CHAPTER III

Integration

§ 1. Integrals and areas under curves

INTEGRATION, in its most limited sense, is just the reverse of differentiation. However integration is also a way of calculating areas and when this understood there is seen to be a much deeper link between the two operations—much deeper, that is, than the elementary observation that one is the reverse of the other. This point will be explained when we deal with what is called the *fundamental theorem of calculus*, cf. 3.5 below.

So, to begin with, we shall consider the problem of finding the area of region under a curve, cf. fig. 33.

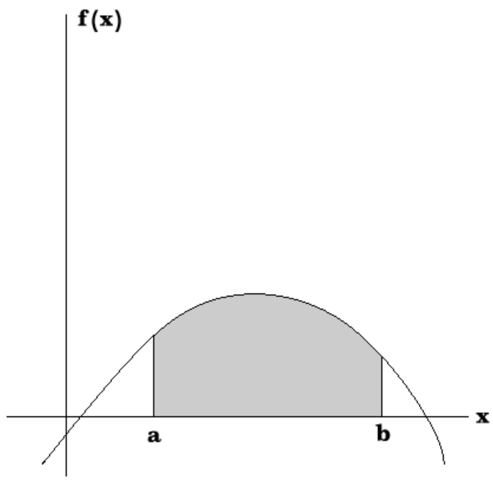

Fig. 33: The area under a curve

The shaded region in fig. 33 is the area that we wish to find: it lies between the vertical lines $x = a$ and $x = b$ and the notation used for this area is that it is denoted by

$$\int_a^b f(x)\,dx \tag{3.1}$$

and eq. 3.1 is referred to as "the integral of f from a to b".

What we do to calculate this area is to use rectangles as shown in fig. 34

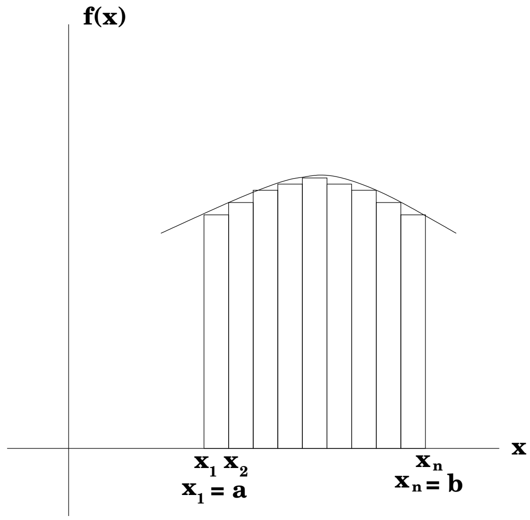

Fig. 34: Rectangles being used to calculate the area

Fig. 34 shows a series of $n-1$ rectangles whose collective area *approximates* the area between $x=a$ and $x=b$. The idea is that, as the number of these rectangles goes to infinity, the approximation becomes *exact.*

Now since fig. 34 shows that the first rectangle has base $(x_2 - x_1)$ and height $f(x_1)$ and the i^{th} rectangle has base $(x_{i+1} - x_i)$ and height $f(x_i)$ and so on. Hence the the area of all the rectangles is the sum

$$\sum_{i=1}^{n-1} f(x_i)(x_{i+1} - x_i) \tag{3.2}$$

Then to define the integral of f from a to b we take the limit as $n \to \infty$ giving

Definition (The integral $\int_a^b f(x)\,dx$)

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n-1} f(x_i)(x_{i+1} - x_i) \tag{3.3}$$

Having defined $\int_a^b f(x)\,dx$ the really important thing is the result one obtains when $f(x)$ is already the derivative of some other function, say

$$f(x) = \frac{dF(x)}{dx}, \qquad \text{for some function } F(x) \tag{3.4}$$

This result is what is given in the fundamental theorem of calculus which we now quote.

Theorem (The fundamental theorem of calculus) *If a function of the form dF/dx is integrated then its integral is given by the formula*

$$\int_a^b \frac{dF(x)}{dx}\,dx = F(b) - F(a) \tag{3.5}$$

We shall sketch a proof of this theorem but first a piece of notation: the quantity $F(b) - F(a)$ is often denoted by $[F(x)]_a^b$ that is

$$[F(x)]_a^b = F(b) - F(a) \tag{3.6}$$

where $F(x)$ is any function. Now for the sketch of the proof.

Proof: First we adjust the bases of the rectangles in fig. 34 to have the same size—which we denote by Δx—that is we have

$$(x_1 - x_2) = (x_2 - x_3) = \cdots = (x_n - x_{n-1}) = \Delta x \tag{3.7}$$

Using this this sum in $\sum_{i=1}^{n-1} f(x_i)(x_{i+1} - x_i)$ of the integral definition 3.3 simplifies: we find that

$$\sum_{i=1}^{n-1} f(x_i)(x_{i+1} - x_i) = \sum_{i=1}^{n-1} f(x_i)\Delta x \tag{3.8}$$

But since $f = dF/dx$ we must now consider the sum

$$\sum_{i=1}^{n-1} \frac{dF(x_i)}{dx}\Delta x \tag{3.9}$$

Now the definition of the derivative is $dF(x_i)/dx$ is

$$\begin{aligned} \frac{dF(x_i)}{dx} &= \lim_{h\to 0} \frac{F(x_i + h) - F(x_i)}{h} \\ &= \lim_{\Delta x\to 0} \frac{F(x_i + \Delta x) - F(x_i)}{\Delta x}, \quad \text{on setting } h = \Delta x \end{aligned} \tag{3.10}$$

But

$$\begin{aligned} \Delta x &= (x_{i+1} - x_i) \\ \Rightarrow x_i + \Delta x &= x_i + x_{i+1} - x_i \\ &= x_{i+1} \end{aligned} \tag{3.11}$$

and using this fact in 3.10 we obtain

$$\frac{dF(x_i)}{dx} = \lim_{\Delta x \to 0} \frac{F(x_{i+1}) - F(x_i)}{\Delta x} \tag{3.12}$$

Putting together 3.3, 3.11 and 3.12 we have

$$\begin{aligned} \int_a^b \frac{dF(x)}{dx}\, dx &= \lim_{n\to\infty} \sum_{i=1}^{n-1} \lim_{\Delta x \to 0} \frac{F(x_{i+1}) - F(x_i)}{\Delta x} \Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n-1} (F(x_{i+1}) - F(x_i)), \quad \text{since } \Delta x \text{ cancels} \end{aligned} \tag{3.13}$$

But if we write out the sum $\sum_{i=1}^{n-1}(F(x_{i+1}) - F(x_i))$ we saw that all, except for two terms, cancel for we have

$$\begin{aligned} \sum_{i=1}^{n-1}(F(x_{i+1}) - F(x_i)) = (F(x_n) &- F(x_{n-1})) \\ + (F(x_{n-1}) &- F(x_{n-2})) \\ + (F(x_{n-2}) &- F(x_{n-3})) \\ \vdots \qquad &\qquad \vdots \\ + (F(x_3) &- F(x_2)) \\ + (F(x_2) &- F(x_1)) \end{aligned} \tag{3.14}$$

If we now look at 3.14 we see that all the terms on the RHS cancel except for the first and the last and so

$$\begin{aligned} \sum_{i=1}^{n-1}(F(x_{i+1}) - F(x_i)) &= F(x_n) - F(x_1) \\ &= F(b) - Fa), \quad \text{recalling from fig. 34 that } \begin{cases} x_n = b \\ x_1 = a \end{cases} \end{aligned} \tag{3.15}$$

So putting this result back into 3.12 we see that we have achieved an informal proof of the *fundamental theorem of calculus* which asserts—cf. 3.5—that

$$\int_a^b \frac{dF(x)}{dx}\, dx = F(b) - F(a) \tag{3.16}$$

§§ 1.1 Some notation and terminology

Some perfectly straightforward terminology can now be given: If we integrate any function f from a to b—that is calculate

$$\int_a^b f(x)\, dx \tag{3.17}$$

then the numbers a and b are called the *limits of integration*. One sometimes omits the limits of integration and writes

$$\int f(x)\,dx \tag{3.18}$$

The notation $\int f(x)\,dx$ simply denotes a function F whose derivative is f; hence we have

$$\int f(x)\,dx = F(x), \quad \text{means that} \quad \frac{dF(x)}{dx} = f(x) \tag{3.19}$$

F is then called the *integral*[1] *of* f.

For example if $f(x) = x^3$ then we could write

$$\int x^3\,dx = \frac{x^4}{4}, \quad \text{because} \quad \frac{d}{dx}\left(\frac{x^4}{4}\right) = x^3 \tag{3.20}$$

However notice that, if C is any constant, we could also write

$$\int x^3\,dx = \frac{x^4}{4} + C, \quad \text{because} \quad \frac{d}{dx}\left(\frac{x^4}{4} + C\right) = x^3 \tag{3.21}$$

Such a constant C is called a *constant of integration*.

This example shows that the integral F of any function is not unique since one can add a constant of integration to F; more concisely one can say: if F is an integral of f then so is $F + C$ for any constant C.

