det\begin{pmatrix} 6 & 3 \\ 1 & 3 \end{pmatrix} = -15$$

$$C_{12} = (-1)^{1+2} M_{12} = -15.$$

and

$$M_{32} = \det\begin{pmatrix} 2 & \blacksquare & 7 \\ 6 & \blacksquare & 3 \\ \blacksquare & \blacksquare & \blacksquare \end{pmatrix} = \det\begin{pmatrix} 2 & 7 \\ 6 & 3 \end{pmatrix} = -36$$

$$C_{32} = (-1)^{3+2} M_{32} = 36.$$

returning to our expansion for the determinant we have

$$\det(\mathbf{A}) = 4C_{12} + 5C_{32} = 4(-15) + 5(36) = -60 + 180 = 120.$$

which is what we had found earlier.

Note 7.4. When asked to obtain the determinant of a 3×3 or larger matrix, it is advantageous to choose a row or column which has the most zero entries.

Example 7.4.4 (a 4×4 determinant)**.**

Find the determinant of

$$\mathbf{B} = \begin{pmatrix} 2 & 4 & 7 & -1 \\ 2 & 0 & 3 & 4 \\ 1 & 5 & 8 & 1 \\ 0 & 0 & 0 & -10 \end{pmatrix}$$

Solution:
We can find the determinant of this matrix by performing a cofactor expansion about any row or column. The choice that leads to the least amount of computation is the 4th row as it contains three zero entries. This is a little better than expanding about the second column with contains two zeros.

The determinant expanded about the 4th row is given by

$$\begin{aligned} \det(\mathbf{B}) &= b_{41}C_{41} + b_{42}C_{42} + b_{43}C_{43} + b_{44}C_{44} \\ &= 0\,C_{41} + 0\,C_{42} + 0\,C_{43} + (-10)\,C_{44} \\ &= -10C_{44} \end{aligned}$$

all we require is a single cofactor C_{44}. To find this we require the minor, M_{44}

$$M_{44} = \det \begin{pmatrix} 2 & 4 & 7 & -1 \\ 2 & 0 & 3 & 4 \\ 1 & 5 & 8 & 1 \\ 0 & 0 & 0 & -10 \end{pmatrix} = \det \begin{pmatrix} 2 & 4 & 7 & \blacksquare \\ 2 & 0 & 3 & \blacksquare \\ 1 & 5 & 8 & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{pmatrix} = \det \begin{pmatrix} 2 & 4 & 7 \\ 2 & 0 & 3 \\ 1 & 5 & 8 \end{pmatrix}$$

This is still a 3×3 determinant which requires its own cofactor expansion! (expanding about the 2nd row)

$$\det \begin{pmatrix} 2 & 4 & 7 \\ 2 & 0 & 3 \\ 1 & 5 & 8 \end{pmatrix} = 2(-1)^{2+1}\det \begin{pmatrix} \blacksquare & 4 & 7 \\ \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & 5 & 8 \end{pmatrix} + 0(-1)^{2+2}\det \begin{pmatrix} 2 & \blacksquare & 7 \\ \blacksquare & \blacksquare & \blacksquare \\ 1 & \blacksquare & 8 \end{pmatrix} + 3(-1)^{2+3}\det \begin{pmatrix} 2 & 4 & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \\ 1 & 5 & \blacksquare \end{pmatrix}$$

$$= -2\det \begin{pmatrix} 4 & 7 \\ 5 & 8 \end{pmatrix} + 0\det \begin{pmatrix} 2 & 7 \\ 1 & 8 \end{pmatrix} - 3\det \begin{pmatrix} 2 & 4 \\ 1 & 5 \end{pmatrix} = -12.$$

We now have $M_{44} = -12$ for the matrix $\mathbf{B}$. The cofactor

$$C_{44} = (-1)^{4+4}M_{44} = -12$$

from our cofactor expansion about the 4th row of matrix $\mathbf{B}$ we have

$$\det(\mathbf{B}) = -10C_{44} = (-10)(-12) = 120.$$

Note 7.5. Had we decided to cofactor expand the determinant of $\mathbf{B}$ across the first row we would have had to compute four 3×3 determinants!

7.4.2 Properties of the determinant

We have seen that getting the determinant of a matrix can be a time consuming process, however there are a number of important properties concerning the determinant of a matrix that can sometimes reduce the amount of work that is needed to compute a determinant.

> **The determinant of the Identity matrix** is always equal to 1, regardless of the size of the identity matrix.

Example 7.4.5.

$$\det(\mathbf{I}_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1 \qquad \det(\mathbf{I}_4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 1$$

> Given two matrices $\mathbf{A}$ and $\mathbf{B}$ such that all the entries in $\mathbf{B}$ are identical to all the entries of $\mathbf{A}$ except for one row of entries where the entries are a constant c times the corresponding entries in $\mathbf{A}$, then
>
> $$\det(\mathbf{B}) = c\det(\mathbf{A})$$

Example 7.4.6. Given the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 4 & -1 \\ 3 & 0.5 & 7 & 0 \\ -1 & 2 & 0.25 & 0 \\ 3 & 4 & 5 & -3 \end{pmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{pmatrix} 1 & 2 & 4 & -1 \\ 6 & 1 & 14 & 0 \\ -1 & 2 & 0.25 & 0 \\ 3 & 4 & 5 & -3 \end{pmatrix}$$

and that the $\det(\mathbf{A}) = -30$. Find $\det(\mathbf{B})$.

