Solutions to 2018-19 EE106 Exam

a

P.1

(a) The i^{th} term of this series (where i goes from 1 to ∞) is

$S_i = \frac{x^{2(i-1)}}{3^{i-1}}$.

Thus, $S_{i+1}/S_i = \frac{(x^{2i}/3^i)}{(x^{2(i-1)}/3^{i-1})} = \frac{x^2}{3}$. The ratio test says that if $\lim_{i\to\infty}\left|\frac{S_{i+1}}{S_i}\right| < 1$, then the series will converge. Here,

$\lim_{i\to\infty}\left|\frac{S_{i+1}}{S_i}\right| = \lim_{i\to\infty}\left|\frac{x^2}{3}\right| = \frac{|x|^2}{3}$

so $\frac{|x|^2}{3} < 1 \Rightarrow \boxed{|x| < \sqrt{3}}$ as desired. [5 marks]

(b) $\frac{x^2-2x}{2-\cos(x)}$ is defined and continuous at $x=\pi$, so the limit is obtained by substituting π in for x:

$$\lim_{x\to\pi}\left(\frac{x^2-2x}{2-\cos(x)}\right) = \frac{\pi^2-2\pi}{2-\cos(\pi)} = \boxed{\frac{\pi^2-2\pi}{3}}$$ [5 marks]

(c) The derivative of a function $f(x)$, denoted $\frac{df}{dx}(x)$ or $f'(x)$, is

$$\boxed{f'(x) = \frac{df}{dx}(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)}$$ [5 marks]

If $f(x) = -x^2+10x+3$, then

$f(x+h) = -(x+h)^2+10(x+h)+3 = -(x^2+2xh+h^2)+10(x+h)+3$

$= -x^2+10x+3-2xh+10h-h^2$

so $f(x+h)-f(x) = -2xh+10h-h^2 = h(-2x+10+h)$.

Thus,

$f'(x) = \lim_{h\to 0}\frac{h(-2x+10+h)}{h} = \lim_{h\to 0}(-2x+10+h) = \boxed{-2x+10}$ [5 marks]

(d) The critical points are all points a where $f'(x)=0$. So first we need $f'(x)$:

$f'(x) = \frac{d}{dx}\left(x^2(4-x^2)\right) = \frac{d}{dx}(4x^2-x^4) = 8x-4x^3 = 4x(2-x^2)$

$f'(a)=0$ gives $4a(2-a^2)=0$. Thus, either $4a=0$ or $2-a^2=0$.

Thus, the three critical points are $\boxed{0, \sqrt{2} \text{ and } -\sqrt{2}}$ [4 marks]

To find if each of these is a max, a min or undetermined, we need $f''(x)$.

This is

$f''(x) = \frac{d}{dx}(8x-4x^3) = 8-12x^2$

At $x=0$, $f''(0)=8>0$, so this is a $\boxed{\text{minimum.}}$ [2 marks]

At $x=\sqrt{2}$, $f''(\sqrt{2}) = 8-12\cdot 2 = -16 < 0$, so this is a $\boxed{\text{maximum}}$ [2 marks]

At $x=-\sqrt{2}$, $f''(-\sqrt{2}) = 8-12\cdot 2 = -16 < 0$, so this is also a $\boxed{\text{maximum}}$ [2 marks]

(e) We know that any DE of the form $\frac{dy}{dx} = \alpha y$ for a constant α has the solution $y(x) = Ce^{\alpha x}$, where C is any constant. Thus, $\boxed{y(x) = Ce^{9x}}$ is a soln. (But we just want a soln, so if an answer like $-e^{9x}$, $5e^{9x}$ or $\frac{11}{17}e^{9x}$ is given, that's fine.) [5 marks]

(f) The area under this curve is $\int_1^3 f(x)\,dx$, so

$$\int_1^3 f(x)\,dx = \int_1^3 \left(x^2 - \frac{8}{x^3}\right) dx = \left[\frac{1}{3}x^3 + \frac{4}{x^2}\right]_1^3 = (9 + 4/9) - (1/3 + 4)$$

$$= \boxed{46/9}$$ [5 marks]

