CHAPTER III
Integration

§ 1. Integrals and areas under curves

INTEGRATION, in its most limited sense, is just the reverse of differentiation. However integration is also a way of calculating areas and when this understood there is seen to be a much deeper link between the two operations—much deeper, that is, than the elementary observation that one is the reverse of the other. This point will be explained when we deal with what is called the *fundamental theorem of calculus*, cf. 3.5 below.

So, to begin with, we shall consider the problem of finding the area of region under a curve, cf. fig. 33.

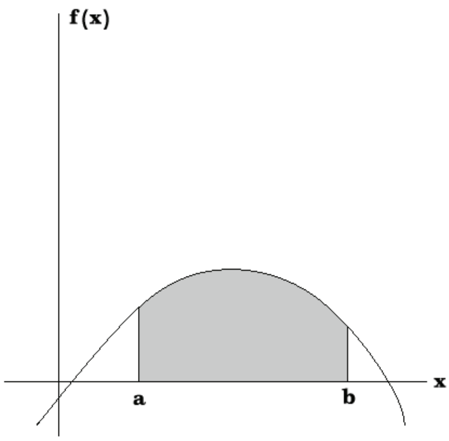

Fig. 33: The area under a curve

The shaded region in fig. 33 is the area that we wish to find: it lies between the vertical lines $x = a$ and $x = b$ and the notation used for this area is that it is denoted by

$$\int_a^b f(x)\,dx \tag{3.1}$$

and eq. 3.1 is referred to as "the integral of f from a to b".

What we do to calculate this area is to use rectangles as shown in fig. 34

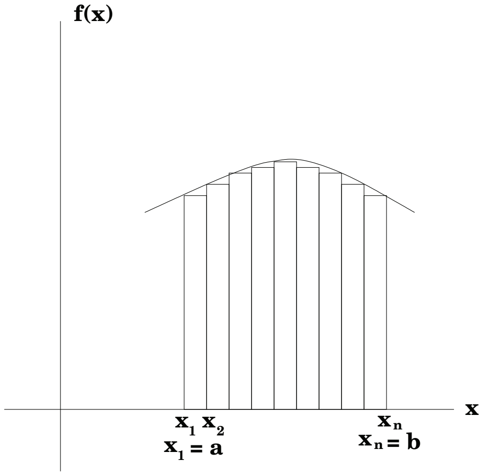

Fig. 34: Rectangles being used to calculate the area

Fig. 34 shows a series of $n-1$ rectangles whose collective area *approximates* the area between $x=a$ and $x=b$. The idea is that, as the number of these rectangles goes to infinity, the approximation becomes *exact*.

Now since fig. 34 shows that the first rectangle has base (x_2-x_1) and height $f(x_1)$ and the i^{th} rectangle has base $(x_{i+1}-x_i)$ and height $f(x_i)$ and so on. Hence the the area of all the rectangles is the sum

$$\sum_{i=1}^{n-1} f(x_i)(x_{i+1}-x_i) \tag{3.2}$$

Then to define the integral of f from a to b we take the limit as $n\to\infty$ giving

Definition (The integral $\int_a^b f(x)\,dx$)

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n-1} f(x_i)(x_{i+1}-x_i) \tag{3.3}$$

Having defined $\int_a^b f(x)\,dx$ the really important thing is the result one obtains when $f(x)$ is already the derivative of some other function, say

$$f(x) = \frac{dF(x)}{dx}, \qquad \text{for some function } F(x) \tag{3.4}$$

This result is what is given in the fundamental theorem of calculus which we now quote.

Theorem (The fundamental theorem of calculus) *If a function of the form dF/dx is integrated then its integral is given by the formula*

$$\int_a^b \frac{dF(x)}{dx}\,dx = F(b) - F(a) \tag{3.5}$$

We shall sketch a proof of this theorem but first a piece of notation: the quantity $F(b) - F(a)$ is often denoted by $[F(x)]_a^b$ that is

$$[F(x)]_a^b = F(b) - F(a) \tag{3.6}$$

where $F(x)$ is any function. Now for the sketch of the proof.

