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MATHEMATICAL PHYSICS

EE112

Engineering Mathematics II

Lines and Planes

Prof. D. M. Heffernan and Mr. S. Pouryahya

4 Equations of Lines and Planes

This section focuses on formulating the equations of lines, planes and their intersections. Vectors are a powerful way of representing such quantities as they are easily extended to as many dimensions a particular problem may require with little or no adjustment to the basic formulae. We will be looking at examples in 3D space throughout this section; this is done for the sake of simplicity in visualising examples and brevity of notation.

4.1 The Equation of a Line

Beginning with the equation of a line we will be interested in three forms all of which are representations of the same line

- Equation of a line in vector form
- Equation of a line in parametric form
- Equation of a line in symmetric form

4.1.1 Equation of a Line in Vector Form

We seek the equation of a line that passes through the points

$$p_1(x_1, y_1, z_1) \quad \text{and} \quad p_2(x_2, y_2, z_2)$$

The associated position vector with the point p_1 is

$$\mathbf{r}_1 = x_1\,\mathbf{i} + y_1\,\mathbf{j} + z_1\,\mathbf{k}$$

the point p_2 has an associated position vector

$$\mathbf{r}_2 = x_2\,\mathbf{i} + y_2\,\mathbf{j} + z_2\,\mathbf{k}.$$

Let the point $p(x, y, z)$ be any point on the line

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

Given two points p_1 and p_2 we denote the vector starting at the point p_1 and ending at the point p_2 as

$$\overrightarrow{p_1p_2}$$

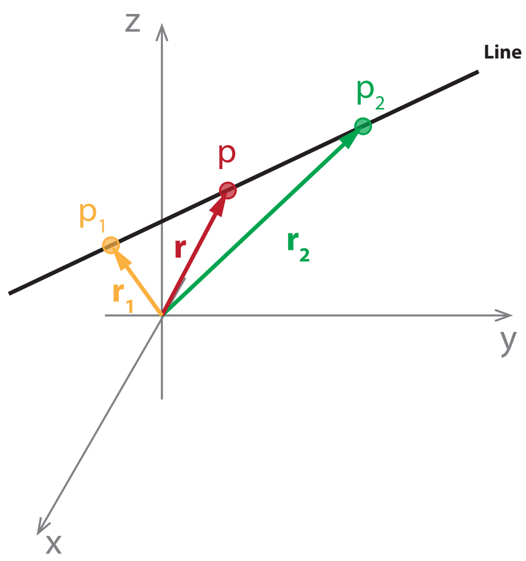

Figure 1: The line passing through the points p_1 and p_2 with associated position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$. The point p is an arbitrary point on the line with associated vector $\mathbf{r}$

Notice above that the vector $\overrightarrow{p_1 p}$ is parallel to $\overrightarrow{p_1 p_2}$, reformulating this in terms of the associated position vectors we have

$$\mathbf{r} - \mathbf{r}_1 \quad \text{is parallel to} \quad \mathbf{r}_2 - \mathbf{r}_1$$

Since these vectors are parallel the can be expressed as scalar multiples of one another; that is for some scalar t we have

$$\mathbf{r} - \mathbf{r}_1 = t(\mathbf{r}_2 - \mathbf{r}_1)$$

It is standard practice in textbooks to shorten the notation at this point and introduce a quantity known as the *direction vector*, we let

$$\begin{aligned} \mathbf{a} = \mathbf{r}_2 - \mathbf{r}_1 &= (x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j} + (z_2 - z_1)\,\mathbf{k} \\ &= \langle (x_2 - x_1), (y_2 - y_1), z_2 - z_1) \rangle \end{aligned}$$

Using this notation we can write

$$\begin{aligned} \mathbf{r} - \mathbf{r}_1 &= t(\mathbf{r}_2 - \mathbf{r}_1) \\ &= t\mathbf{a} \end{aligned}$$

which brings us to our result

Important Formula 4.1 (vector equation of a line)**.**
The vector equation of the line joining the points p_1 and p_2, which have respective position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ is given by

$$\mathbf{r} = \mathbf{r}_1 + t\mathbf{a}$$

where $\mathbf{a}$ is the direction vector of the line, given by

$$\mathbf{a} = \mathbf{r}_2 - \mathbf{r}_1$$

4.1.2 Equation of a line in parametric form

Beginning with our result from the previous section and restricting ourselves to vectors in $\mathbb{R}^3$ for brevity of notation we have, the equation of line that passes through the points p_1 and p_2 given by,

$$\mathbf{r} = \mathbf{r}_1 + t\mathbf{a}$$

using the angled bracket notation we have

$$\begin{aligned}\langle x, y, z\rangle &= \langle x_1, y_1, z_1\rangle + t\langle a_1, a_2, a_3\rangle \\ &= \langle x_1 + ta_1, y_1 + ta_2, z_1 + ta_3\rangle\end{aligned}$$

comparing the components of the vectors together we have the parametric form of the line