(g) This is most easily done via substitution: let $u = x^2 + 36$. Then $\frac{dy}{dx} = 2x$, so $du = \frac{du}{dx}dx = 2x\,dx$. Thus,

$$\int \frac{2x}{x^2+36}\,dx = \int \frac{1}{u}\,du = \ln(u) + C$$

where C is an arbitrary constant. Putting in for u gives the answer:

$$\boxed{\int \frac{2x}{x^2+36}\,dx = \ln(x^2+36) + C}$$ [10 marks]

P.2

(a) l'Hôpital's rule states that the limit of $\frac{f(x)}{g(x)}$ as $x \to a$ in either the case that f and g both go to zero as $x \to a$, or f and g both diverge as $x \to a$, is the same as $\lim_{x \to a} \frac{f'(x)}{g'(x)}$. [5 marks]

$\frac{x}{\sinh(x)\cosh(3x)}$ has the form $f(x)/g(x)$ where $f(x) = x$ and $g(x) = \sinh(x)\cosh(3x)$. $\lim_{x \to 0} f(x) = \lim_{x \to 0} \sinh(x)\cosh(3x) = 0$, so l'Hôpital's rule applies here. $f'(x) = 1$ and $g'(x) = \cosh(x)\cosh(3x) + 3\sinh(x)\sinh(3x)$, so $\frac{f'(x)}{g'(x)} = \frac{1}{\cosh(x)\cosh(3x) + 3\sinh(x)\sinh(3x)}$

As $x \to 0$, $\cosh(x)\cosh(3x) \to 1$ and $\sinh(x)\sinh(3x) \to 0$, so the limit exists and is 1. Thus,

$$\boxed{\lim_{x \to 0}\left(\frac{x}{\sinh(x)\cosh(3x)}\right) = 1}$$ [5 marks]

(b) Taylor's Theorem states that if a function $f(x)$ and all its derivatives exist in a region around a point a, then we can write $f(x)$ as a power series of the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

where $f^{(n)}(x)$ is the "n"th derivative of $f(x)$. [5 marks]

Here, $f(x) = \frac{1}{\sqrt{1-2x}}$, so we need to find few derivatives:

$$f^{(1)}(x) = \frac{d}{dx}\left((1-2x)^{-1/2}\right) = -\frac{1}{2}(1-2x)^{-3/2}(-2) = (1-2x)^{-3/2} = \frac{1}{(1-2x)^{3/2}}$$

$$f^{(2)}(x) = \frac{d}{dx}\left((1-2x)^{-3/2}\right) = -\frac{3}{2}(1-2x)^{-5/2}(-2) = \frac{3}{(1-2x)^{5/2}}$$

$$f^{(3)}(x) = \frac{d}{dx}\left(3(1-2x)^{-5/2}\right) = 3\cdot\left(-\frac{5}{2}\right)(1-2x)^{-7/2}(-2) = \frac{15}{(1-2x)^{7/2}}$$

and so on. At $x=0$, each of these is nonzero:

$$f(0)=1,\quad f^{(1)}(0)=1,\quad f^{(2)}(0)=3,\quad f^{(3)}(0)=15$$

so the first four nonzero terms in $\frac{1}{\sqrt{1-2x}}$ are the $n=0$, $n=1$, $n=2$ and $n=3$ terms:

$$\frac{1}{\sqrt{1-2x}} = \frac{f^{(0)}(0)}{0!}(x-0)^0 + \frac{f^{(1)}(0)}{1!}(x-0)^1 + \frac{f^{(2)}(0)}{2!}(x-0)^2 + \frac{f^{(3)}(0)}{3!}(x-0)^4 + \dots$$

$$= 1 + x + \frac{3}{2}x^2 + \frac{15}{6}x^3 + \dots$$

$$\boxed{= 1 + x + \frac{3}{2}x^2 + \frac{5}{2}x^3 + \dots}$$

[10 marks]