Finally to distinguish integrals with and without limits one has the following terminology. An integral *with limits* such as

$$\int_a^b f(x)\,dx \tag{3.22}$$

is called a *definite integral*, while an integral *without limits* like

$$\int f(x)\,dx \tag{3.23}$$

is called an *indefinite integral*.

§§ 1.2 Some examples

[1] Alternative terms are often used for F. An integral of F is sometimes called a *primitive of* F or an *anti-derivative of* F—we shall use neither term since the word *integral* is the more common usage.

Example *The area under the parabola $f(x) = x^2$ between $x = 3$ and $x = 9$.*
With the fundamental theorem of calculus under our belt it is a simple matter to calculate the area shown in the fig. 35.

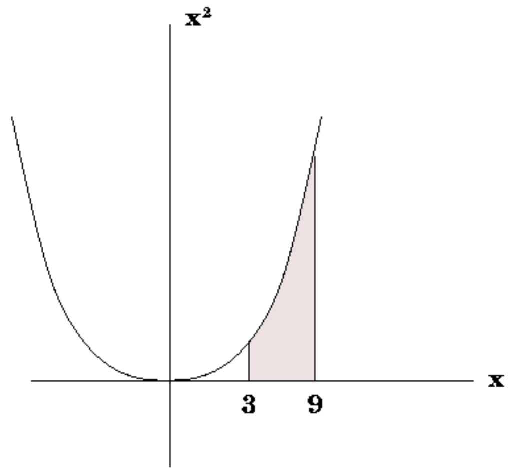

Fig. 35: The area under the parabola between $x = 3$ and $x = 9$

The area A that we want is given by

$$A = \int_3^9 x^2 \, dx \tag{3.24}$$

and so we can straightaway compute that

$$\begin{aligned} \int_3^9 x^2 \, dx &= \left[\frac{x^3}{3}\right]_3^9 \\ &= \left[\frac{9^3}{3} - \frac{3^3}{3}\right] \\ &= \left[\frac{729}{3} - \frac{27}{3}\right] \\ &= 234 \end{aligned} \tag{3.25}$$

and so we have our area A. That example was a definite integral so next we consider an indefinite integral.
Example *The integral*

$$\int \sin^2(x) \cos(x) \, dx \tag{3.26}$$

So this time we just want a function F that satisfies

$$\frac{dF}{dx} = \sin^2(x) \cos(x) \tag{3.27}$$

After some tinkering with various combinations of trigonometric functions we should be able to find that a solution is

$$F(x) = \frac{\sin^3(x)}{3} \tag{3.28}$$

which certainly does the trick. Thus we can write

$$\int \sin^2(x)\cos(x)\,dx = \frac{\sin^3(x)}{3} \tag{3.29}$$

and if we want to be really precise we add on a constant of integration and write the *most general* statement of the result which is

$$\int \sin^2(x)\cos(x)\,dx = \frac{\sin^3(x)}{3} + C, \qquad \text{for any constant } C \tag{3.30}$$

The next example illustrates a very important point. It is that if the area being calculated *lies under* the x-axis, rather than above it, then the area A calculated by the integral will be *negative*.

Example *Areas under the x-axis count negatively in integrals*

$$\int_{\pi/2}^{3\pi/2} \cos(x)\,dx \tag{3.31}$$

Fig. 36 shows the graph of $\cos(x)$ between $x = \pi/2$ and $3\pi/2$; this can be seen to be an interval on which $\cos(x)$ is *negative*.

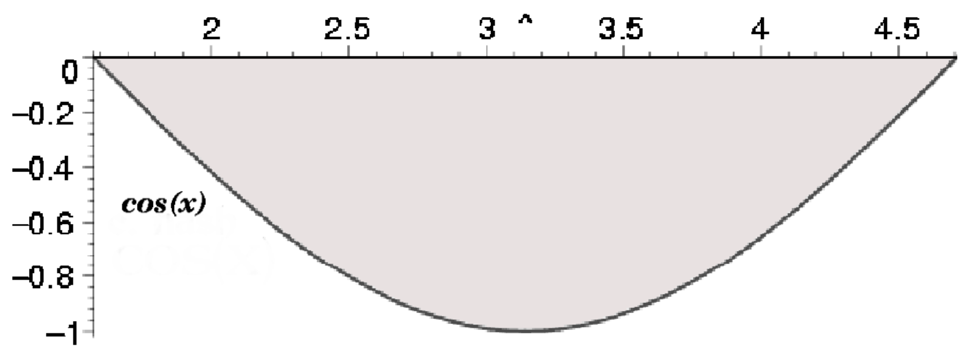

Fig. 36: The graph of $\cos(x)$ for the interval $[\pi/2, 3\pi/2]$

The integral we want is $\int_{\pi/2}^{3\pi/2} \cos(x)\,dx$ and we compute that

$$\begin{aligned}
\int_{\pi/2}^{3\pi/2} \cos(x)\,dx &= [\sin(x)]_{\pi/2}^{3\pi/2} \\
&= [\sin(3\pi/2) - (\sin(\pi/2))] \\
&= [-1 - 1], \quad \text{since} \begin{cases} \sin(\pi/2) = 1 \\ \sin(3\pi/2) = -1 \end{cases} \\
&= -2
\end{aligned} \tag{3.32}$$

Hence the integral has the value

$$-2 \tag{3.33}$$

and is negative as claimed. The *actual shaded area* shown in fig. 36 is of course

$$+2 \tag{3.34}$$

One should not be disturbed by this: all that is happening is that an integral $\int_a^b f(x)\,dx$ can have both *negative and positive* contributions depending on whether the function f is *negative or positive* for a given part of the integration interval $[a, b]$.

This can lead to an integral being zero because the positive and negative contributions exactly cancel. This happens in the next example.

Example *The integral*

$$\int_0^{2\pi} \sin(x)\,dx \tag{3.35}$$

First let's get the computation of the integral out of the way. We find that

$$\begin{aligned}\int_0^{2\pi} \sin(x)\,dx &= -\left[\cos(x)\right]_0^{2\pi} \\ &= -\left[\cos(2\pi) - \cos(0)\right] \\ &= -\left[1 - 1\right], \quad \text{since} \begin{cases} \cos(2\pi) = 1 \\ \cos(0) = 1 \end{cases} \\ &= 0\end{aligned} \tag{3.36}$$

So the integral does indeed vanish. Now if we look at fig. 37 we see two shaded areas of differing densities and it is clear that what has happened is that these two areas have simply cancelled.

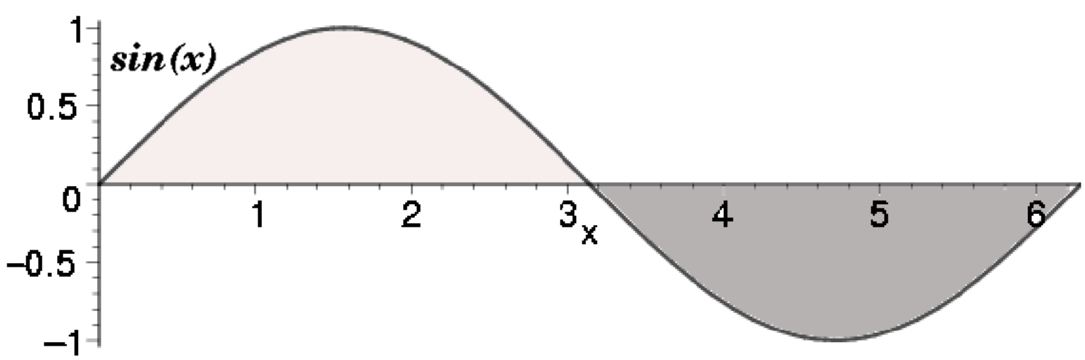

Fig. 37: The two cancelling areas for $\sin(x)$ on the interval $[0, 2\pi]$

If one really wants to prove this then one can calculate the two areas using separate integrals. In other words one calculates the two integrals

$$\int_0^{\pi} \sin(x)\,dx, \int_{\pi}^{2\pi} \sin(x)\,dx \tag{3.37}$$

One then readily verifies that the first integral is positive and equal to

$$+2 \tag{3.38}$$

while the second is equal to

$$-2 \tag{3.39}$$

A useful fact about a definite integral such as

$$\int_a^c f(x)\,dx \tag{3.40}$$

is that one can choose a number b between a and c and split the integral up into two pieces. What one obtains is just

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx, \quad a < b < c \tag{3.41}$$

All that we are doing is observing that the area given by $\int_a^c f(x)\,dx$ is the sum of the two smaller areas $\int_a^b f(x)\,dx$ and $\int_b^c f(x)\,dx$.

It is also useful to observe that if one *interchanges* a *and* b in the integral $\int_a^b f(x)\,dx$ then the integral *changes sign*. In other words we have

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \tag{3.42}$$

This fact is evident already in the statement 3.5 of the fundamental theorem of calculus where we see at once that the RHS changes sign if a and b are interchanged.

§ 2. Difficult integrals...?