Important Formula 4.2 (parametric form of the line)**.** Given two points on a line p_1 and p_2 in $\mathbb{R}^3$ the parametric form of the line is given by

$$\begin{aligned}x &= x_1 + ta_1 \\ y &= y_1 + ta_2 \\ z &= z_1 + ta_3\end{aligned}$$

where a_1, a_2 and a_3 are the components of the direction vector

$$a_1 = x_2 - x_1, \quad a_2 = y_2 - y_1 \quad \text{and} \quad a_3 = z_2 - z_1$$

4.1.3 Equation of a line in symmetric form

The symmetric form of the line is written by solving for the parameter t in the parametric form of the line, we have the parametric form

$$x = x_1 + ta_1, \quad y = y_1 + ta_2, \quad z = z_1 + ta_3$$

solving each of these for t we have

$$t = \frac{x - x_1}{a_1}, \quad t = \frac{y - y_1}{a_2}, \quad t = \frac{z - z_1}{a_3}$$

and we have the symmetric form

Important Formula 4.3 (symmetric form of the line)**.** Given two points on a line p_1 and p_2 in $\mathbb{R}^3$ the symmetric form of the line is given by

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{a_2} = \frac{z - z_1}{a_3}$$

where a_1, a_2 and a_3 are the components of the direction vector

$$a_1 = x_2 - x_1, \quad a_2 = y_2 - y_1 \quad \text{and} \quad a_3 = z_2 - z_1$$

4.1.4 Worked example

Example 4.1.1.
Find the vector equation, parametric form and symmetric form for the line passing through the points $(2, -1, 8)$ and $(5, 6, -3)$.

Solution:
Begin by writing down the associated position vectors to both points

$$\mathbf{r}_1 = 2\,\mathbf{i} - \mathbf{j} + 8\,\mathbf{k}, \qquad \mathbf{r}_2 = 5\,\mathbf{i} + 6\,\mathbf{j} - 3\,\mathbf{k}$$

now the *direction vector* can be found

$$\begin{aligned} \mathbf{a} = \mathbf{r}_2 - \mathbf{r}_1 &= 5\,\mathbf{i} + 6\,\mathbf{j} - 3\,\mathbf{k} - (2\,\mathbf{i} - \mathbf{j} + 8\,\mathbf{k}) \\ &= 3\,\mathbf{i} + 7\,\mathbf{j} - 11\,\mathbf{k} \\ &= \langle 3, 7, -11 \rangle \end{aligned}$$

Using the formula

$$\mathbf{r} = \mathbf{r}_1 + t\mathbf{a}$$

we have

$$\langle x, y, z \rangle = \langle 2, -1, 8 \rangle + t\langle 3, 7, -11 \rangle$$

example continued ...

and finally have the vector form

$$\langle x, y, z\rangle = \langle 2+3t, -1+7t, 8-11t\rangle$$

To obtain the the parametric form (also referred to as the component form) we compare the components and obtain

The parametric form

$$\begin{aligned} x &= 2+3t \\ y &= -1+7t \\ z &= 8-11t \end{aligned}$$

Finally solving each of these equations for t and equating will yield the symmetric form

$$t = \frac{x-2}{3}, \quad t = \frac{y+1}{7} \quad t = \frac{z-8}{-11}$$

The symmetric form

$$\frac{x-2}{3} = \frac{y+1}{7} = \frac{8-z}{11}$$

4.2 The Plane and Intersections

The aim here will be to specify the equation of a plane in vector notation and then examine the intersection between a plane and a line (resulting in a point) and the intersection of two planes (resulting in a line[1]).