Solution:
In this problem we are given that the determinant

$$\det(\mathbf{A}) = -30 \qquad \text{check this for yourself!.}$$

Since the matrix $\mathbf{B}$ is identical to $\mathbf{A}$ except for the entries in one of its rows, all the entries on the second row of $\mathbf{B}$ are equal to 2 times the corresponding entries of $\mathbf{A}$. We can use our shortcut to state that

$$\det(\mathbf{B}) = 2 \times \det(\mathbf{A}) = -60.$$

The determinant of a product of square matrices is equal to the product of the determinants. That is, given $\mathbf{A}$ and $\mathbf{B}$ which are both square and of the same order,

$$\det(\mathbf{AB}) = \det(\mathbf{A}) \times \det(\mathbf{B})$$

Example 7.4.7. Given the matrices

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{B} \begin{pmatrix} 0 & 5 \\ 4 & 7 \end{pmatrix}$$

Show that $\det(\mathbf{AB}) = \det(\mathbf{A}) \times \det(\mathbf{B})$.

Solution:

We have

$$\det(\mathbf{A}) = (2)(4) - (3)(1) = 5$$
$$\det(\mathbf{B}) = (0)(7) - (5)(4) = -20$$

this tells us that $\det(\mathbf{A})\det(\mathbf{B}) = -100$
We also want the determinant of $\mathbf{AB}$

$$\mathbf{AB} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 0 & 5 \\ 4 & 7 \end{pmatrix} = \begin{pmatrix} 12 & 31 \\ 16 & 33 \end{pmatrix}$$

$$\det(\mathbf{AB}) = \det \begin{pmatrix} 12 & 31 \\ 16 & 33 \end{pmatrix} = (12)(33) - (31)(16) = -100$$

and we have shown that $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$ for this example.

If the i^{th}-row of a matrix $\mathbf{A}$ is the sum of the i^{th}-row of a matrix $\mathbf{B}$ and the i^{th}-row of matrix $\mathbf{C}$ and all the other corresponding entries of matrix $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are equal between the matrices, then

$$\det(\mathbf{A}) = \det(\mathbf{B}) + \det(\mathbf{C})$$

Note 7.6. For general matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ this is not true, ie $\det(\mathbf{A}) \neq \det(\mathbf{B}) + \det(\mathbf{C})$, it is only in specific cases such as this one that this equality holds.

Example 7.4.8. Taking

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & -2 \\ 0 & 2 & 12 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 & 0 & 2 \\ -4 & 9 & -12 \\ 0 & 2 & 12 \end{pmatrix} \qquad \text{and} \quad \mathbf{C} = \begin{pmatrix} 1 & 0 & 2 \\ 7 & -8 & 10 \\ 0 & 2 & 12 \end{pmatrix}$$

and given that $\det(\mathbf{B}) = 116$ and $\det(\mathbf{C}) = -88$, find $\det(\mathbf{A})$.

Solution:
From the matrices we can see that all three matrices are equal apart from a single row. For that particular row the matrix $\mathbf{A}$ is equal to the sum of the corresponding rows of the matrices $\mathbf{B}$ and $\mathbf{C}$. Hence we say that

$$\det(\mathbf{A}) = \det(\mathbf{B}) + \det(\mathbf{C}) = 116 - 88 = 28.$$

(as an exercise it is worthwhile verifying this by getting the determinant of $\mathbf{A}$ explicitly).

> The determinant of a matrix with an entire row/column of zeros is always zero.

Example 7.4.9.

$$\det\begin{pmatrix} 7 & 5 & -3 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & -12 & 7 & 8 \\ -3 & 0 & 4 & 9 \end{pmatrix} = 0 \qquad \det\begin{pmatrix} 0 & 2 & 3 & 7 \\ 0 & -3 & 0.6 & 17 \\ 0 & 4 & 6 & 0 \\ 0 & -0.25 & 2 & 1 \end{pmatrix} = 0$$

> The determinant of matrix is equal to the determinant of the transpose of the matrix. That is
>
> $$\det(\mathbf{A}) = \det(\mathbf{A}^\top)$$

> Given the matrix $\mathbf{A}$ the determinant of the matrix $\mathbf{A}^k$ is related to the determinant of $\mathbf{A}$ in the following manner
>
> $$\det(\mathbf{A}^k) = \big(\det(\mathbf{A})\big)^k$$
>
> for $k > 0$.

Example 7.4.10. Given that

$$\mathbf{A} = \begin{pmatrix} 6 & -2 & 1 \\ 0 & 1 & 3 \\ -1 & 4 & 2 \end{pmatrix}$$

find the value of $\det(\mathbf{A}^4)$.