Unlike derivatives, integrals can often be impossible to compute in terms of well known functions [2] and then one has to resort to numerical methods to deal with them.

[2] This may seem a slightly vague statement since the phrase “well known functions” is certainly not precise. However mathematicians do have a more precise set of functions in mind. This sets consists of all functions that can be obtained by addition, multiplication, division and composition of: polynomials, trigonometric functions and their inverses, and the functions ln and exp. These functions are sometimes then referred to as the *elementary functions*.

For example, there is no elementary function F which satisfies

$$\frac{dF(x)}{dx} = e^{-x^2} \tag{3.43}$$

In everyday language we say *we can't do the integral*

$$\int e^{-x^2}\, dx \tag{3.44}$$

Another example of an integral *"we can't do"* is

$$\int \sqrt{x} \sin(x)\, dx \tag{3.45}$$

Despite this we *can do* the very similar integral

$$\int x \sin(x)\, dx \tag{3.46}$$

for differentiation readily verifies the correctness of the statement that

$$\int x \sin(x)\, dx = \sin(x) - x\cos(x) \tag{3.47}$$

Thus a small change in the function to be integrated can make a big difference.

§ 3. Some well known integrals

It is always useful to have a list of integrals for functions that one comes across often. For this reason we provide a short table of integrals on p. 86.

§ 4. Integration techniques

There are two main techniques for doing integrals and these are
(a) Integration by change of variables or substitution.
(b) Integration by parts. We shall study both of these methods beginning with (a).

§§ 4.1 Integration by change of variable or integration by substitution

Next we illustrate the technique of integration by *change of variable* (also called integration by *substitution*). Consider then the following example.

Example *The integral*

$$\int_0^{\pi/2} \sin^5(x)\cos(x)\, dx \tag{3.48}$$

$f(x)$	$\int f(x)\,dx$
$x^n,\ (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$
$\dfrac{1}{x}$	$\ln(x)$
e^x	e^x
a^x	$\dfrac{a^x}{\ln(a)}$
$\ln(x)$	$x\ln(x) - x$
$\sin(x)$	$-\cos(x)$
$\cos(x)$	$\sin(x)$
$\tan(x)$	$-\ln(\cos(x)$
$\sec(x)$	$\ln(\sec(x) + \tan(x))$
$\mathrm{cosec}(x)$	$-\ln(\mathrm{cosec}(x) + \cot(x))$
$\cot(x)$	$\ln(\sin(x))$
$\sec^2(x)$	$\tan(x)$
$\sec(x)\tan(x)$	$\sec(x)$
$\sinh(x)$	$\cosh(x)$
$\cosh(x)$	$\sinh(x)$
$\tanh(x)$	$\ln(\cosh(x))$
$\dfrac{1}{1+x^2}$	$\arctan(x)$
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin(x)$
$-\dfrac{1}{\sqrt{1-x^2}}$	$\arccos(x)$

A table of useful integrals

We change variable from x to u where

$$u = \sin(x) \tag{3.49}$$

So that we have

$$\begin{aligned}\frac{du}{dx} &= \cos(x)\\ \Rightarrow du &= \cos(x)dx\\ \Rightarrow dx &= \frac{du}{\cos(x)}\end{aligned} \tag{3.50}$$

So now, if we *temporarily* omit the limits of integration, we can substitute this information into the integral giving us the equation

$$\begin{aligned}\int \sin^5(x)\cos(x)\,dx &= \int u^5\cos(x)\frac{du}{\cos(x)}\\ &= \int u^5 du\end{aligned} \tag{3.51}$$

Finally to restore the limit of integration we notice that, since $u = \sin(x)$, we have

$$\begin{aligned}x = 0 &\Rightarrow u = 0\\ x = \frac{\pi}{2} &\Rightarrow u = \sin(\pi/2) = 1\end{aligned} \tag{3.52}$$

Hence our limits in terms of u are 0 and 1 and, restoring the limits to both integrals, we have the equality

$$\begin{aligned}\int_0^{\pi/2} \sin^5(x)\cos(x)\,dx &= \int_0^1 u^5\,du\\ &= \left[\frac{u^6}{6}\right]_0^1\\ &= \left[\frac{1}{6} - 0\right]\\ &= \frac{1}{6}\end{aligned} \tag{3.53}$$

and our integral has been completed. We move on.

Example *The integral*

$$\int \frac{dx}{x\ln(x)} \tag{3.54}$$

This time we set

$$\begin{aligned}u &= \ln(x)\\ \Rightarrow \frac{du}{dx} &= \frac{1}{x}\\ \Rightarrow dx &= x\,du\end{aligned} \tag{3.55}$$

Now we put this information into our integral and find that

$$\begin{aligned} \int \frac{dx}{x\ln(x)} &= \int \frac{xdu}{xu} \\ &= \int \frac{du}{u} \\ &= \ln(u) \\ &= \ln(\ln(x)) \end{aligned} \tag{3.56}$$

and so we have established that

$$\int \frac{dx}{x\ln(x)} = \ln(\ln(x)) \tag{3.57}$$

Time for our next integral which is

Example *The integral*

$$\int_0^a \sqrt{a^2 - x^2}\, dx \tag{3.58}$$

The trick here is to use a trigonometric substitution or change of variable; the one that works for this case[3] is

$$\begin{aligned} x &= a\sin(\theta) \\ \Rightarrow dx &= a\cos(\theta)\, d\theta \end{aligned} \tag{3.59}$$

We must also change the limits of integration to accommodate the new variable θ; to this end note that

$$\begin{aligned} x = 0 &\Rightarrow \theta = 0 \\ x = 1 &\Rightarrow \theta = \frac{\pi}{2} \end{aligned} \tag{3.60}$$

[3] For other closely related integrals such as

$$\int \left(a^2 \mp x^2\right)^{\mp 1/2}\, dx$$

one should also try a trigonometric substitution such as

$$x = a\sin(\theta),\ x = a\cos(\theta),\ x = a\tan(\theta)$$

$x = a\tan(\theta)$ being the one to use when $(a^2 + x^2)$, rather than $(a^2 - x^2)$, occurs in the integrand.

so the new limits are 0 and $\pi/2$ and we obtain

$$\begin{aligned}
\int_0^a \sqrt{a^2 - x^2}\,dx &= \int_0^{\pi/2} \sqrt{a^2 - a^2\sin(\theta)}\,a\cos(\theta)\,d\theta \\
&= \int_0^{\pi/2} \sqrt{a^2(1-\sin^2(\theta))}\,a\cos(\theta)\,d\theta \\
&= \int_0^{\pi/2} a\cos(\theta)a\cos(\theta)\,d\theta, \quad \text{using } (1-\sin^2(\theta)) = \cos^2(\theta) \\
&= a^2\int_0^{\pi/2} \cos^2(\theta)\,d\theta \\
&= a^2\int_0^{\pi/2} \frac{1+\cos(2\theta)}{2}\,d\theta, \quad \text{using } \cos^2(\theta) = \frac{1+\cos(2\theta)}{2} \\
&= a^2\left[\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right]_0^{\pi/2} = a^2\left[\frac{\pi}{4} + \frac{\sin(\pi)}{4}\right] = \frac{a^2\pi}{4}, \quad \text{since } \sin(\pi) = 0
\end{aligned} \tag{3.61}$$

Now we consider

Example *The integral*

$$\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx \tag{3.62}$$

This time we shall try

$$\begin{aligned}
u &= \sqrt{e^x+1} \\
\Rightarrow du &= \frac{1}{2}\left(e^x+1\right)^{-1/2} e^x dx
\end{aligned} \tag{3.63}$$

Also

$$\begin{aligned}
u &= \sqrt{e^x+1} \\
\Rightarrow e^x &= u^2 - 1 \\
&\Rightarrow \begin{cases} e^x + 1 = u^2 \\ e^{2x} = (u^2-1)^2 \\ dx = \dfrac{2u}{u^2-1}du \end{cases}
\end{aligned} \tag{3.64}$$

Our integral now displays the following transformation

$$\begin{aligned}
\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx &= \int \frac{(u^2-1)^2}{u}\frac{2u}{u^2-1}\,du \\
&= \int 2(u^2-1)\,du \\
&= \frac{2}{3}u^3 - 2u
\end{aligned} \tag{3.65}$$

But $u = \sqrt{e^x + 1}$ so we have deduced that

$$\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx = \frac{2}{3}\left(e^x+1\right)^{3/2} - 2\left(e^x+1\right)^{1/2} \tag{3.66}$$

We are now ready for the other main integration technique which is integration by parts.