4.2.1 The equation of a plane

There are two pieces of information one requires in order to uniquely specify a plane; these are

1. A point on the plane, denoted here as p_0
2. A normal vector to the plane, denoted $\mathbf{n}$

We begin by taking $p(x, y, z)$ to be an arbitrary point on the plane. The vector from the point p_0 to the point p denoted as $\overrightarrow{p_0p}$ is a vector on the plane,

$$\overrightarrow{p_0p} = (x - x_0)\,\mathbf{i} + (y - y_0)\,\mathbf{j} + (z - z_0)\,\mathbf{k}$$

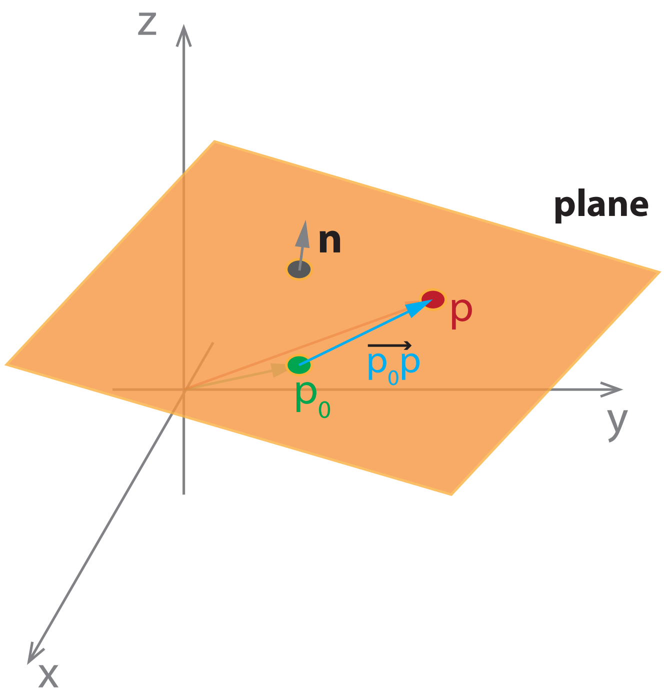

Figure 2: The plane given a normal vector $\mathbf{n}$ and point p_0, the point p is an arbitrary point on the plane.

We have that $\mathbf{n}$ is normal to the plane and thus it must also be perpendicular to $\overrightarrow{p_0p}$, we can express this relationship in terms of the dot product

[1]these are true only if we are working in $\mathbb{R}^3$, in higher dimensions the intersections between *hyperplanes* results in objects that are themselves *hypersurfaces spanned* by vectors.

$$\mathbf{n} \cdot \overrightarrow{p_0 p} = 0$$

This generates the equation of the plane; taking the normal vector to be

$$\mathbf{n} = n_1\,\mathbf{i} + n_2\,\mathbf{j} + n_3\,\mathbf{k}$$

we have the equation of the plane given by

Important Formula 4.4 (equation of a plane)**.**

Given a point on the plane $p_0(x_0, y_0, z_0)$ and a normal vector to the plane $\mathbf{n} = n_1\,\mathbf{i} + n_2\,\mathbf{j} + n_3\,\mathbf{k}$ the equation of the plane is given by

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

Equivalently one can write the equation of the plane as

Formula 4.1.

$$n_1 x + n_2 y + n_3 z = D$$

where $D = n_1 x_0 + n_2 y_0 + n_3 z_0$

4.2.2 Worked example

Example 4.2.1.

Find the equation of the plane which passes through the three points

$$p_1(0,0,1), \quad p_2(2,0,0) \quad \text{and} \quad p_3(0,3,0)$$

Solution:
In order to write down the equation of the plane we require a unit normal vector to the plane. Although we are not given this directly we can find a normal vector by taking the cross product between two vectors that are on the plane.

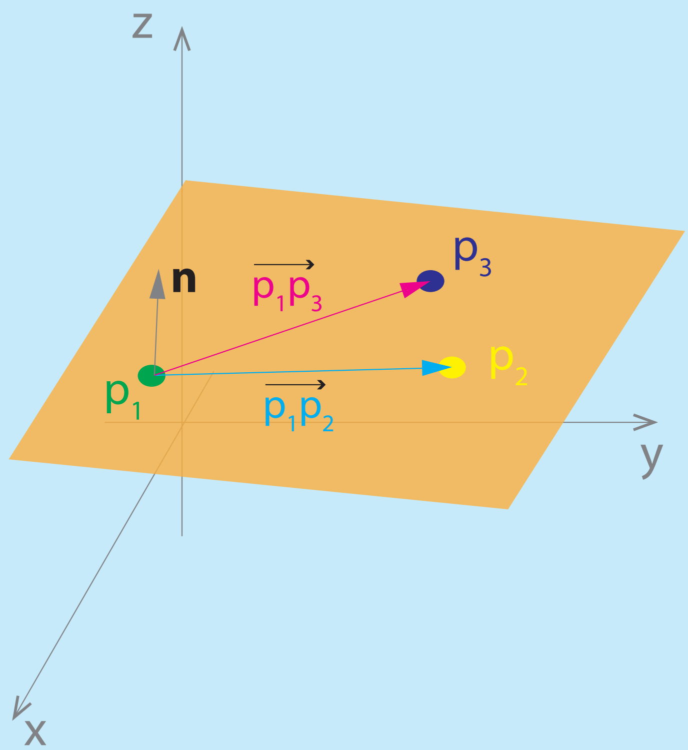

Figure 3: Constructing a normal vector to the plane from three points on the plane.