Solution:
We could multiply the matrix $\mathbf{A}$ by itself 4 times and find the determinant of the resulting matrix, which would be an impressive exercise on its own! However this problem can be complete without resorting such a length computation. First we find

$$\begin{aligned}
\det(\mathbf{A}) &= \det\begin{pmatrix} 6 & -2 & 1 \\ 0 & 1 & 3 \\ -1 & 4 & 2 \end{pmatrix} \qquad \text{expanding about the first column} \\
&= 6(-1)^{1+1}\det\begin{pmatrix} \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & 1 & 3 \\ \blacksquare & 4 & 2 \end{pmatrix} + 0(-1)^{2+1}\det\begin{pmatrix} \blacksquare & -2 & 1 \\ \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & 4 & 2 \end{pmatrix} + -1(-1)^{3+1}\det\begin{pmatrix} \blacksquare & -2 & 1 \\ \blacksquare & 1 & 3 \\ \blacksquare & \blacksquare & \blacksquare \end{pmatrix} \\
&= -56.
\end{aligned}$$

Now,

$$\det\left(\mathbf{A}^4\right) = \left(\det(\mathbf{A})\right)^4 = (-56)^4 = 9834496$$

Given the matrix $\mathbf{A}$ which is *triangular* the determinant of the matrix $\mathbf{A}$ is simply the *product of all the entries on the main diagonal.* That is, if $\mathbf{A}$ is a triangular matrix of order n then

$$\det(\mathbf{A}) = \prod_{i=1}^{n} a_{ii}$$

Example 7.4.11.

$$\det\begin{pmatrix} 2 & 3 & -1 & 12 \\ 0 & -1 & 0 & 4 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 2 \end{pmatrix} = (2)(-1)(5)(2) = -20$$

$$\det\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 10 & 2 & 0 & 0 & 0 \\ 55 & 72 & -1 & 0 & 0 \\ 61 & 99 & 1 & 3 & 0 \\ -44 & 51 & 43 & 12 & -1 \end{pmatrix} = (1)(2)(-1)(3)(-1) = 6$$

7.5 Type of Matrices: Inverse matrix and Invertible matrices

One of the most important uses of a determinant is in the computation of an *inverse* matrix which is the subject of the next section of this course. Here we define what we mean by an inverse matrix.

Let $\mathbf{A}$ be an $n \times n$ matrix. If there exists an $n \times n$ matrix $\mathbf{B}$ such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$$

where $\mathbf{I}_n$ is the $n \times n$ identity matrix, then the matrix $\mathbf{A}$ is said to be nonsingular or invertible.

The matrix $\mathbf{B}$ is said to be the inverse of $\mathbf{A}$, and is usually denoted $\mathbf{A}^{-1}$.

Example 7.5.1. The matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

is invertible since the matrix

$$\mathbf{B} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

is its inverse (show this!).

Note 7.7. Not every square matrix is invertible!

Example 7.5.2. Show that the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

is not invertible.

Solution:
We want to show that there exists no matrix

$$\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

such that

$$\mathbf{AB} = \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Lets assume there does exist an inverse to the matrix $\mathbf{A}$, this would mean

$$\mathbf{AB} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} b_{11} + b_{21} & b_{12} + b_{22} \\ 0 & 0 \end{pmatrix}$$

is equal to the identity matrix, which is impossible as

$$\begin{pmatrix} b_{11} + b_{21} & b_{12} + b_{22} \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

as that would require us to have $0 = 1$ which is a contradiction. As such the matrix $\mathbf{A}$ does not have an inverse and is said to be *singular* or *noninvertible*.

Two questions that we will attempt to answer in the next section of the notes are

- How do we know if a general square matrix is invertible?
- Given an invertible matrix how to we find its inverse?

Finding the inverse of a matrix takes a little more work, however at this point we are able to define yet another special type of matrix.

7.5.1 Orthogonal matrix

An important class of matrices for engineering and science are known as *orthogonal* matrices. These matrices often arise when we are working with problems that involve vectors in a particular basis that we wish to transform into another basis. The transition matrix from one basis to another basis is an *orthogonal* matrix.

At this point however we are only interested in the mathematical definition of what an orthogonal matrix is.

A matrix $\mathbf{A}$ is said to be orthogonal if its transpose is equal to its inverse, that is if

$$\mathbf{A}^\top = \mathbf{A}^{-1}$$

Example 7.5.3. Show that the matrix

$$\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is an orthogonal matrix.

Solution:
The transpose of the matrix $\mathbf{A}$

$$\mathbf{A}^\top = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

now we want to check if $\mathbf{A}^\top = \mathbf{A}^{-1}$

$$\begin{aligned} \mathbf{A}\mathbf{A}^\top &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \end{aligned}$$

and thus,

$$\mathbf{A}^\top = \mathbf{A}^{-1}$$

Example 7.5.4. Other examples of orthogonal matrices include

$$\mathbf{A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Show that these are in fact orthogonal matrices!