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THE NATIONAL UNIVERSITY OF IRELAND MAYNOOTH

MATHEMATICAL PHYSICS

EE112

Engineering Mathematics II

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5 Scalars and Vectors

5.1 The Scalar

Quantities that can be completely specified by a number and unit and that therefore have *magnitude* only are called scalars.
Examples of physicals quantities that are scalars are:

- mass
- length
- time
- density
- energy
- temperature.

5.2 The Vector

A vector is a *directed line segment*. To describe a vector we must specify both its *length* and its *direction*.

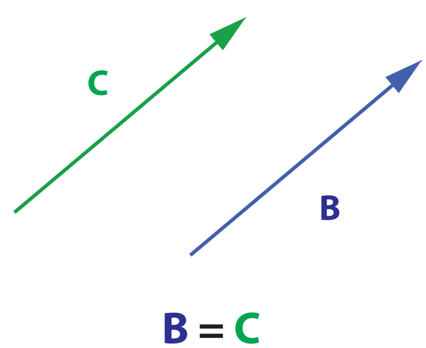


Figure 1: If two vectors have the same length and the same direction they are equal.

The length of a vector is called its *magnitude*. The magnitude of the vector $\mathbf{A}$ is written $|\mathbf{A}|$ or simply A

$$A \equiv |\mathbf{A}|$$

If the length of a vector is one unit, we call it a *unit vector*,

$$\hat{\mathbf{A}} = \frac{\mathbf{A}}{|\mathbf{A}|}$$

Hence,

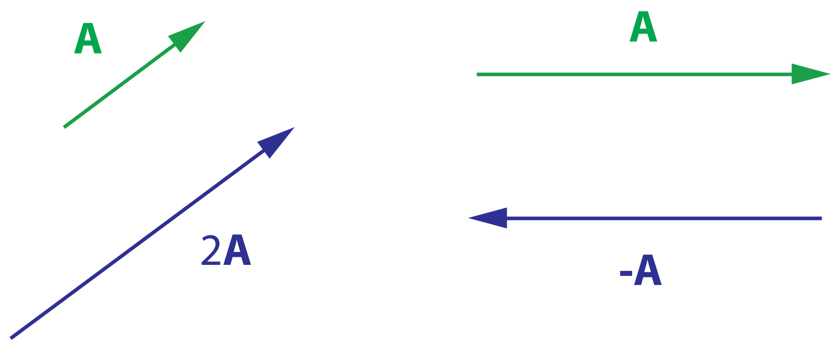
$$\mathbf{A} = A\hat{\mathbf{A}}$$

5.3 Multiplication of a Vector by a Scalar

If we multiply a vector $\mathbf{A}$ by a positive scalar α, the result is a new vector $\mathbf{C} = \alpha\mathbf{A}$. The vector $\mathbf{C}$ is parallel to $\mathbf{A}$ and its length is α times greater.
Hence,

$$\hat{\mathbf{C}} = \hat{\mathbf{A}} \quad \text{and} \quad C = \alpha A.$$

The result of multiplying a vector by -1 is a new vector pointing in the opposite direction (antiparallel) to the original.



(a) Multiplication by a positive scalar. (b) Multiplication by -1.

Figure 2: Multiplying a vector by a scalar.

5.4 Addition of Two Vectors

Addition of two vectors has the simple geometric interpretation show in Figure 3.

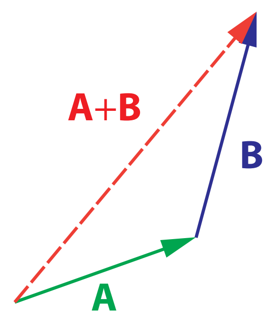


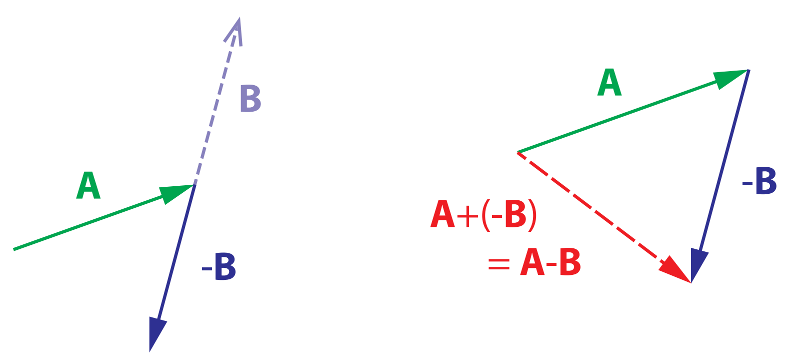
Figure 3: Addition of the vector $\mathbf{B}$ to the vector $\mathbf{A}$.

Note 5.1 (Addition between vectors)**.**
<u>Rule:</u> To add $\mathbf{B}$ to $\mathbf{A}$, place the tail of $\mathbf{B}$ at the head of $\mathbf{A}$. The sum $\mathbf{A}+\mathbf{B}$ is the vector from the tail of $\mathbf{A}$ to the head of $\mathbf{B}$.

5.5 Subtraction Between Two Vectors

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$$

To subtract $\mathbf{B}$ from $\mathbf{A}$ we multiply $\mathbf{B}$ by -1 and add the result to $\mathbf{A}$.



(a) Find $-\mathbf{B}$.

(b) Sum the vectors to find $\mathbf{A} - \mathbf{B}$

Figure 4: Subtracting $\mathbf{B}$ from $\mathbf{A}$ is equivalent to adding -$\mathbf{B}$ to $\mathbf{A}$

An equivalent way to construct $\mathbf{A} - \mathbf{B}$ is to use the following rule illustrated by Figure 5

Note 5.2 (Subtraction between vectors - Alternative)**.**
<u>Rule:</u> To subtract $\mathbf{B}$ from $\mathbf{A}$, place the head of $\mathbf{B}$ at the head of $\mathbf{A}$. Then $\mathbf{A} - \mathbf{B}$ is the vector extending from the tail of $\mathbf{A}$ to the tail of $\mathbf{B}$.

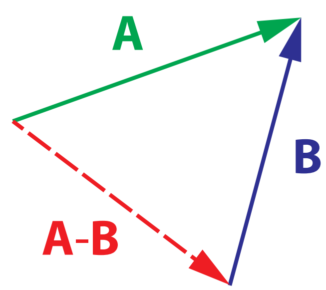


Figure 5: Subtraction between two vectors.