One can construct the vectors

$$\overrightarrow{p_1p_2} \quad \text{and} \quad \overrightarrow{p_1p_3}$$

to obtain two vectors that are on the plane. Taking the cross product between these vectors yields a normal vector

$$\mathbf{n} = \overrightarrow{p_1p_2} \times \overrightarrow{p_1p_3}$$

example continued . . .

for this example we have

$$\overrightarrow{p_1p_2} = 2\,\mathbf{i} + 0\,\mathbf{j} - 1\,\mathbf{k} \qquad \overrightarrow{p_1p_3} = 0\,\mathbf{i} + 3\,\mathbf{j} - 1\,\mathbf{k}$$

and hence we have a normal vector

$$\begin{aligned}\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} &= \mathbf{i}\begin{vmatrix} 0 & -1 \\ 3 & -1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} \\ &= \big((0)(-1) - (-1)(3)\big)\mathbf{i} - \big((2)(-1) - (-1)(0)\big)\mathbf{j} + \big((2)(3) - (0)(0)\big)\mathbf{k} \\ &= 3\,\mathbf{i} + 2\,\mathbf{j} + 6\,\mathbf{k}.\end{aligned}$$

Now let $p(x, y, z)$ be an arbitrary point on the plane; we seek a vector from a point on the plane to this arbitrary point, the vector $\overrightarrow{p_1p}$ will do

$$\overrightarrow{p_1p} = (x - 0)\,\mathbf{i} + (y - 0)\,\mathbf{j} + (z - 1)\,\mathbf{k} = x\,\mathbf{i} + y\,\mathbf{j} + (z - 1)\,\mathbf{k}$$

Now the plane must satisfy the condition

$$\mathbf{n} \cdot \overrightarrow{p_1p} = 0 \quad \Rightarrow \quad (3\,\mathbf{i} + 2\,\mathbf{j} + 6\,\mathbf{k}) \cdot (x\,\mathbf{i} + y\,\mathbf{j} + (z - 1)\,\mathbf{k}) = 0$$

finally we have the equation of the plane

$$3x + 2y + 6(z - 1) = 0$$

which can be written as

$$3x + 2y + 6z = 6$$

4.2.3 Worked Example: Intersection between a line and a plane.

Example 4.2.2.
Find the point where the line

$$x = \frac{8}{3} + 2t, \quad y = -2t, \quad z = 1 + t$$

intersects the plane

$$3x + 2y + 6z = 6$$

Solution:

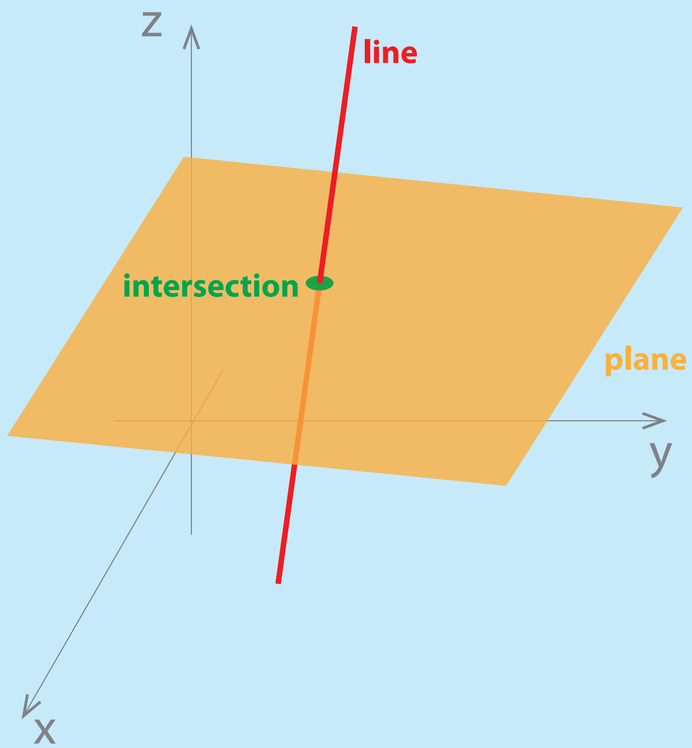

Figure 4: Intersection between a line and a plane in $\mathbb{R}^3$.