§§ 4.2 Integration by parts

All integrations by parts rest on a clever use of the same formula. This formula is very simply obtained: one just integrates the formula for the derivative of a product. More precisely we begin with the formula

$$\frac{d}{dx}\left(f(x)g(x)\right) = f(x)\frac{dg(x)}{dx} + \frac{df(x)}{dx}g(x) \tag{3.67}$$

which we rewrite as

$$f(x)\frac{dg(x)}{dx} = \frac{d}{dx}\left(f(x)g(x)\right) - \frac{df(x)}{dx}g(x) \tag{3.68}$$

and then we integrate both sides from a to b yielding

$$\begin{aligned} \int_a^b f(x)\frac{dg(x)}{dx}\,dx &= \int_a^b \frac{d}{dx}\left(f(x)g(x)\right)\,dx - \int_a^b \frac{df(x)}{dx}g(x)\,dx \\ \Rightarrow \int_a^b f(x)\frac{dg(x)}{dx}\,dx &= \left[f(x)g(x)\right]_a^b - \int_a^b \frac{df(x)}{dx}g(x)\,dx \\ \text{Or more compactly} \qquad \int_a^b fg'\,dx &= [fg]_a^b - \int_a^b f'g\,dx \end{aligned} \tag{3.69}$$

This last formula is the one we want and we emphasise its importance by quoting it as a theorem.

Theorem (Integration by parts) *If f and g are two differentiable functions then*

$$\begin{aligned} &\int_a^b fg'\,dx = [fg]_a^b - \int_a^b f'g\,dx \\ \textit{or} \qquad &\int fg'\,dx = fg - \int f'g\,dx, \qquad \textit{without limits} \end{aligned} \tag{3.70}$$

Now we need some examples of integration by parts in action. We begin with something simple.

Example *The integral*

$$\int xe^x \, dx \tag{3.71}$$

The method of integration by parts then consists of equating the integrand to fg' and then using 3.70. So we write

$$xe^x = fg' \tag{3.72}$$

and immediately we are faced with the task of deciding which part of xe^x should we equate to f and which to g'. The answer to this problem is that one proceeds partially by trial and error and partially by previous experience. This time we choose to set

$$f = x \tag{3.73}$$

which forces

$$g' = e^x \tag{3.74}$$

With this choice for f and g we use 3.70 giving us

$$\begin{aligned} \int xe^x \, dx &= xe^x - \int 1 \cdot e^x \, dx \\ &= xe^x - e^x \end{aligned} \tag{3.75}$$

and so we are finished.

A favourite trick in integration by parts is to set g' equal to 1 and then use the formula 3.70. This can be seen at work in our next calculation.

Example *The integral*

$$\int \ln(x) \, dx \tag{3.76}$$

We set

$$g' = 1 \tag{3.77}$$

We must therefore set $f = \ln(x)$ and 3.70 gives

$$\begin{aligned} \int \ln(x) \cdot 1 \, dx &= \ln(x) \cdot x - \int \frac{1}{x} \cdot x \, dx \\ &= x \ln(x) - x \end{aligned} \tag{3.78}$$

and so we have our integral and the result agrees, as it must, with that quoted in our integral table on p. 86.

A second popular trick is to try to express $\int f\,dx$ in terms of itself and solve the resulting formula. We illustrate this next.

Example *The integral*

$$\int \frac{\ln(x)}{x}\,dx \tag{3.79}$$

Setting $f = \ln(x)$, and thus $g' = 1/x$, we obtain the equation

$$\begin{aligned} \int \frac{\ln(x)}{x}\,dx &= \ln(x)\ln(x) - \int \frac{1}{x}\ln(x)\,dx \\ \Rightarrow 2\int \frac{\ln(x)}{x}\,dx &= \ln(x)\ln(x) \\ \text{or} \qquad \int \frac{\ln(x)}{x}\,dx &= \frac{\ln^2(x)}{2} \end{aligned} \tag{3.80}$$

and we note the appearance of $\int \ln(x)/x\,dx$ on both sides of the equation in the first line of 3.80. Another integral of this type is

Example *The integral*

$$\int \sin^2(x)\,dx \tag{3.81}$$

Choosing $f = \sin(x)$, and therefore $g' = \sin(x)$ also, gives

$$\begin{aligned} \int \sin^2(x)\,dx &= \sin(x)(-\cos(x)) - \int \cos(x)(-\cos(x))\,dx \\ &= -\sin(x)\cos(x) + \int \cos^2(x)\,dx \\ &= -\sin(x)\cos(x) + \int (1-\sin^2(x))\,dx, \quad \text{using } \cos^2(x) = 1-\sin^2(x) \\ \Rightarrow 2\int \sin^2(x)\,dx &= -\sin(x)\cos(x) + \int dx \\ \Rightarrow \int \sin^2(x)\,dx &= \frac{x - \sin(x)\cos(x)}{2} \end{aligned} \tag{3.82}$$

§ 5. A little more integration

We shall finish this chapter with a bit more integration practice.

Example *The integral*

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx \tag{3.83}$$

We use the substitution $u = \sqrt{x}$ and so find that

$$\begin{aligned} \frac{du}{dx} &= \frac{1}{2}x^{-1/2} \\ \Rightarrow dx &= 2\,x^{1/2}du \end{aligned} \tag{3.84}$$

Hence we obtain

$$\begin{aligned} \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx &= \int e^u \frac{2\,x^{1/2}du}{\sqrt{x}} \\ &= 2\int e^u\,du \\ &= 2e^u = 2e^{\sqrt{x}} \end{aligned} \tag{3.85}$$

and so we have shown that

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2e^{\sqrt{x}} \tag{3.86}$$

Example *The integral*

$$\int x^2 e^x\,dx \tag{3.87}$$

We can do this by parts but we will apply the method twice to get to the end. Setting

$$g' = e^x \tag{3.88}$$

we get

$$\int x^2 e^x\,dx = x^2 e^x - \int 2xe^x\,dx \tag{3.89}$$

This leaves us with the integral $\int xe^x\,dx$ still to do; so for this integral we again set

$$g' = e^x \tag{3.90}$$

and obtain

$$\begin{aligned} \int xe^x\,dx &= xe^x - \int 1\cdot e^x\,dx \\ &= xe^x - e^x \end{aligned} \tag{3.91}$$

Substituting 3.91 into 3.89 we find the result that

$$\int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x \tag{3.92}$$

Example *The integral*

$$\int \frac{e^x}{e^{2x} + 2e^x + 1} \, dx \tag{3.93}$$

For this integral note that the denominator of the integrand is a perfect square—that is

$$e^{2x} + 2e^x + 1 = (e^x + 1)^2 \tag{3.94}$$

This means that we can write

$$\int \frac{e^x}{e^{2x} + 2e^x + 1} \, dx = \int \frac{e^x}{(e^x + 1)^2} \, dx \tag{3.95}$$

and now if we set

$$u = e^x + 1 \tag{3.96}$$

we find that

$$\begin{aligned} \int \frac{e^x}{(e^x + 1)^2} \, dx &= \int \frac{e^x}{u^2} \frac{du}{e^x} \\ &= \int \frac{du}{u^2} \\ &= -\frac{1}{u} = -\frac{1}{e^x + 1} \end{aligned} \tag{3.97}$$

So we have deduced that

$$\int \frac{e^x}{e^{2x} + 2e^x + 1} \, dx = -\frac{1}{e^x + 1} \tag{3.98}$$

Example *The integral*

$$\int \sqrt{x} \ln(x) \, dx \tag{3.99}$$

For this example we use integration by parts and set

$$g' = \sqrt{x} \tag{3.100}$$

This yields the equation

$$\begin{aligned}\int \sqrt{x}\ln(x)\,dx &= \frac{2}{3}x^{3/2}\ln(x) - \int \frac{2}{3}x^{3/2}\frac{1}{x}\,dx \\ &= \frac{2}{3}x^{3/2}\ln(x) - \frac{2}{3}\int x^{1/2}\,dx \\ &= \frac{2}{3}x^{3/2}\ln(x) - \left(\frac{2}{3}\right)^2 x^{3/2}\end{aligned} \tag{3.101}$$

So we have found that

$$\int \sqrt{x}\ln(x)\,dx = \frac{2}{3}x^{3/2}\left(\ln(x) - \frac{2}{3}\right) \tag{3.102}$$

CHAPTER IV

Miscellaneous topics

§ 1. Volumes, surfaces and curves

WE shall use this chapter to discuss an assorted list of topics most of them applications of calculus. We begin with an important use of integration to calculate *volumes*, *surface areas* and *lengths of curves* rather than just the *area* under a curve. We shall deal first with volumes. To this end we introduce what is called a *volume of revolution* and then calculate its volume.

§§ 1.1 Volumes of revolution and their volumes

A volume of revolution is made by rotating a curve through one complete revolution—i.e. 2π radians—about an axis. In fig. 38 we show what is needed to understand what is going on. The first part of the figure simply show the graph of the curve $f(x)$ while the second illustration shows the solid obtained by rotating this curve once about the x-axis. Such a solid is called a *volume of revolution*. We shall now see how to calculate the volume of this solid.

All we have to do is to divide the solid up into cylinders or disks: fig. 38 shows a typical disk, with *centre* x, *radius* $f(x)$ and thickness dx.