Proof: First we adjust the bases of the rectangles in fig. 34 to have the same size—which we denote by Δx—that is we have

$$(x_1 - x_2) = (x_2 - x_3) = \cdots = (x_n - x_{n-1}) = \Delta x \tag{3.7}$$

Using this this sum in $\sum_{i=1}^{n-1} f(x_i)(x_{i+1} - x_i)$ of the integral definition 3.3 simplifies: we find that

$$\sum_{i=1}^{n-1} f(x_i)(x_{i+1} - x_i) = \sum_{i=1}^{n-1} f(x_i)\Delta x \tag{3.8}$$

But since $f = dF/dx$ we must now consider the sum

$$\sum_{i=1}^{n-1} \frac{dF(x_i)}{dx}\Delta x \tag{3.9}$$

Now the definition of the derivative is $dF(x_i)/dx$ is

$$\begin{aligned}\frac{dF(x_i)}{dx} &= \lim_{h\to 0}\frac{F(x_i + h) - F(x_i)}{h}\\ &= \lim_{\Delta x\to 0}\frac{F(x_i + \Delta x) - F(x_i)}{\Delta x}, \quad \text{on setting } h = \Delta x\end{aligned} \tag{3.10}$$

But

$$\begin{aligned}\Delta x &= (x_{i+1} - x_i)\\ \Rightarrow x_i + \Delta x &= x_i + x_{i+1} - x_i\\ &= x_{i+1}\end{aligned} \tag{3.11}$$

and using this fact in 3.10 we obtain

$$\frac{dF(x_i)}{dx} = \lim_{\Delta x \to 0} \frac{F(x_{i+1}) - F(x_i)}{\Delta x} \tag{3.12}$$

Putting together 3.3, 3.11 and 3.12 we have

$$\begin{aligned} \int_a^b \frac{dF(x)}{dx}\, dx &= \lim_{n\to\infty} \sum_{i=1}^{n-1} \lim_{\Delta x \to 0} \frac{F(x_{i+1}) - F(x_i)}{\Delta x} \Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n-1} (F(x_{i+1}) - F(x_i)), \quad \text{since } \Delta x \text{ cancels} \end{aligned} \tag{3.13}$$

But if we write out the sum $\sum_{i=1}^{n-1}(F(x_{i+1}) - F(x_i))$ we saw that all, except for two terms, cancel for we have

$$\begin{aligned} \sum_{i=1}^{n-1}(F(x_{i+1}) - F(x_i)) = (F(x_n) &- F(x_{n-1})) \\ &+ (F(x_{n-1}) - F(x_{n-2})) \\ &+ (F(x_{n-2}) - F(x_{n-3})) \\ &\quad \vdots \qquad\qquad \vdots \\ &+ (F(x_3) - F(x_2)) \\ &+ (F(x_2) - F(x_1)) \end{aligned} \tag{3.14}$$

If we now look at 3.14 we see that all the terms on the RHS cancel except for the first and the last and so

$$\begin{aligned} \sum_{i=1}^{n-1}(F(x_{i+1}) - F(x_i)) &= F(x_n) - F(x_1) \\ &= F(b) - Fa), \quad \text{recalling from fig. 34 that } \begin{cases} x_n = b \\ x_1 = a \end{cases} \end{aligned} \tag{3.15}$$

So putting this result back into 3.12 we see that we have achieved an informal proof of the *fundamental theorem of calculus* which asserts—cf. 3.5—that

$$\int_a^b \frac{dF(x)}{dx}\, dx = F(b) - F(a) \tag{3.16}$$

§§ 1.1 Some notation and terminology

Some perfectly straightforward terminology can now be given: If we integrate any function f from a to b—that is calculate

$$\int_a^b f(x)\, dx \tag{3.17}$$

then the numbers a and b are called the *limits of integration.* One sometimes omits the limits of integration and writes

$$\int f(x)\,dx \tag{3.18}$$

The notation $\int f(x)\,dx$ simply denotes a function F whose derivative is f; hence we have

$$\int f(x)\,dx = F(x), \qquad \text{means that} \qquad \frac{dF(x)}{dx} = f(x) \tag{3.19}$$

F is then called the *integral*[1] *of* f.