5.6 The Algebra of Vectors

There are three main laws concerned with additionthese are

Formula 5.1 (Commutative law)**.**

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

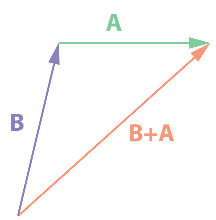
Formula 5.2 (Associative law)**.**

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$$

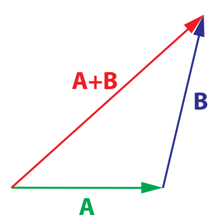
Formula 5.3 (Distributive law)**.**

$$\alpha(\beta\mathbf{A}) = (\alpha\beta)\mathbf{A}, \qquad (\alpha + \beta)\mathbf{A} = \alpha\mathbf{A} + \beta\mathbf{A}, \qquad \alpha(\mathbf{A} + \mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B}$$

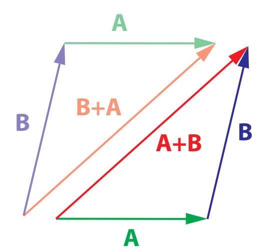
One can often use simple diagrams to show relationships between vectors. An example being a diagrammatic representation of the commutative law of vector addition



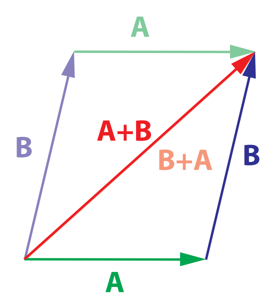
(a) The addition of **B** to **A** ie **A**+**B**



(b) The addition of **A** to **B** ie **B**+**A**



(c) Slide theses figures toward each other.



(d) Both figures superimposed ontop of each other.

Figure 6: The commutative law of vector addition.

5.7 A Non Vector Non Scalar!

Not all quantities that have both a magnitude and direction are vectors take for example (finite) rotations of an object about two axes.

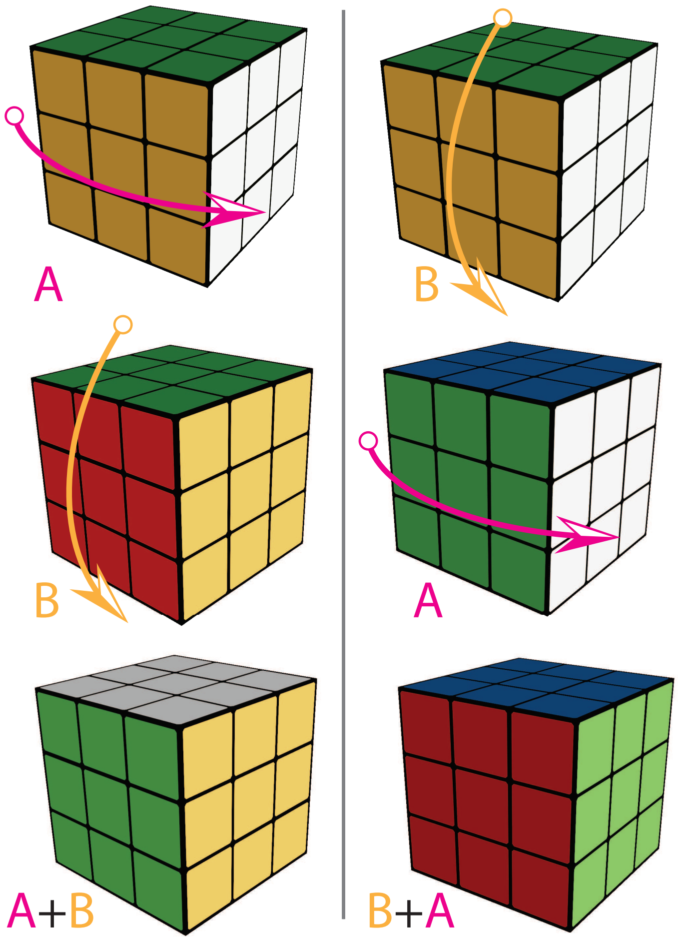
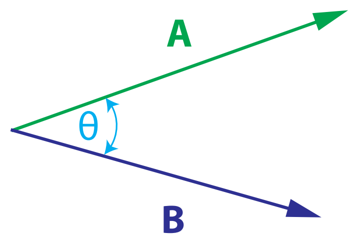


Figure 7: Non vectors: rotations of the Rubix cube. On the left we have a 90° rotation A followed by the 90° rotation B, while on the right we have B followed by A.

ADVANCED ASIDE 5.1. Formally vectors are defined as elements of a *vector space*. The elements of a vector space, the vectors, must adhere to a number of conditions which can be found in any text dealing with vector algebra. These necessary properties include *commutativity* of vector addition, which is simply written if $\mathbf{A}$ and $\mathbf{B}$ are elements in a vector space then $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ which is precisely the property that fails in our proposed vectors in figure above.

5.8 Scalar Product ("Dot" Product) Between Vectors

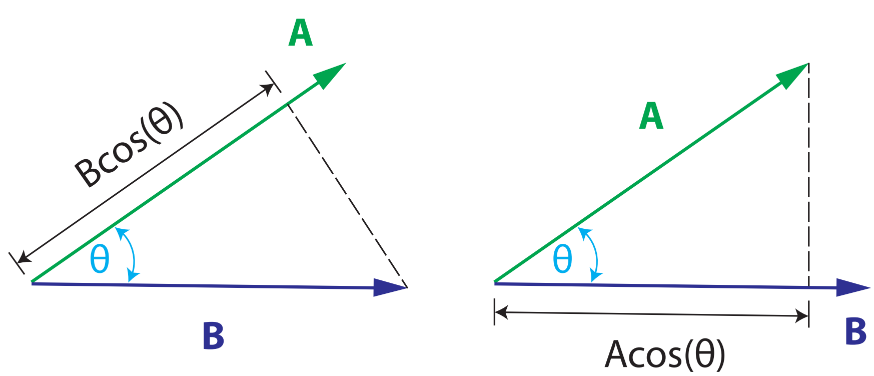
The scalar or dot product is a way of combining two vectors to get a scalar.



Important Formula 5.1 (Scalar Product).

$$\mathbf{A} \cdot \mathbf{B} = AB\cos(\theta)$$

Here θ is the angle between $\mathbf{A}$ and $\mathbf{B}$, $A \equiv |\,\mathbf{A}\,|$ and $B \equiv |\,\mathbf{B}\,|$.



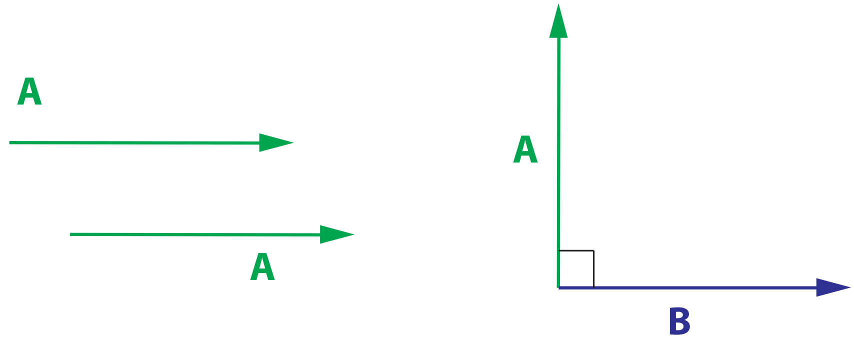
(a) The projection of **B** along **A**

(b) The projection of **A** along **B**

Figure 8: Equivalent projections.