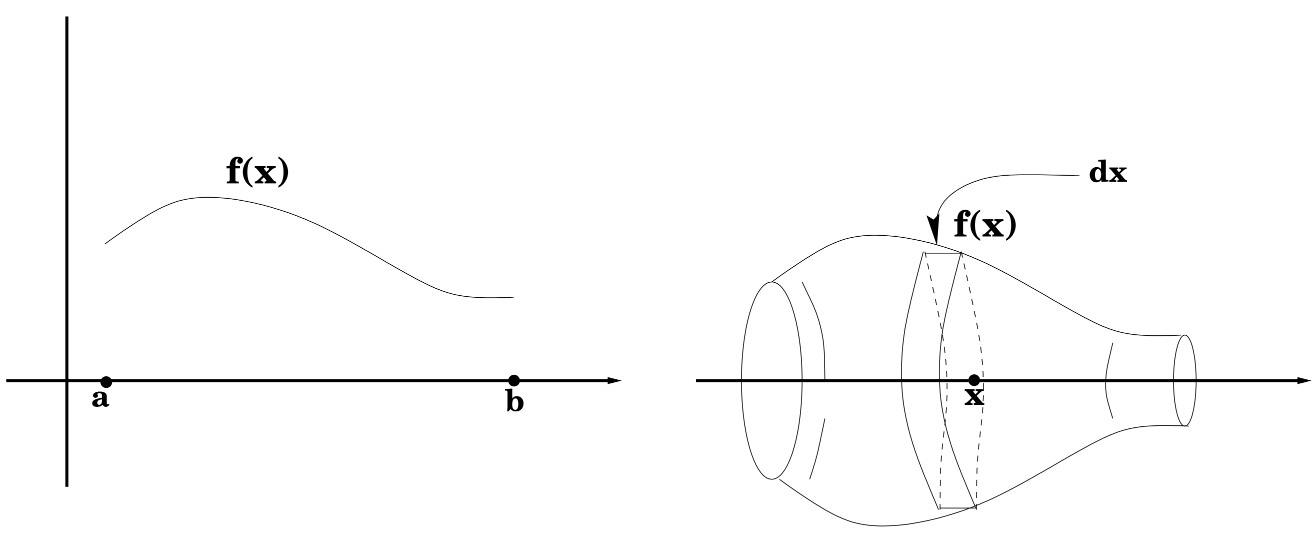

Fig. 38: A volume of revolution made from the function $f(x)$

Now a disk of radius r and thickness h is just a cylinder and so has volume

$$\pi r^2 h \tag{4.1}$$

Hence the disk shown in fig. 38 has volume

$$\pi(f(x))^2 dx \tag{4.2}$$

The entire volume V of the solid will be obtained by summing over all disks as their thickness dx tends to zero; but this is just the integral

$$\int_a^b \pi f^2(x)\, dx \tag{4.3}$$

where a and b are the points where the curve $f(x)$ begins and ends. Thus we have a formula for the volume V of the surface of revolution namely

$$V = \int_a^b \pi f^2(x)\, dx \tag{4.4}$$

Let us use this formula in an actual calculation.

Example *The volume of an ellipsoid of revolution*

An ellipse centered at the origin has the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{4.5}$$

and is shown in fig. 39.

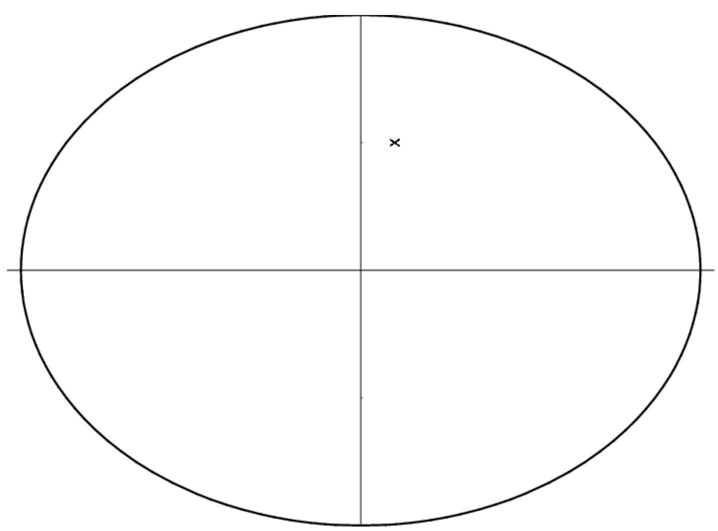

Fig. 39: The ellipse $x^2/a^2 + y^2/b^2 = 1$

If we take *half* of this ellipse—as shown in fig. 40—and rotate it about the x axis we obtain a solid called an *ellipsoid of revolution*.

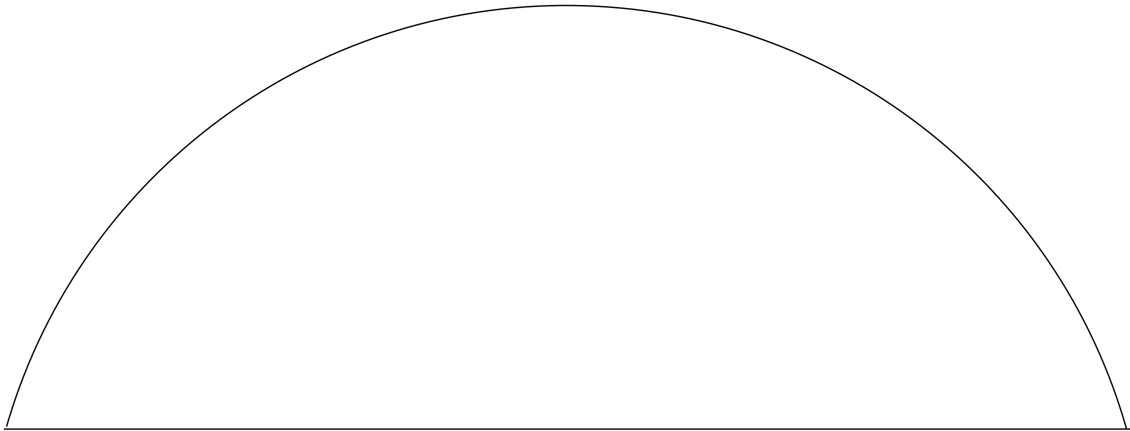

Fig. 40: The upper half of the ellipse $x^2/a^2 + y^2/b^2 = 1$

Now we need to find the appropriate function $f(x)$ so that we can use formula 4.4; but $f(x)$ is just the quantity y so this will come at once from the equation of the ellipse: We have

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1 \\ \Rightarrow y^2 &= b^2\left(1 - \frac{x^2}{a^2}\right) \qquad (4.6) \\ \Rightarrow y &= b\sqrt{1 - \frac{x^2}{a^2}} \end{aligned}$$

Hence $f(x)$ is given by the equation

$$f(x) = b\sqrt{1 - \frac{x^2}{a^2}} \qquad (4.7)$$

Now we can use formula 4.4 to compute the volume V of this ellipsoid. This gives us that

$$\begin{aligned} V &= \int_a^b \pi f^2(x)\,dx \\ &= \pi \int_{-a}^{a} b^2\left(1 - \frac{x^2}{a^2}\right) dx \\ &= \pi b^2 \left[x - \frac{x^3}{3a^2}\right]_{-a}^{a} \qquad (4.8) \\ &= \pi b^2 \left[a - \frac{a^3}{3a^2} - (-a) + \frac{(-a)^3}{3a^2}\right] \\ &= \pi b^2 \left[\frac{4a}{3}\right] \end{aligned}$$

So the volume of the ellipsoid of revolution is given by

$$V = \frac{4\pi ab^2}{3} \tag{4.9}$$

Notice that, since a circle is a special case of an ellipse, we should be able to reproduce the formula for the volume of a sphere by making our ellipse a circle. This will provide a check on our calculation and so we shall do it. All we have to do is to set

$$b = a \tag{4.10}$$

and then the equation of the ellipse becomes

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{a^2} &= 1 \\ \Rightarrow x^2 + y^2 &= a^2 \end{aligned} \tag{4.11}$$

i.e. the ellipse has a become a circle of radius a. Such a circle would, if we rotated its upper half, produce a sphere of radius a which we know has a volume

$$\frac{4\pi a^3}{3} \tag{4.12}$$

But if we set $b = a$ in the formula 4.9 for the volume of our ellipsoid we obtain

$$\frac{4\pi a^3}{3} \tag{4.13}$$

which is indeed the volume of the sphere, and our check has succeeded.

§§ **1.2 Surfaces of revolution and their areas**

We can also use integration to find the area of the *surface* of one of these solids. Such a surface is called, not surprisingly, a *surface of revolution*.

Now to calculate the area of this surface one can again use disks but this time they the disks are not small cylinders of height dx but *skew disks* of *slanted height dl*. An example of a skew disk is shown in fig. 41.