P.3

(a) We know that the DE $\frac{d^2y}{dx^2} + ky = 0$, if k is positive, has the solutions $\cos(\sqrt{k}x)$ and $\sin(\sqrt{k}x)$. Here, $k=144$, so $\sqrt{k}=12$, giving $\boxed{\cos(12x) \text{ and } \sin(12x)}$ as two independent solns. [5 marks each]

(b) As in P.1e, we know $\frac{dN}{dt} = -kN$ has a soln $N(t) = Ce^{-kt}$ for any constant C. Here, however, we also know that the initial number of ^{96}Tc atoms is N_0, i.e. $N(0)=N_0$. Since $N(0)=C$, this gives $\boxed{N(t)=N_0e^{-kt}}$ [10 marks]

The half-life $t_{1/2}$ is the time it takes for half of the initial number of atoms to decay, i.e. that t such that $N(t) = \frac{1}{2}N(0) = \frac{1}{2}N_0$. Thus, $N(t_{1/2}) = N_0e^{-kt_{1/2}} = \frac{1}{2}N_0$. Solving for $t_{1/2}$ gives

$$t_{1/2} = \frac{\ln(2)}{k} \approx \frac{0.6932}{58.8\,y^{-1}} = 1.179\times10^{-2}\,y$$

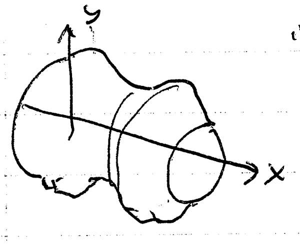
$$= (1.179\times10^{-2})(365\,d) = \boxed{4.3\,d}$$

[5 marks]

so half the ^{96}Tc will be gone in about four-and-a-half days.

P. 4

(a) If a curve is given by $y=f(x)$, then the volume of revolution on the interval $[a,b)$ is



$$V = \int_a^b \pi [f(x)]^2 \, dx$$

Thus, if $f(x) = 4\sqrt{\frac{x}{x^2+1}}$, then the volume between $x=0$ and $x=1$ is

$$V = \int_0^1 \pi \left[4\sqrt{\frac{x}{x^2+1}}\right]^2 dx = \int_0^1 16\pi \frac{x}{x^2+1} \, dx$$

Using $u = x^2+1$, we see $du = \frac{du}{dx}dx = 2x\,dx$, so $x\,dx = \frac{1}{2}du$.

Thus, since $u=1$ at $x=0$ and $u=2$ at $x=1$,

$$V = \int_1^2 16\pi \frac{(\frac{1}{2}du)}{(u)} = 8\pi \int_1^2 \frac{1}{u} du = 8\pi \ln(u) \Big|_1^2$$

$$= 8\pi [\ln(2) - \ln(1)] = \boxed{8\pi \ln(2)}$$ [10 marks]

(or 17.42, if you like numbers.)

(b) The integration by parts technique says that if you have an integral of the form $\int u\,dv$ for some functions $u(x)$ and $v(x)$, then this is equivalent to $uv - \int v\,du$. [5 marks]

Here we have $\int x \sinh(2x)\,dx$. Let's pick $u = x$ and $dv = \sinh(2x)\,dx$; thus, v is whatever function has derivative $\sinh(2x)$, i.e. $\int \sinh(2x)\,dx$. This is $\frac{1}{2}\cosh(2x)$, so $v = \frac{1}{2}\cosh(2x)$. $du = \frac{du}{dx}dx = dx$ here, so

$$\int x\sinh(2x)\,dx = (x)(\tfrac{1}{2}\cosh(2x)) - \int (\tfrac{1}{2}\cosh(2x))(dx)$$

$$= \tfrac{1}{2}x\cosh(2x) - \tfrac{1}{2}\int \cosh(2x)\,dx$$

The integral of $\cosh(2x)$ is $\frac{1}{2}\sinh(2x) + C$, where C is an arbitrary constant, so we have the final result

$$\boxed{\int x\sinh(2x)\,dx = \tfrac{1}{2}x\cosh(2x) - \tfrac{1}{4}\sinh(2x) - \tfrac{1}{2}C}$$ [10 marks]