Note 5.3 (Projection). $B\cos(\theta)$ is the projection of $\mathbf{B}$ along $\mathbf{A}$, denoted $\text{proj}_{\mathbf{A}}\mathbf{B}$, and $A\cos(\theta)$ is the projection of $\mathbf{A}$ along $\mathbf{B}$, denoted $\text{proj}_{\mathbf{B}}\mathbf{A}$,
Hence,

$$\mathbf{A} \cdot \mathbf{B} = A(\text{proj}_{\mathbf{A}}\mathbf{B}) = B(\text{proj}_{\mathbf{B}}\mathbf{A})$$



(a) Vector **A** is parallel to itself, Hence $\theta = 0°$

(b) Vector **A** is perpendicular to **B**, Hence $\theta = 90°$

Figure 9: Special Cases.

Note 5.4 (Scalar product of a vector with itself)**.**

$$\begin{aligned}\mathbf{A} \cdot \mathbf{A} &= AA\cos(\theta) \\ &= A^2\end{aligned}$$

$$\Rightarrow A = \sqrt{\mathbf{A} \cdot \mathbf{A}}$$

Note 5.5 (Scalar product between perpendicular vectors)**.**

$$\begin{aligned}\mathbf{A} \cdot \mathbf{B} &= AB\cos(\theta) \\ &= AB\cos(90°) \\ &= AB(0) = 0\end{aligned}$$

Thus if $A \neq 0$ and $B \neq 0$ and $\mathbf{A} \cdot \mathbf{B} = 0$ one can conclude that the vectors **A** and **B** are perpendicular.

5.8.1 Important properties concerning the scalar product.

Formula 5.4 (Dot product distributive law)**.**

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$$

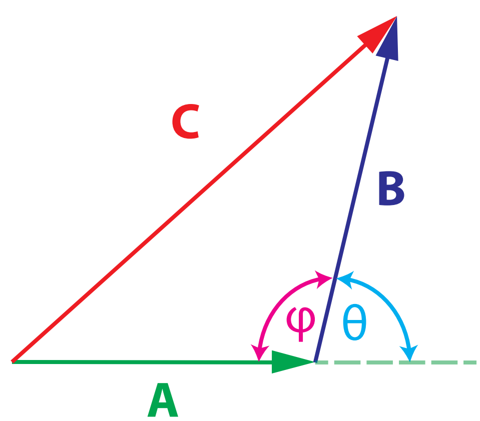
Formula 5.5 (Commutative property)**.**

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$$

Formula 5.6 (Factoring scalars)**.**

$$\mathbf{A} \cdot (\alpha \mathbf{B}) = \alpha \mathbf{A} \cdot \mathbf{B} = (\alpha \mathbf{A}) \cdot \mathbf{B}$$

5.9 Deriving the Law of Cosines Using the Dot Product



Consider the vector

$$\mathbf{C} = \mathbf{A} + \mathbf{B}$$

dotting this vector with itself,

$$
\begin{aligned}
\mathbf{C}\cdot\mathbf{C} &= (\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B})\\
&= \underbrace{\mathbf{A}\cdot\mathbf{A}}_{A^2} + \mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{A} + \underbrace{\mathbf{B}\cdot\mathbf{B}}_{B^2}
\end{aligned}
$$

we now have,

$$
\begin{aligned}
C^2 &= A^2 + B^2 + AB\cos(\theta) + BA\cos\theta\\
&= A^2 + B^2 + 2AB\cos(\theta).
\end{aligned}
$$

This result can be written in terms of the internal angle ϕ by making use of the fact that

$$\phi = 180^\circ - \theta \quad \Rightarrow \theta = 180^\circ - \phi$$

Since,

$$\cos(\theta) = \cos(180^\circ - \phi) = -\cos(\phi)$$

we have the trigonometric result (known as the law of cosines)

$$\boxed{C^2 = A^2 + B^2 - 2AB\cos(\phi)}$$

ASIDE 5.1.

$$
\begin{aligned}
\cos(A-B) &= \cos(A)\cos(B) + \sin(A)\sin(B)\\
\cos(180^\circ - \phi) &= \cos(180^\circ)\cos(\phi) + \sin(180^\circ)\sin(\phi)
\end{aligned}
$$

noting that

$$\cos(180^\circ) = -1 \quad \text{and} \quad \sin(180^\circ) = 0$$

we have,

$$\cos(180^\circ - \phi) = -\cos(\phi)$$

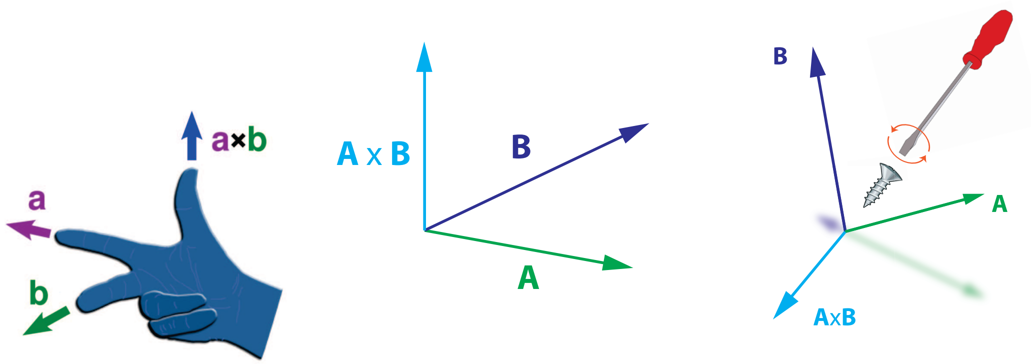
5.10 Vector Product ("Cross" Product) Between Vectors

The cross product of two vectors in $\mathbb{R}^3$ $\mathbf{A}$ and $\mathbf{B}$ is the vector

Important Formula 5.2 (Cross product)**.**

$$\mathbf{A} \times \mathbf{B} = AB\sin(\theta)\hat{\mathbf{n}}$$

where θ is the angle between the vectors $\mathbf{A}$ and $\mathbf{B}$ and $\hat{\mathbf{n}}$ is a *unit* vector perpendicular to the plane containing $\mathbf{A}$ and $\mathbf{B}$. The direction that $\hat{\mathbf{n}}$ points is given by the *right hand rule* which also known as the *right hand screwdriver rule*



(a) The right hand rule. (b) $\mathbf{A} \times \mathbf{B}$ is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$ (c) The screwdriver rule.

Figure 10: The direction of $\mathbf{A} \times \mathbf{B}$.

5.10.1 What is $\mathbf{B} \times \mathbf{A}$?

Begin by examining the length of the vector $\mathbf{B} \times \mathbf{A}$

$$\begin{aligned} |\mathbf{B} \times \mathbf{A}| &= BA\sin(\theta) \\ &= AB\sin(\theta) = |\mathbf{A} \times \mathbf{B}| \end{aligned}$$

thus the length of the vector $\mathbf{B} \times \mathbf{A}$ is the same as $\mathbf{A} \times \mathbf{B}$. All that is needed now is to find the direction in which the vector $\mathbf{B} \times \mathbf{A}$ is pointing: use the right-hand rule
notice that resulting vectors due to the cross product in Figure 11 is pointing in the opposite direction to that of Figure 10. This means we have

$$\hat{\boldsymbol{n}}_1 = -\hat{\mathbf{n}}$$

and hence,

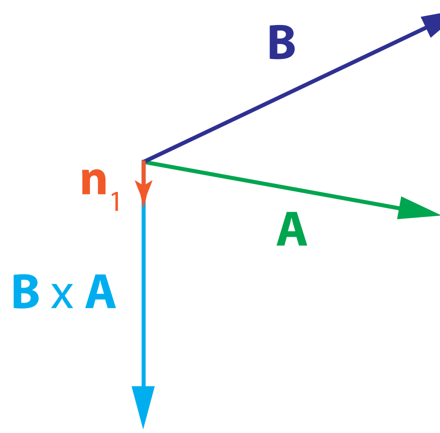


Figure 11: The cross product $\mathbf{B} \times \mathbf{A}$. The vector $\hat{\mathbf{n}}_1$ is a unit vector pointing in the direction of $\mathbf{B} \times \mathbf{A}$.