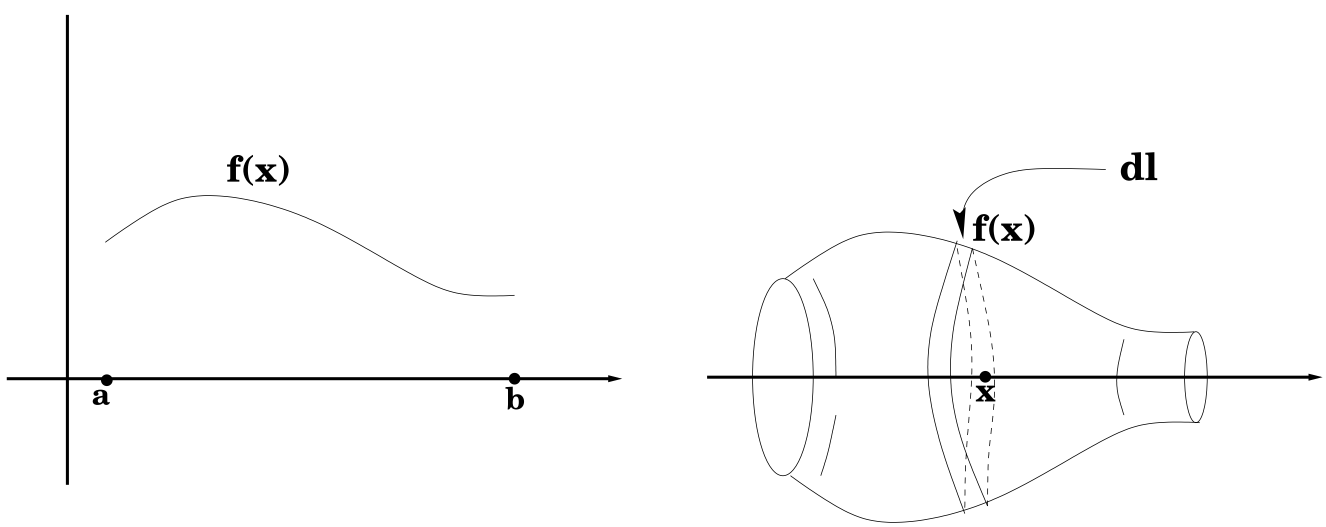

Fig. 41: A surface of revolution and its division into *skew* disks

The reader should carefully compare figs. 38 and 41 until he or she notices that the disk of 38 have *straight sides* of height dx while those of 41 have *slanted sides* of length dl.

The important point is that dl is, in general, not equal to dx because dl points along the direction of the tangent to f at x while dx is always parallel to the x-axis. But if the tangent makes an angle θ with the x-axis at the point x then, by definition of the derivative, we know that

$$f'(x) = \tan(\theta) \tag{4.14}$$

It is also true that dx and dl are at an angle θ relative to one another so that we have

$$dx = dl\cos(\theta) \tag{4.15}$$

Now we are ready to calculate the area: a typical skew disk of radius $f(x)$ and skew side of slanted length dl has *surface area*

$$\begin{aligned} &2\pi f(x)dl \\ &= 2\pi f(x)\frac{dx}{\cos(\theta)}, \qquad \text{using 4.15} \end{aligned} \tag{4.16}$$

But we can relate $\cos(\theta)$ to $f'(x)$ by using 4.14. We have the fact that

$$\begin{aligned} 1 + \tan^2(\theta) &= \sec^2(\theta) \\ &= \frac{1}{\cos^2(\theta)} \\ \Rightarrow \frac{1}{\cos(\theta)} &= \sqrt{1+\tan^2(\theta)} \\ &= \sqrt{1+f'(x)^2}, \qquad \text{using 4.14} \end{aligned} \tag{4.17}$$

So our surface area formula now becomes

$$2\pi f(x)\frac{dx}{\cos(\theta)} = 2\pi f(x)\sqrt{1+f'(x)^2}\,dx \tag{4.18}$$

The final task to get the surface area S of the surface of revolution is to sum over all such skew disks and this gives us an integral. The resulting expression for S

$$S = \int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,dx \tag{4.19}$$

It is now time to look at an example.

Example *The surface area of the ellipsoid*

The calculation is rather straightforward since we have already had the experience of the volume computation.

Recall that 4.7 says that

$$\begin{aligned} f(x) &= b\sqrt{1-\frac{x^2}{a^2}} \\ \Rightarrow f'(x) &= b\left(\frac{1}{2}\right)\left(1-\frac{x^2}{a^2}\right)^{-1/2}\left(\frac{-2x}{a^2}\right) \\ &= \frac{-bx}{a^2}\frac{1}{\sqrt{1-\frac{x^2}{a^2}}} \\ \Rightarrow 1+f'(x)^2 &= 1+\frac{b^2x^2}{a^4}\frac{1}{1-\frac{x^2}{a^2}} \\ &= \frac{a^4+(b^2-a^2)x^2}{a^4(1-\frac{x^2}{a^2})} \end{aligned} \tag{4.20}$$

Substituting this information back into the expression 4.19 for the area S gives

$$S = 2\pi\int_{-a}^{a}\frac{b}{a^2}\sqrt{a^4+(b^2-a^2)x^2}\,dx \tag{4.21}$$

Now to speed up the calculation [1] we set

$$b = a \tag{4.23}$$

so that we are now computing the surface area of a sphere. With this simplification we find that

$$\begin{aligned} S &= 2\pi a\int_{-a}^{a}dx \\ &= 2\pi a\,[x]_{-a}^{a} \\ &= 4\pi a^2 \end{aligned} \tag{4.24}$$

[1] We don't want to bother the reader with the expression for the surface area when $b \neq a$, but for the curious we give it anyway in this footnote. If we use the fact that

$$\int\sqrt{1-x^2}\,dx = \frac{1}{2}\left\{x\sqrt{1-x^2}+\arcsin(x)\right\}, \quad \text{(integrate by parts twice to get this)}$$

then we find that we can do the integral for S after a suitable change of variable. After doing this, and putting in the limits, we find that

$$S = 2\pi\left\{b^2+\frac{a^2b}{\sqrt{a^2-b^2}}\arcsin\left(\frac{\sqrt{a^2-b^2}}{a}\right)\right\} \tag{4.22}$$

and this is indeed the surface area of a sphere of radius a.

We can now turn to the matter of calculating the length of an arbitrary curve.

§§ 1.3 Curves and their lengths

If $f(x)$ is any curve, such as that depicted in fig. 41, then we can easily derive an integral formula for the length of f. If the endpoints of the curve are, as in fig. 41, at $x = a$ and $x = b$ then the length L of the curve is given by

$$L = \int_a^b dl \tag{4.25}$$

where dl is an infinitesimal piece of the curve exactly as it is in 41. But 4.15 gives the relation between dx and dl as

$$dx = dl\cos(\theta) \tag{4.26}$$

and 4.17 tells us that

$$\cos(\theta) = \frac{1}{\sqrt{1 + f'(x)^2}} \tag{4.27}$$

Hence $dl = \sqrt{1 + f'(x)^2}\,dx$ giving us our final formula for the curve length which is

$$L = \int_a^b \sqrt{1 + f'(x)^2}\,dx \tag{4.28}$$

We are now ready to do a calculation.

Example *The length of an arc of a circle*

If the circle has radius A then its equation is

$$\begin{aligned} x^2 + y^2 &= A^2 \\ \Rightarrow y &= \sqrt{A^2 - x^2} \end{aligned} \tag{4.29}$$

This means we set

$$f(x) = \sqrt{A^2 - x^2} \tag{4.30}$$

and so

$$f'(x) = -\frac{x}{\sqrt{A^2 - x^2}} \tag{4.31}$$

and thus

$$\begin{aligned} L &= \int_a^b \sqrt{1 + \frac{x^2}{A^2 - x^2}}\,dx \\ &= \int_a^b \sqrt{\frac{x^2 - A^2 + x^2}{A^2 - x^2}}\,dx \\ &= A\int_a^b \frac{dx}{\sqrt{A^2 - x^2}} \end{aligned} \tag{4.32}$$

Let us take a circle of unit radius so that $A = 1$ and then we have

$$\begin{aligned} L &= \int_a^b \frac{dx}{\sqrt{1-x^2}} \\ &= [\arcsin(x)]_a^b\,, \quad \text{using our table of integrals} \end{aligned} \tag{4.33}$$

Finally let us decide to compute the length of a quadrant of this unit circle—cf. fig 42; this means that we must choose the interval $[a, b]$ to be given by

$$[a, b] = [0, 1] \tag{4.34}$$

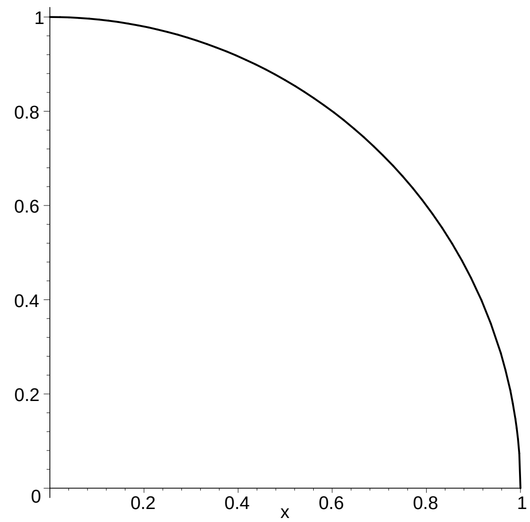

Fig. 42: A quadrant of a circle of radius 1

We then find that

$$\begin{aligned} L &= [\arcsin(x)]_0^1 \\ &= [\arcsin(1) - \arcsin(0)] \\ &= \left[\frac{\pi}{2} - 0\right] \\ &= \frac{\pi}{2} \end{aligned} \tag{4.35}$$

and so $L = \pi/2$; a fact which we can easily deduce without integration since the arc is precisely a quarter of the circle's circumference.