For example if $f(x) = x^3$ then we could write

$$\int x^3\,dx = \frac{x^4}{4}, \qquad \text{because} \qquad \frac{d}{dx}\left(\frac{x^4}{4}\right) = x^3 \tag{3.20}$$

However notice that, if C is any constant, we could also write

$$\int x^3\,dx = \frac{x^4}{4} + C, \qquad \text{because} \qquad \frac{d}{dx}\left(\frac{x^4}{4} + C\right) = x^3 \tag{3.21}$$

Such a constant C is called a *constant of integration.*

This example shows that the integral F of any function is not unique since one can add a constant of integration to F; more concisely one can say: if F is an integral of f then so is $F + C$ for any constant C.

Finally to distinguish integrals with and without limits one has the following terminology. An integral *with limits* such as

$$\int_a^b f(x)\,dx \tag{3.22}$$

is called a *definite integral*, while an integral *without limits* like

$$\int f(x)\,dx \tag{3.23}$$

is called an *indefinite integral.*

§§ **1.2 Some examples**

[1] Alternative terms are often used for F. An integral of F is sometimes called a *primitive of* F or an *anti-derivative of* F—we shall use neither term since the word *integral* is the more common usage.

Example *The area under the parabola $f(x) = x^2$ between $x = 3$ and $x = 9$.*
With the fundamental theorem of calculus under our belt it is a simple matter to calculate the area shown in the fig. 35.

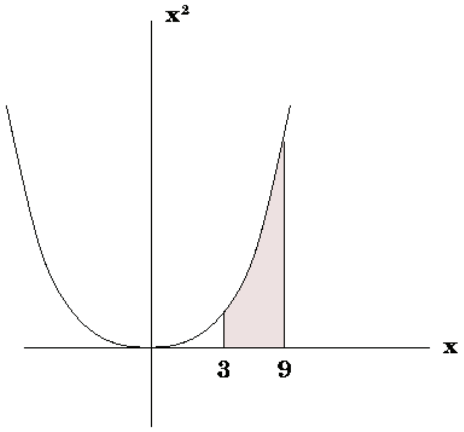

Fig. 35: The area under the parabola between $x = 3$ and $x = 9$

The area A that we want is given by

$$A = \int_3^9 x^2\, dx \tag{3.24}$$

and so we can straightaway compute that

$$\begin{aligned}\int_3^9 x^2\, dx &= \left[\frac{x^3}{3}\right]_3^9 \\ &= \left[\frac{9^3}{3} - \frac{3^3}{3}\right] \\ &= \left[\frac{729}{3} - \frac{27}{3}\right] \\ &= 234\end{aligned} \tag{3.25}$$

and so we have our area A. That example was a definite integral so next we consider an indefinite integral.
Example *The integral*

$$\int \sin^2(x)\cos(x)\, dx \tag{3.26}$$

So this time we just want a function F that satisfies

$$\frac{dF}{dx} = \sin^2(x)\cos(x) \tag{3.27}$$

After some tinkering with various combinations of trigonometric functions we should be able to find that a solution is

$$F(x) = \frac{\sin^3(x)}{3} \tag{3.28}$$

which certainly does the trick. Thus we can write

$$\int \sin^2(x)\cos(x)\,dx = \frac{\sin^3(x)}{3} \tag{3.29}$$

and if we want to be really precise we add on a constant of integration and write the *most general* statement of the result which is

$$\int \sin^2(x)\cos(x)\,dx = \frac{\sin^3(x)}{3} + C, \quad \text{for any constant } C \tag{3.30}$$

The next example illustrates a very important point. It is that if the area being calculated *lies under* the x-axis, rather than above it, then the area A calculated by the integral will be *negative.*

Example *Areas under the x-axis count negatively in integrals*

$$\int_{\pi/2}^{3\pi/2} \cos(x)\,dx \tag{3.31}$$

Fig. 36 shows the graph of $\cos(x)$ between $x = \pi/2$ and $3\pi/2$; this can be seen to be an interval on which $\cos(x)$ is *negative.*