Important Formula 5.3.

$$\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}$$

5.10.2 Important Properties Concerning the Vector Product.

Formula 5.7 (Cross product non-commutative property)**.**

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

Formula 5.8 (Distributive property)**.**

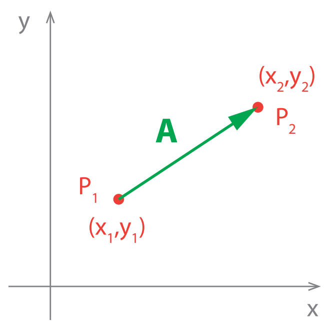
$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$$
$$(\mathbf{A} + \mathbf{B}) \times \mathbf{C} = \mathbf{A} \times \mathbf{C} + \mathbf{B} \times \mathbf{C}$$

Formula 5.9 (Factoring scalars)**.**

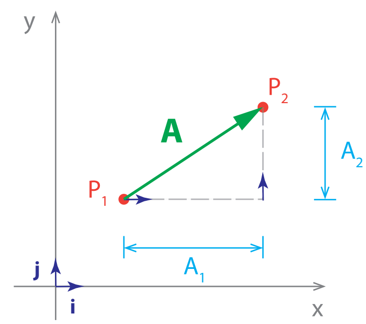
$$\mathbf{A} \times (\alpha \mathbf{B}) = \alpha \mathbf{A} \times \mathbf{B} = (\alpha \mathbf{A}) \times \mathbf{B}$$

where α is a scalar.

5.11 Vectors in $\mathbb{R}^2$



(a) Two points p_1 and p_2 in $\mathbb{R}^2$ defining a vector.



(b) Vectors **i** and **j** are unit vectors in the positive x and y directions respectively.

Figure 12: A vector in $\mathbb{R}^2$.

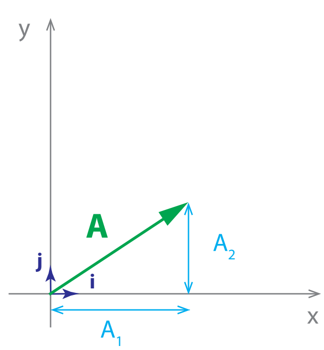
The line joining the point $p_1(x_1, y_1)$ to the point $p_2(x_2, y_2)$ is the vector

$$\mathbf{A} = (x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j}$$

where **i** is a unit vector along the x-axis and **j** is a unit vector along the y-axis. Defining $A_1 = x_2 - x_1$ and $A_2 = y_2 - y_1$ one can simply write the vector **A** as

$$\mathbf{A} = A_1\,\mathbf{i} + A_2\,\mathbf{j}$$

In general any vector on the plane can be written as



$$\mathbf{A} = A_1\,\mathbf{i} + A_2\,\mathbf{j} \quad \text{or simply} \quad \mathbf{A} = \langle A_1, A_2 \rangle$$

Note 5.6. One can find the *vector components* A_1 and A_2 in terms of the angle the the vector makes with respect to the x-axis, θ and the length of the vector,A. Using a little trigonometry one should find

$$A_1 = A\cos(\theta), \qquad A_2 = A\sin(\theta)$$

One should also find the relationships

$$A = \sqrt{A_1^2 + A_2^2} \qquad \tan(\theta) = \frac{A_2}{A_1}$$

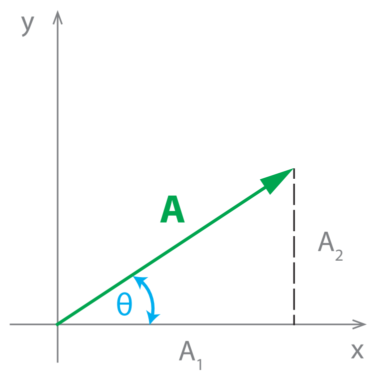
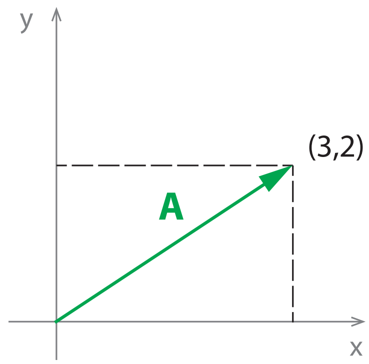
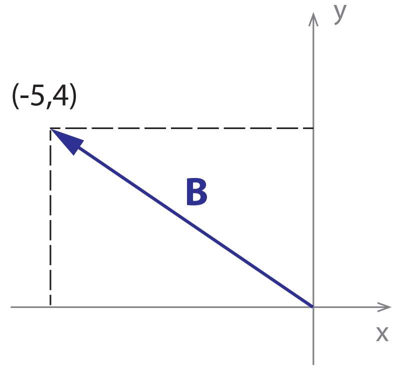


Figure 13: Constructing a right angled triangle using the vector **A** as the diagonal.



(a) Graph for Example 5.11.1



(b) Graph for Example 5.11.2

Example 5.11.1.

$$\mathbf{A} = 3\,\mathbf{i} + 2\,\mathbf{j}$$

Here,

$$A_1 = 3, \qquad A_2 = 2.$$

Now,

$$A = \sqrt{3^2 + 2^2} = \sqrt{13}$$

and

$$\tan(\theta) = \frac{2}{3} \Rightarrow \theta = \arctan\left(\frac{2}{3}\right)$$

Example 5.11.2.

$$\mathbf{B} = -5\,\mathbf{i} + 4\,\mathbf{j}$$

Find the length of the vector and the angle the vector makes with the x-axis.