At the intersection point both the equation of the line and the equation of the plane must be simultaneously satisfied.

example continued ...

Hence we can substituted the parametric equations of the line into our equation for the plane

$$
\begin{aligned}
3\left(\frac{8}{3}+2t\right)+2\left(-2t\right)+6\left(1+t\right)&=6\\
8+6t-4t+\cancel{6}+6t&=\cancel{6}\\
&\Rightarrow 8t=-8
\end{aligned}
$$

we have found that at the intersection point we must have

$$t=-1.$$

From the parametric form of the line we can find this point

$$
\begin{aligned}
&x=\frac{8}{3}+2(-1)=\frac{2}{3}\\
&y=-2(-1)=2\\
&z=1+(-1)=0
\end{aligned}
$$

and we have found that the intersection point occurs at

$$\left(\frac{2}{3},\,2,\,0\right)$$

4.2.4 Worked Example: Intersection between two planes.

Example 4.2.3.

Find the line of intersection between the two planes

$$2x - 3y + 4z = 1$$
$$x - y - z = 5$$

Solution:

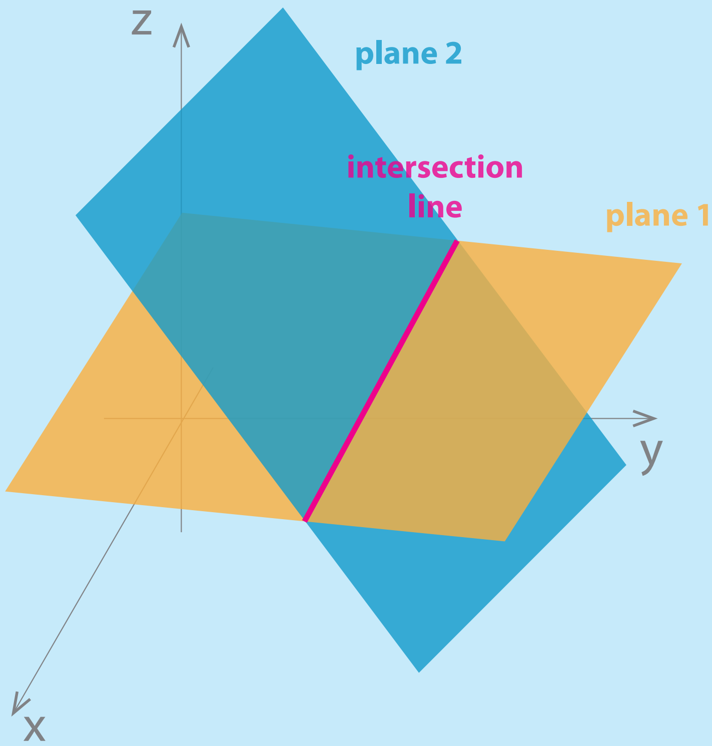

Figure 5: Intersection of two planes in $\mathbb{R}^3$.

Notice that we have two equations (the planes) and three unknowns (x, y and z). This means there will not be *one* unique point that will satisfy the equations but rather a collection of points (the line) that will satisfy the system. We begin by letting

$$z = t$$

and proceed to solve the given equations for x and y. This will result in a parametric equation for the intersection line.

example continued ...

The equations of our intersecting planes

$$\begin{aligned}2x - 3y + 4z &= 1\\ x - y - z &= 5\end{aligned}$$

become

$$\begin{aligned}2x - 3y &= 1 - 4t\\ x - y &= 5 + t\end{aligned}$$

when we let $z = t$. This is a set of two simultaneous equations in two unknowns which we can solve.

$$\begin{aligned}2x - 3y &= 1 - 4t\\ 3x - 3y &= 15 + 3t\end{aligned}$$

Subtracting the first equation from the second yields

$$x = 14 + 7t.$$

We can now find y by substituting this value of x into

$$\begin{aligned}x - y &= 5 + t\\ &\Rightarrow 14 + 7t - y = 5 + t\end{aligned}$$

and we have

$$y = 9 + 6t.$$

Finally we can write the parametric equations of the line of intersection between the two planes

$$x = 14 + 7t, \qquad y = 9 + 6t \quad \text{and} \quad z = t$$