Example *Find the length of the curve* $f(x) = 1/x + x^3/12$ *between* $x = 2$ *and* $x = 5$

The curve $f(x)$ is shown in fig. 43.

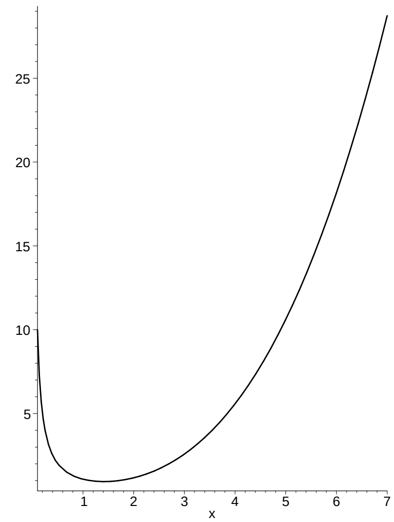

Fig. 43: The curve $f(x) = 1/x + x^3/12$

With

$$f(x) = \frac{1}{x} + \frac{x^3}{12} \tag{4.36}$$

we find that

$$f'(x) = -\frac{1}{x^2} + \frac{3x^2}{12} \tag{4.37}$$

and this gives us

$$\begin{aligned} 1 + f'(x)^2 &= 1 + \left(-\frac{1}{x^2} + \frac{3x^2}{12}\right)^2 \\ &= 1 + \frac{1}{x^4} - \frac{1}{2} + \frac{x^4}{16} \\ &= \frac{1}{x^4} + \frac{1}{2} + \frac{x^4}{16} \\ &= \left(\frac{1}{x^2} + \frac{x^2}{4}\right)^2 \end{aligned} \tag{4.38}$$

This means that the length L in question is given by

$$\begin{aligned} L &= \int_2^5 \sqrt{\left(\frac{1}{x^2}+\frac{x^2}{4}\right)^2}\,dx \\ &= \int_2^5 \left(\frac{1}{x^2}+\frac{x^2}{4}\right)dx \\ &= \left[-\frac{1}{x}+\frac{x^3}{12}\right]_2^5 \\ &= \left[-\frac{1}{5}+\frac{5^3}{12}+\frac{1}{2}-\frac{2^3}{12}\right] \\ &= \frac{201}{20} \end{aligned} \tag{4.39}$$

and so L is found.

Example *The length of a piece of a parabola*

For this example we take

$$f(x) = x^2 \tag{4.40}$$

which is a parabola and is displayed in fig. 44.

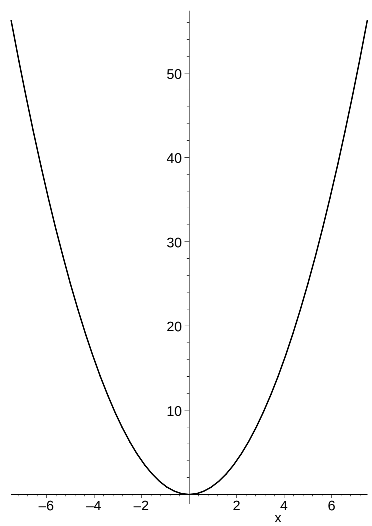

Fig. 44: The parabola $f(x) = x^2$

We readily compute that

$$\begin{aligned} f'(x) &= 2x \\ \Rightarrow \sqrt{1+f'(x)^2} &= \sqrt{1+4x^2} \end{aligned} \tag{4.41}$$

Hence the length L of the section of this parabola between $x = a$ and $x = b$ is given by

$$L = \int_a^b \sqrt{1 + 4x^2}\, dx \tag{4.42}$$

If we choose, say, $a = 0$ and $b = 7$ then we find that

$$L = \int_0^7 \sqrt{1 + 4x^2}\, dx \tag{4.43}$$

Now if we simply *accept* that

$$\int \sqrt{1 + 4x^2}\, dx = \frac{1}{2}x\sqrt{1 + 4x^2} + \frac{1}{4}\,\text{arcsinh}(2x) \tag{4.44}$$

a fact that we do *not* expect the reader to be able to derive. Then we find that

$$\begin{aligned} L &= \left[\frac{1}{2}x\sqrt{1 + 4x^2} + \frac{1}{4}\,\text{arcsinh}(2x)\right]_0^7 \\ &= \left[\frac{1}{2}7\sqrt{1 + 4 \cdot 49} + \frac{1}{4}\,\text{arcsinh}(14) - 0\right] \\ &= 49.95821036 \end{aligned} \tag{4.45}$$

where we used a calculator to evaluate the final expression.

§ 2. The Mean value of a function

Suppose we have a function f(x) whose value we measure n times thereby obtaining the series of values

$$f(x_1), f(x_2), \ldots f(x_n) \tag{4.46}$$

The mean, or average, of these n measurements is

$$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \tag{4.47}$$

We shall denote[2] this mean by $< f >$ so that

$$< f >= \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \tag{4.48}$$

[2] The mean is also sometimes denoted by $\bar{f}$.

If n becomes very large—or indeed tends to infinity—then the resulting mean, which we shall still denote by $< f >$, can be expressed as an integral. We shall now explain this.

Suppose that the possible values of x all lie within the interval $[a, b]$ and suppose that we consider, for a moment the integral of f over this interval. Returning to 3.3 for the definition of an integral we have [3]

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} (f(x_1) + f(x_2) + \cdots + f(x_n))\,\Delta x \tag{4.49}$$

But since the interval $[a, b]$ is divided into precisely n pieces, each of size Δx, we have

$$\begin{aligned} n\Delta x &= b - a \\ \Rightarrow \Delta x &= \frac{(b-a)}{n} \end{aligned} \tag{4.50}$$

Substituting this into our expression for the integral gives

$$\begin{aligned} \int_a^b f(x)\,dx &= \lim_{n\to\infty} (f(x_1) + f(x_2) + \cdots + f(x_n))\,\frac{(b-a)}{n} \\ &= \lim_{n\to\infty} \frac{(f(x_1) + f(x_2) + \cdots + f(x_n))}{n}(b-a) \\ &=< f > (b-a) \end{aligned} \tag{4.51}$$

In other words we have shown that the mean $< f >$ of f over the interval $[a, b]$ is just $\int_a^b f\,dx$ divided by the length of the interval; that is

$$< f >= \frac{\displaystyle\int_a^b f(x)\,dx}{b-a} \tag{4.52}$$

The importance of this way of defining the mean is that it is very well suited to quantities that vary *continuously* with time such as the voltage output from an electrical device. Here is an example.

Example *The mean voltage produced by an oscillator*

An oscillator produces a current output $I(t)$ given by

$$I(t) = a\cos(10\omega t) + b\cos(\omega t) \tag{4.53}$$

where t stands for time—cf. fig. 45.

[3] Note when comparing with 3.3 that we have used n instead of $n-1$, and written Δx for $(x_{i+1} - x_i)$

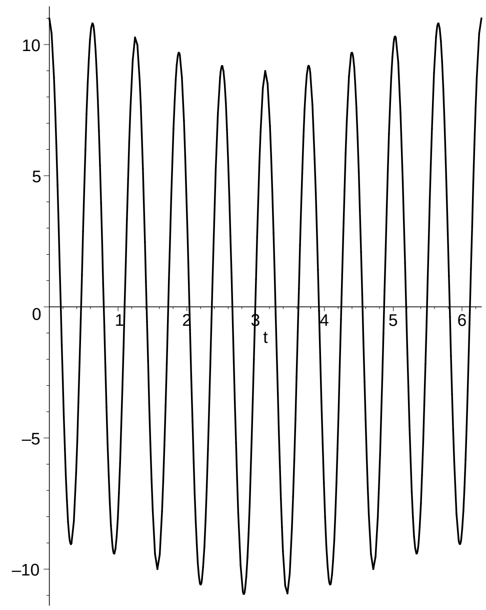

Fig. 45: The oscillator current $I(t) = a\cos(10\omega t) + b\cos(\omega t)$ for $a = 10$, $b = 1$, $\omega = 1$

Notice that $I(T)$ is a sum of a very rapidly varying term $a\cos(10\omega t)$ and a term of slower variation $b\cos(\omega t)$. The term of rapid variation has a periodic time T_{rapid} given by

$$T_{rapid} = \frac{2\pi}{10\omega} \tag{4.54}$$

and we shall now calculate the mean of $I(t)$ over this period. We have

$$< I >= \frac{1}{T_{rapid}} \int_0^{T_{rapid}} (a\cos(10\omega t) + b\cos(\omega t))\ dt, \quad T_{rapid} = \frac{2\pi}{10\omega} \tag{4.55}$$