Solution:

$$\begin{aligned} B &= \sqrt{(-5)^2 + 4^2} \\ &= \sqrt{25 + 16} \\ &= \sqrt{41} \end{aligned}$$

and

$$\tan(\theta) = \frac{4}{-5} = -\frac{4}{5} \Rightarrow \theta = \arctan\left(-\frac{4}{5}\right)$$

5.11.1 The position vector in $\mathbb{R}^2$

If we are given the point p_1 as the origin $(0,0)$ and the point p_2 an arbitrary point (x,y), the resulting vector between these two point is called the *position vector* in $\mathbb{R}^2$. This vector is typically denoted by $\mathbf{r}$

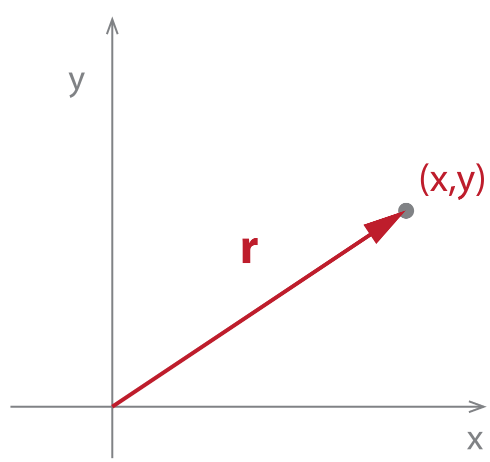


Figure 14: The position vector in $\mathbb{R}^2$

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} \qquad \text{or} \qquad \mathbf{r} = \langle x, y \rangle$$

5.12 Vectors in $\mathbb{R}^3$

In $\mathbb{R}^3$ the vector $\mathbf{A}$ between two points is specified by the ordered triple

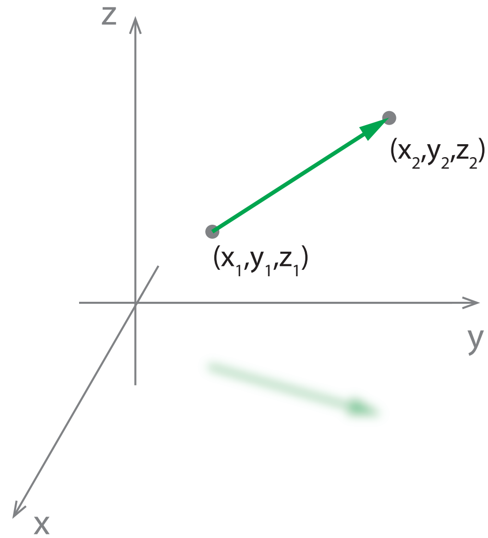


Figure 15: A vector in $\mathbb{R}^3$ specified by two points.

$$\begin{aligned}\mathbf{A} &\equiv A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k} \\ &\equiv \langle A_1, A_2, A_3\rangle\end{aligned}$$

where,

$$A_1 = (x_2 - x_1), \quad A_2 = (y_2 - y_1), \quad A_3 = (z_2 - z_1)$$

Theorem 5.12.1 (Vectors as ordered triples of real numbers (in $\mathbb{R}^3$))**.**
Given a fixed Cartesian coordinate system, each vector is uniquely determined by an ordered triple of corresponding components. Conversely, to each ordered triple of real numbers (A_1, A_2, A_3) there exists precisely one vector

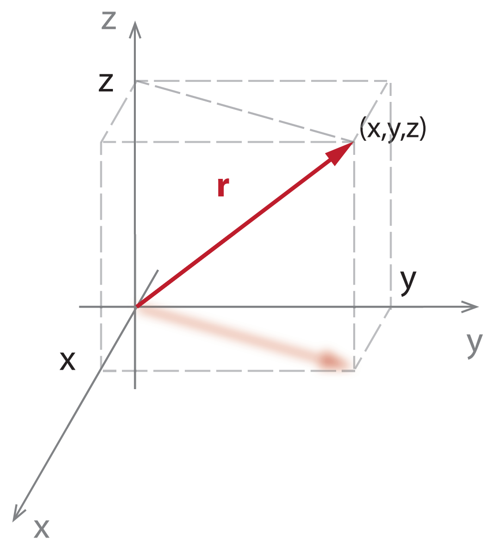
$$\boldsymbol{A} = A_1\,\boldsymbol{i} + A_2\,\boldsymbol{j} + A_3\,\boldsymbol{k} \quad (\equiv \langle A_1, A_2, A_3\rangle).$$

The origin itself defines the zero vector $\boldsymbol{O} = \langle 0, 0, 0\rangle$.

The length of the vector $\mathbf{A}$ is given by

$$|\,\mathbf{A}\,| = \sqrt{A_1^2 + A_2^2 + A_3^2}$$

5.12.1 The position vector in $\mathbb{R}^3$.



The position vector in $\mathbb{R}^3$ in Cartesian coordinates is simply given by

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

The length or *magnitude* is given by

$$r = \sqrt{x^2 + y^2 + z^2}$$

5.12.2 Addition and equality between vectors in $\mathbb{R}^3$.

Given two vectors in $\mathbb{R}^3$

$$\mathbf{A} = A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k} \quad \text{and} \quad \mathbf{B} = B_1\,\mathbf{i} + B_2\,\mathbf{j} + B_3\,\mathbf{k}$$

we say that

$$\mathbf{A} = \mathbf{B} \iff A_1 = B_1, \quad A_2 = B_2 \quad \text{and} \quad A_3 = B_3.$$

The sum of the vectors is given by

$$\boxed{\begin{aligned} \mathbf{A} + \mathbf{B} = A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k} + \mathbf{B} &= B_1\,\mathbf{i} + B_2\,\mathbf{j} + B_3\,\mathbf{k} \\ &= (A_1 + B_1)\,\mathbf{i} + (A_2 + B_2)\,\mathbf{j} + (A_3 + B_3)\,\mathbf{k} \end{aligned}}$$

similarly the difference,

$$\mathbf{A} - \mathbf{B} = (A_1 - B_1)\,\mathbf{i} + (A_2 - B_2)\,\mathbf{j} + (A_3 - B_3)\,\mathbf{k}$$

5.13 The Dot Product in Cartesian Form

We begin by examing the dot product between the unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, recall that these are vectors of unit length that point in the positive x, y and z directions respectively,

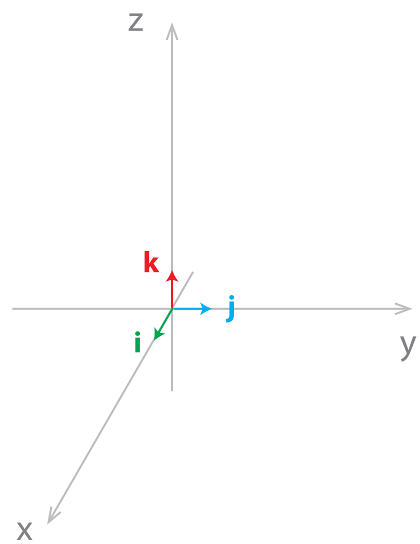


Figure 16: Figure showing $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ unit vectors.

$$\begin{aligned}\mathbf{i}\cdot\mathbf{i} &= |\,\mathbf{i}\,|\,|\,\mathbf{i}\,|\cos(0^\circ)\\ &= (1)(1)(1)\\ &= 1.\end{aligned}$$

Similarly, $\mathbf{j}\cdot\mathbf{j} = 1$ and $\mathbf{k}\cdot\mathbf{k} = 1$. Turning attention to the other possible combination of products one finds that

$$\begin{aligned}\mathbf{i}\cdot\mathbf{j} &= |\,\mathbf{i}\,|\,|\,\mathbf{j}\,|\cos(90^\circ)\\ &= (1)(1)(0)\\ &= 0\end{aligned}$$

similarly due to the symmetry between the unit vectors $\mathbf{i}\cdot\mathbf{k} = 0$ and $\mathbf{j}\cdot\mathbf{k} = 0$. Thus in summary we have

Note 5.7.

$$\begin{aligned}&\mathbf{i}\cdot\mathbf{i} = 1 && \mathbf{j}\cdot\mathbf{j} = 1 && \mathbf{k}\cdot\mathbf{k} = 1\\ &\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{i} = 0 && \mathbf{i}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0 && \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{j} = 0\end{aligned}$$

With these relations it is possible to further examine the quantity $\mathbf{A}\cdot\mathbf{B}$ in cartesian form. Beginning with the vectors

$$\begin{aligned}\mathbf{A} &= A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k}\\ \mathbf{B} &= B_1\,\mathbf{i} + B_2\,\mathbf{j} + B_3\,\mathbf{k}\end{aligned}$$

now,

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= (A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k})\cdot(B_1\,\mathbf{i} + B_2\,\mathbf{j} + B_3\,\mathbf{k})\\ &= A_1B_1\mathbf{i}\cdot\mathbf{i} + A_1B_2\cancelto{0}{\mathbf{i}\cdot\mathbf{j}} + A_1B_3\cancelto{0}{\mathbf{i}\cdot\mathbf{k}} + A_2B_1\cancelto{0}{\mathbf{j}\cdot\mathbf{i}} + A_2B_2\mathbf{j}\cdot\mathbf{j} + A_2B_3\cancelto{0}{\mathbf{j}\cdot\mathbf{k}}\\ &\quad + A_3B_1\cancelto{0}{\mathbf{k}\cdot\mathbf{i}} + A_3B_2\cancelto{0}{\mathbf{k}\cdot\mathbf{j}} + A_3B_3\mathbf{k}\cdot\mathbf{k}\end{aligned}$$

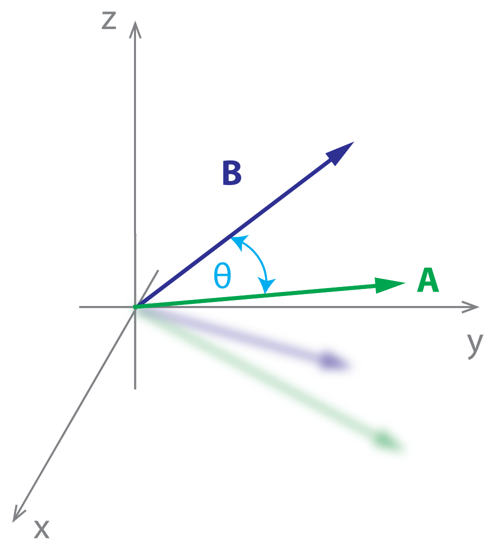
Hence,

Important Formula 5.4.

$$\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$$

5.13.1 The angle between two vectors in $\mathbb{R}^3$.

The angle between two vectors, θ, can be found by using the dot product,



by definition

$$\mathbf{A} \cdot \mathbf{B} = |\,\mathbf{A}\,|\,|\,\mathbf{B}\,|\cos(\theta)$$

we have also shown that in cartesian coordinates

$$\mathbf{A} \cdot \mathbf{B} = A_1 B_1 + A_2 B_2 + A_3 B_3$$

thus,

$$|\,\mathbf{A}\,|\,|\,\mathbf{B}\,|\cos(\theta) = A_1 B_1 + A_2 B_2 + A_3 B_3$$

Formula 5.10.

$$\theta = \arccos\left(\frac{A_1 B_1 + A_2 B_2 + A_3 B_3}{|\,\mathbf{A}\,|\,|\,\mathbf{B}\,|}\right)$$

5.13.2 Dot product worked examples.

Example 5.13.1 (A dot product in $\mathbb{R}^2$.)**.**
Given,

$$\mathbf{A} = 2\,\mathbf{i} + 3\,\mathbf{j}, \quad \mathbf{B} = \mathbf{i} + 2\,\mathbf{j}$$

find $\mathbf{A} \cdot \mathbf{B}$.

Solution:

$$\begin{aligned}
\mathbf{A} \cdot \mathbf{B} &= (2\,\mathbf{i} + 3\,\mathbf{j}) \cdot (\,\mathbf{i} + 2\,\mathbf{j}) \\
&= 2\mathbf{i} \cdot \mathbf{i} + 4\cancelto{0}{\mathbf{i} \cdot \mathbf{j}} + 3\cancelto{0}{\mathbf{j} \cdot \mathbf{i}} + 6\mathbf{j} \cdot \mathbf{j} \\
&= 2 + 6 \\
&= 8
\end{aligned}$$

Example 5.13.2 (A dot prodcut in $\mathbb{R}^3$.)**.**
If

$$\mathbf{A} = 2\,\mathbf{i} - 3\,\mathbf{j} + \mathbf{k}, \quad \mathbf{B} = \mathbf{i} + \mathbf{j} - 2\,\mathbf{k}.$$

Find $\mathbf{A} \cdot \mathbf{B}$

Solution:

$$\begin{aligned}
\mathbf{A} \cdot \mathbf{B} &= (2\,\mathbf{i} - 3\,\mathbf{j} + \mathbf{k}) \cdot (\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}) \\
&= 2\mathbf{i} \cdot \mathbf{i} + 2\cancelto{0}{\mathbf{i} \cdot \mathbf{j}} - 4\cancelto{0}{\mathbf{i} \cdot \mathbf{k}} - 3\cancelto{0}{\mathbf{j} \cdot \mathbf{i}} - 3\mathbf{j} \cdot \mathbf{j} + 6\cancelto{0}{\mathbf{j} \cdot \mathbf{k}} \\
&\quad + \cancelto{0}{\mathbf{k} \cdot \mathbf{i}} + \cancelto{0}{\mathbf{k} \cdot \mathbf{j}} - 2\mathbf{k} \cdot \mathbf{k} \\
&= 2 - 3 - 2 \\
&= -3
\end{aligned}$$

Example 5.13.3.

$$\mathbf{A} = \mathbf{i} - 2\,\mathbf{j} + \mathbf{k}, \quad \mathbf{B} = 2\,\mathbf{i} + 3\,\mathbf{j} - 2\,\mathbf{k}.$$

Find $\mathbf{A} \cdot \mathbf{B}$

Solution:
It is more convenient to use the result

$$\mathbf{A} \cdot \mathbf{B} = A_1 B_1 + A_2 B_2 + A_3 B_3$$

to find the dot product. In this example we have

$$\begin{aligned}\mathbf{A} \cdot \mathbf{B} &= (1)(2) + (-2)(3) + (1)(-2) \\ &= 2 - 6 - 2 \\ &= -6\end{aligned}$$

Example 5.13.4.

$$\mathbf{A} = 2\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}.$$

Find the length of $\mathbf{A}$ using the dot product.

Solution:
We can take the dot product of $\mathbf{A}$ with itself

$$\begin{aligned}\mathbf{A} \cdot \mathbf{A} &= |\,\mathbf{A}\,|\,|\,\mathbf{A}\,| \cos(0^\circ) \\ &= A^2.\end{aligned}$$

we also know that,

$$\begin{aligned}\mathbf{A} \cdot \mathbf{A} &= (2)(2) + (1)(1) + (-2)(-2) \\ &= 4 + 1 + 4 \\ &= 9\end{aligned}$$

now we have that $A^2 = 9 \Rightarrow A = 3$.