But

$$\begin{aligned} \int_0^{T_{rapid}} (a\cos(10\omega t) + b\cos(\omega t))\ dt &= \left[\frac{a}{10\omega}\sin(10\omega t) + \frac{b}{\omega}\sin(\omega t)\right]_0^{T_{rapid}} \\ &= \left[\frac{a}{10\omega}\sin(10\omega T_{rapid}) + \frac{b}{\omega}\sin(\omega T_{rapid}) - 0\right] \\ &= \frac{a}{10\omega}\sin\left(10\omega\frac{2\pi}{10\omega}\right) + \frac{b}{\omega}\sin\left(\omega\frac{2\pi}{10\omega}\right) \\ &= \frac{a}{10\omega}\sin(2\pi) + \frac{b}{\omega}\sin\left(\frac{2\pi}{10}\right) \\ &= 0 + \frac{b}{\omega}\sin\left(\frac{2\pi}{10}\right) \end{aligned} \tag{4.56}$$

Hence we find that

$$\begin{aligned} < I > &= \frac{1}{T_{rapid}} \frac{b}{\omega} \sin\left(\frac{2\pi}{10}\right) \\ &= \frac{10\omega}{2\pi} \frac{b}{\omega} \sin\left(\frac{2\pi}{10}\right) \\ \text{i.e.} \qquad < I > &= \frac{10b}{2\pi} \sin\left(\frac{2\pi}{10}\right) \end{aligned} \tag{4.57}$$

Note that the periodic time T of the *combined* system of both terms is just the periodic time of the slowest varying term; so in this case it is

$$T = \frac{2\pi}{\omega} \tag{4.58}$$

and if we calculate the mean $< I >$ over this time T we shall find that

$$< I >= 0 \tag{4.59}$$

because, during this time T, the graph of $I(T)$ spends exactly the same amount of time above the t axis as below (cf. fig. 45) and this causes the integral $\int_0^T I(t)\,dt$ to vanish.

This vanishing of the mean of a function is illustrated in the next example.

Example *The mean of the simple sinusoidal voltage* $V(t) = a\sin(\omega t)$

In this example we have a variable voltage which $V(t)$ is just a sine wave and so has periodic time $T = 2\pi/\omega$ and we can see at once that its mean is zero. Calculating the mean we have

$$\begin{aligned} < V > &= \frac{1}{T}\int_0^T a\sin(\omega t)\,dt, \quad T = \frac{2\pi}{\omega} \\ &= -\frac{1}{T}\left[\frac{a\cos(\omega t)}{\omega}\right]_0^T \\ &= -\frac{1}{T}\left[\frac{a\cos(\omega T)}{\omega} - \frac{a\cos(0)}{\omega}\right] \\ &= -\frac{\omega}{2\pi}\left[\frac{a\cos(2\pi)}{\omega} - \frac{a}{\omega}\right] \\ &= -\frac{\omega}{2\pi}\left[\frac{a}{\omega} - \frac{a}{\omega}\right] \\ \text{i.e.} \qquad < V > &= 0 \end{aligned} \tag{4.60}$$

§ 3. The Root Mean Square or RMS value of a function

Since the mean of a voltage such as $V(t) = a\sin(\omega t)$ vanishes and so $< V >$ tells us little about such a voltage it is useful to introduce another related mean which gives more information about $V(t)$. This is called the *Root Mean Square value* or the *RMS value* of the function in question.

If f is any function then we shall denotes its RMS value on an interval $[a, b]$ by

$$f_{RMS} \tag{4.61}$$

where the definition of f_{RMS} is that

$$f_{RMS} = \sqrt{< f^2 >}$$

So f_{RMS}^2 is the average of f^2 rather than f. The important point to note then is that since f^2 is *always positive*—being the square of something—then f_{RMS} will never be zero unless f itself is identically zero which is a trivial case.

In any case we have

$$f_{RMS} = \sqrt{< f^2 >} \tag{4.62}$$

and it is time to do an example.

Example *The RMS value of the voltage* $V(t) = V_0 \sin(\omega t)$ *over its period* T

First we note from our previous work above that the period $T = 2\pi/\omega$ and then we calculate

$$< V^2 > \tag{4.63}$$

This is given by

$$\begin{aligned}
< V^2 > &= \frac{1}{T}\int_0^T V^2(t)\,dt \\
&= \frac{1}{T}\int_0^T V_0^2 \sin^2(\omega t)\,dt \\
&= \frac{1}{T}\int_0^T V_0^2 \frac{1}{2}\left(1+\cos(2\omega t)\right)\,dt, \qquad \text{using } \sin^2(t) = \frac{1}{2}\left(1+\cos(2t)\right) \\
&= \frac{1}{T}\frac{V_0^2}{2}\left[t + \frac{1}{2\omega}\sin(2\omega t)\right]_0^T \\
&= \frac{1}{T}\frac{V_0^2}{2}\left[T + \frac{1}{2\omega}\sin(2\omega T) - 0\right] \\
&= \frac{\omega}{2\pi}\frac{V_0^2}{2}\left[\frac{2\pi}{\omega} + \frac{1}{2\omega}\sin(4\pi)\right] \\
&= \frac{\omega}{2\pi}\frac{V_0^2}{2}\left[\frac{2\pi}{\omega} + 0\right] \\
&= \frac{V_0^2}{2}
\end{aligned} \tag{4.64}$$

So $V_{RMS} = \sqrt{< V^2 >}$ is given by

$$V_{RMS} = \frac{V_0}{\sqrt{2}} \tag{4.65}$$

This is a well known result in electrical engineering.

The reader should note that, since $V(t) = V_0 \sin(\omega t)$, then the *largest value* that $V(t)$ can have is when $\sin(\omega t)$ has its largest value which is unity; this in turn means that the maximum value of $V(t)$ is V_0 and V_0 is often called the *peak voltage*. So it is both common and useful to think of V_{RMS} as being $1/\sqrt{2}$ times the peak voltage; it is worth memorising the approximate value of $1/\sqrt{2}$ so we quote it here

$$\frac{1}{\sqrt{2}} = 0.707\ldots \tag{4.66}$$

We see that V_{RMS} is about 70% of the peak voltage V_0. If a country uses an AC system of voltage supply the voltage usually quoted is the RMS value—230 volts for Ireland.

§ 4. Numerical Integration

We have seen already that many integrals are hard to do in closed from and so have to be done numerically. there are many techniques for doing this. Two simple ones which are often introduced at the beginning of a discussion on numerical integration are the *trapezoidal rule* and *Simpson's rule*; the latter being superior to the former. We shall have a very brief look at Simpson's rule.

§§ 4.1 Simpson's rule

Suppose we want to evaluate

$$\int_a^b f(x)\,dx \tag{4.67}$$

then we divide the interval $[a, b]$ up into

$$2n \tag{4.68}$$

intervals so that each has size

$$h = \frac{b-a}{2n} \tag{4.69}$$

Now we select $2n+1$ values of x in $[a, b]$ given by $x_j = a + jh$ for $j = 0, \ldots 2n$, i.e.

$$\begin{aligned} x_0 &= a \\ x_1 &= a + h \\ x_2 &= a + 2h \\ \vdots \quad &\quad \vdots \\ x_{2n-1} &= x + 2(n-1)h \\ x_{2n} &= b \end{aligned} \tag{4.70}$$

With this accomplished *Simpson's rule* gives the following *approximate* value for the integral

$$\int_a^b f(x)\,dx \sim \frac{b-a}{6n}\{f(a) + 2(f(x_1) + f(x_3) + \cdots + f(x_{2n-1})) + 4(f(x_2) + f(x_4) + \cdots + f(x_{2n-2})) + f(b)\} \tag{4.71}$$

and the *error* is proportional to

$$\frac{1}{n^4} \tag{4.72}$$

We simply quote this formula 4.71 as we have no space left to give its (quite straightforward) derivation. We finish by using Simpson's rule in an example.

Example *The integral*

$$\int_0^2 \sqrt{x}\sin(x)\,dx \tag{4.73}$$

First if we use a computer package such as *Maple*, *Mathematica* or *Matlab* then they have very good in built numerical integration routines and using Maple we can ask for, say, 10 significant figures of accuracy and then we obtain the result that

$$\int_0^2 \sqrt{x}\sin(x)\,dx = 1.235911456 \tag{4.74}$$

Now if we use Simpson's rule, and use Maple to evaluate 4.71 for $n = 10, 20, 30, 40, 50, 60$ and 70, we obtain the set of values

$$\begin{array}{c} 1.235915368 \\ 1.235911716 \\ 1.235911508 \\ 1.235911474 \\ 1.235911468 \\ 1.235911459 \\ 1.235911456 \end{array} \tag{4.75}$$

and the reader can see the sort of accuracy of Simpson's rule. We obtain the correct answer to 10 significant figures with $n = 70$. It is now easy to treat many more examples and we leave this task to the reader and bring these lectures to a close here.

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