Example 5.13.5 (Alternative notation and the angle between vectors.)**.**
What is the dot product of and the angle between the vectors $\langle 2, 3, 1\rangle$ and $\langle -1, 5, 1\rangle$.

Solution:
rewriting our vectors using the **i**, **j** and **k** unit vector notation we have

$$\begin{aligned}\mathbf{A} &= 2\,\mathbf{i} + 3\,\mathbf{j} + \mathbf{k}\\ \mathbf{B} &= -1\,\mathbf{i} + 5\,\mathbf{j} + \mathbf{k}\end{aligned}$$

The dot product,

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= (2)(-1) + (3)(5) + (1)(1)\\ &= 14. \qquad (5.13.1)\end{aligned}$$

However we also know that the dot product is related to the angle between two vectors by the formula

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta.$$

This formula involves the length of the vectors **A** and **B** which we begin by fining

$$\begin{aligned}A \equiv |\,\mathbf{A}\,| &= \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}\\ B \equiv |\,\mathbf{B}\,| &= \sqrt{(-1)^2 + 5^2 + 1^2} = \sqrt{27}.\end{aligned}$$

Substituting these into the definition of the dot product yields

$$\mathbf{A}\cdot\mathbf{B} = \sqrt{14}\sqrt{27}\cos(\theta). \qquad (5.13.2)$$

Comparing equations 5.13.1and 5.13.2 we have

$$\begin{aligned}14 &= \sqrt{14}\sqrt{27}\cos(\theta)\\ \Rightarrow \cos(\theta) &= \frac{14}{\sqrt{14}\sqrt{27}}\end{aligned}$$

Hence,

$$\begin{aligned}\theta &= \arccos\left(\frac{14}{\sqrt{14}\sqrt{27}}\right)\\ &= 0.77\text{radians} \quad \text{or} \quad 44.9^\circ\end{aligned}$$

5.14 Cross Product in Cartesian Coordinates.

Like our examination of the dot product, it is useful to begin an examination of the cross product

by looking at the cross product between the **i**, **j** and **k** unit vectors. Using the definition of the cross product one finds

$$\begin{aligned}|\mathbf{i} \times \mathbf{i}| &= |\,\mathbf{i}\,|\,|\,\mathbf{i}\,|\sin(\theta) \\ &= (1)(1)\sin(0^\circ) \\ &= (1)(1)(0) \\ &= 0\end{aligned}$$

thus

$$\mathbf{i} \times \mathbf{i} = \mathbf{0}.$$

Similarly $\mathbf{j} \times \mathbf{j} = 0$ and $\mathbf{k} \times \mathbf{k} = 0$, thus taking the cross product of any one of these unit vectors with themselves is the zero vector. This is in fact true for any vector $\mathbf{A}$, that is $\mathbf{A} \times \mathbf{A} = \mathbf{0}$, since a vector will make an angle $\theta = 0^\circ$ with itself and the definition of the cross product involves a $\sin(\theta)$ terms as a factor.

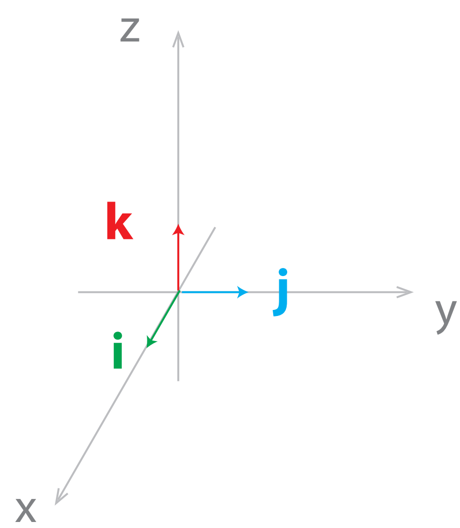


Figure 17: The unit vectors **i**, **j** and **k** in $\mathbb{R}^3$

Looking a the other possible combinations of possible cross products between the **i**, **j** and **k** vectors, it is useful to remind oneself that these are vectors that are of unit length and are parallel to the direction of increase of the Cartesian coordinate system in $\mathbb{R}^3$. This means that the angle between any two of these distinct unit vectors make an angle $\theta = 90^\circ$ with each other. Examining

$$\begin{aligned}|\mathbf{i} \times \mathbf{j}| &= |\,\mathbf{i}\,|\,|\,\mathbf{j}\,|\sin(\theta) \\ &= (1)(1)\sin(90^\circ) \\ &= (1)(1)(1) \\ &= 1\end{aligned}$$

we find that the length of $\mathbf{i} \times \mathbf{j}$ is one. The right hand screwdriver rule tells us that the vector resulting from $\mathbf{i} \times \mathbf{j}$ is pointing in the direction of the vector $\mathbf{k}$. Furthermore since the length of $\mathbf{i} \times \mathbf{j}$ and the length of $\mathbf{k}$ are both equal we have

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}.$$

Recall that the cross product between two vectors $\mathbf{A}$ and $\mathbf{B}$ has the property $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$, we can say that

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}.$$

Similarly one can find that $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$, and $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$.
Thus in summary we have

Note 5.8.

$$\begin{array}{ccc} \mathbf{i} \times \mathbf{i} = \mathbf{0} & \mathbf{j} \times \mathbf{j} = \mathbf{0} & \mathbf{k} \times \mathbf{k} = \mathbf{0} \\ \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \end{array}$$

5.14.1 The cyclic rule diagram.

An easy way to recall these relationships is by remembering that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and using the following diagram

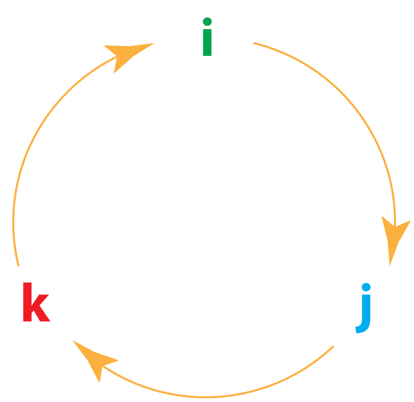


Figure 18: The cross product cycle diagram.
Moving opposite to the direction of the arrows (eg $\mathbf{k} \times \mathbf{j}$) means that the result is -1 times the vector one arrives at ($\mathbf{k} \times \mathbf{j} = -\mathbf{i}$)

5.15 Cross Product of Two Vectors in Cartesian Form

If we are given two vectors $\mathbf{A} = A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k}$ and $\mathbf{B} = B_1\,\mathbf{i} + B_2\,\mathbf{j} + B_3\,\mathbf{k}$ then

Important Formula 5.5.

$$\mathbf{A} \times \mathbf{B} = (A_2 B_3 - A_3 B_2)\,\mathbf{i} - (A_1 B_3 - A_3 B_1)\,\mathbf{j} + (A_1 B_2 - A_2 B_1)\,\mathbf{k}.$$

Here we prove this result,

$$\begin{aligned}
\mathbf{A} \times \mathbf{B} &= (A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k}) \times (B_1\,\mathbf{i} + B_2\,\mathbf{j} + B_3\,\mathbf{k}) \\
&= A_1 B_1 \cancelto{0}{\mathbf{i} \times \mathbf{i}} + A_1 B_2\,\underbrace{\mathbf{i} \times \mathbf{j}}_{\mathbf{k}} + A_1 B_3\,\underbrace{\mathbf{i} \times \mathbf{k}}_{-\mathbf{j}} + A_2 B_1\,\underbrace{\mathbf{j} \times \mathbf{i}}_{-\mathbf{k}} + A_2 B_2 \cancelto{0}{\mathbf{j} \times \mathbf{j}} + A_2 B_3\,\underbrace{\mathbf{j} \times \mathbf{k}}_{\mathbf{i}} \\
&\quad + A_3 B_1\,\underbrace{\mathbf{k} \times \mathbf{i}}_{\mathbf{j}} + A_3 B_2\,\underbrace{\mathbf{k} \times \mathbf{j}}_{-\mathbf{i}} + A_1 B_3 \cancelto{0}{\mathbf{k} \times \mathbf{k}}
\end{aligned}$$

grouping together $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ terms,

$$\begin{aligned}
\mathbf{A} \times \mathbf{B} &= A_1 B_2 \mathbf{k} - A_1 B_3 \mathbf{j} - A_2 B_1 \mathbf{k} + A_2 B_3 \mathbf{i} + A_3 B_1 \mathbf{j} - A_3 B_2 \mathbf{i} \\
&= (A_2 B_3 - A_3 B_2)\,\mathbf{i} + (A_3 B_1 - A_1 B_3)\,\mathbf{j} + (A_1 B_2 - A_2 B_1)\,\mathbf{k} \\
&= (A_2 B_3 - A_3 B_2)\,\mathbf{i} - (A_1 B_3 - A_3 B_1)\,\mathbf{j} + (A_1 B_2 - A_2 B_1)\,\mathbf{k}
\end{aligned}$$

5.18 Direction Cosines

We now briefly turn out attention to the angles a vector makes with the x, y and z axes. The angles the vector $A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k}$ makes with the vector $\mathbf{i}$ (x-axis), $\mathbf{j}$ (y-axis) and $\mathbf{k}$ (z-axis) are called its *direction angles* and are denoted by α, β and γ respectively.

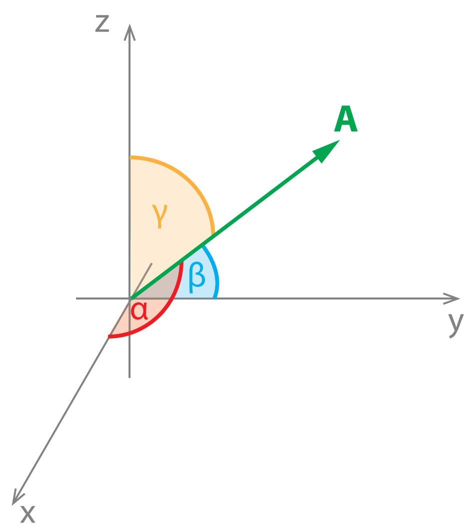


Figure 19: The direction angles of the vector $\mathbf{A}$.

Notice that taking the dot product of the vector $\mathbf{A}$ and the unit vector $\mathbf{i}$ we have

$$\mathbf{A}\cdot\mathbf{i} = |\mathbf{A}||\,\mathbf{i}\,|\cos(\alpha)$$
$$\Rightarrow \cos(\alpha) = \frac{\mathbf{A}\cdot\mathbf{i}}{|\mathbf{A}||\,\mathbf{i}\,|}$$

Similarlily we have

$$\cos(\beta) = \frac{\mathbf{A}\cdot\mathbf{j}}{|\mathbf{A}||\,\mathbf{j}\,|} \quad \text{and} \quad \cos(\gamma) = \frac{\mathbf{A}\cdot\mathbf{k}}{|\mathbf{A}||\,\mathbf{k}\,|}$$

The vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are simply unit vectors and hence $|\,\mathbf{i}\,| = |\,\mathbf{j}\,| = |\,\mathbf{k}\,| = 1$ and we have

Formula 5.14.

$$\cos(\alpha) = \frac{\mathbf{A}\cdot\mathbf{i}}{A}, \quad \cos(\beta) = \frac{\mathbf{A}\cdot\mathbf{j}}{A} \quad \text{and} \quad \cos(\gamma) = \frac{\mathbf{A}\cdot\mathbf{k}}{A}$$

where we have used A as the shorthand notation for the length of $\mathbf{A}$.

Making the further observation that

$$\frac{\mathbf{A}}{A} = \hat{\mathbf{A}} = \frac{1}{A}\,(A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k})$$
$$= \frac{A_1}{A}\,\mathbf{i} + \frac{A_2}{A}\,\mathbf{j} + \frac{A_3}{A}\,\mathbf{k}$$

we have the two results

Formula 5.15.

$$\cos(\alpha) = \hat{\mathbf{A}}\cdot\mathbf{i}, \quad \cos(\beta) = \hat{\mathbf{A}}\cdot\mathbf{j} \quad \text{and} \quad \cos(\gamma) = \hat{\mathbf{A}}\cdot\mathbf{k}$$

and

Formula 5.16.

$$\hat{\mathbf{A}} = \cos(\alpha)\,\mathbf{i} + \cos(\beta)\,\mathbf{j} + \cos(\gamma)\,\mathbf{k}$$

Example 5.18.1.
What are the direction cosines of

$$\mathbf{A} = 2\,\mathbf{i} + 5\,\mathbf{j} + 4\,\mathbf{k}$$

Solution:
Begin by finding the length of $\mathbf{A}$ so that we can write its corresponding unit vector,

$$|\mathbf{A}| = \sqrt{2^2 + 5^2 + 4^2} = \sqrt{45} = 3\sqrt{5}$$

a unit vector in the direction of $\mathbf{A}$ is given by

$$\hat{\mathbf{A}} = \frac{2}{3\sqrt{5}}\,\mathbf{i} + \frac{5}{3\sqrt{5}}\,\mathbf{j} + \frac{4}{3\sqrt{5}}\,\mathbf{k}$$

and by our result we know that

$$\hat{\mathbf{A}} = \cos(\alpha)\,\mathbf{i} + \cos(\beta)\,\mathbf{j} + \cos(\gamma)\,\mathbf{k}$$

and hence we have

$$\cos(\alpha) = \frac{2}{3\sqrt{5}}, \quad \cos(\beta) = \frac{5}{3\sqrt{5}} \quad \text{and} \quad \cos(\gamma) = \frac{4}{3\sqrt{5}}.$$

The directional angels themselves are simply found by taking the inverse cosine

$$\alpha = \arccos\left(\frac{2}{3\sqrt{5}}\right) = 1.27 \text{ rads}, \quad \beta = \arccos\left(\frac{5}{3\sqrt{5}}\right) = 0.73 \text{ rads},$$

$$\text{and} \quad \gamma = \arccos\left(\frac{4}{3\sqrt{5}}\right) = 0.93 \text{ rads}.$$