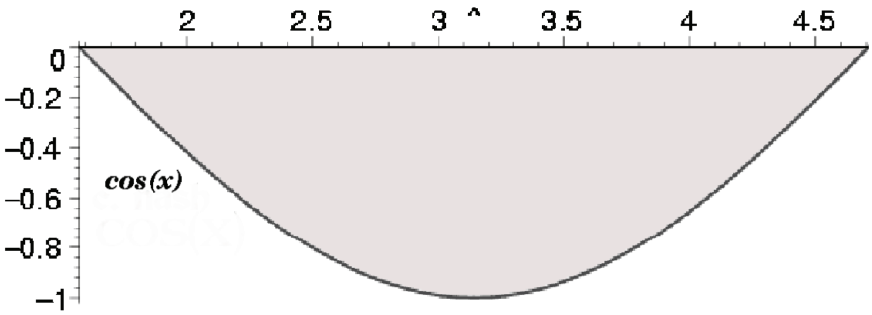

Fig. 36: The graph of $\cos(x)$ for the interval $[\pi/2, 3\pi/2]$

The integral we want is $\int_{\pi/2}^{3\pi/2} \cos(x)\,dx$ and we compute that

$$\begin{aligned}\int_{\pi/2}^{3\pi/2} \cos(x)\,dx &= [\sin(x)]_{\pi/2}^{3\pi/2} \\ &= [\sin(3\pi/2) - (\sin(\pi/2))] \\ &= [-1-1], \quad \text{since } \begin{cases}\sin(\pi/2) = 1 \\ \sin(3\pi/2) = -1\end{cases} \\ &= -2\end{aligned} \tag{3.32}$$

Hence the integral has the value

$$-2 \tag{3.33}$$

and is negative as claimed. The *actual shaded area* shown in fig. 36 is of course

$$+2 \tag{3.34}$$

One should not be disturbed by this: all that is happening is that an integral $\int_a^b f(x)\,dx$ can have both *negative and positive* contributions depending on whether the function f is *negative or positive* for a given part of the integration interval $[a, b]$.

This can lead to an integral being zero because the positive and negative contributions exactly cancel. This happens in the next example.

Example *The integral*

$$\int_0^{2\pi} \sin(x)\,dx \tag{3.35}$$

First let's get the computation of the integral out of the way. We find that

$$\begin{aligned}\int_0^{2\pi} \sin(x)\,dx &= -\left[\cos(x)\right]_0^{2\pi} \\ &= -\left[\cos(2\pi) - \cos(0)\right] \\ &= -\left[1 - 1\right], \quad \text{since } \begin{cases}\cos(2\pi) = 1 \\ \cos(0) = 1\end{cases} \\ &= 0\end{aligned} \tag{3.36}$$

So the integral does indeed vanish. Now if we look at fig. 37 we see two shaded areas of differing densities and it is clear that what has happened is that these two areas have simply cancelled.

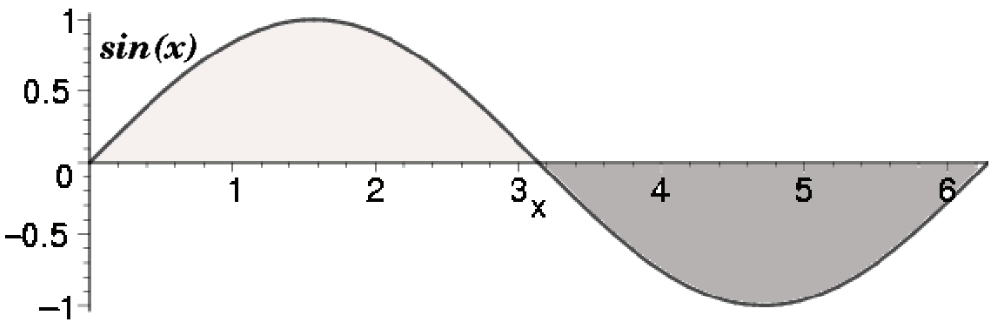

Fig. 37: The two cancelling areas for $\sin(x)$ on the interval $[0, 2\pi]$

If one really wants to prove this then one can calculate the two areas using separate integrals. In other words one calculates the two integrals

$$\int_0^{\pi} \sin(x)\, dx, \quad \int_{\pi}^{2\pi} \sin(x)\, dx \tag{3.37}$$

One then readily verifies that the first integral is positive and equal to

$$+2 \tag{3.38}$$

while the second is equal to

$$-2 \tag{3.39}$$

A useful fact about a definite integral such as

$$\int_a^c f(x)\, dx \tag{3.40}$$

is that one can choose a number b between a and c and split the integral up into two pieces. What one obtains is just

$$\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx, \quad a < b < c \tag{3.41}$$

All that we are doing is observing that the area given by $\int_a^c f(x)\, dx$ is the sum of the two smaller areas $\int_a^b f(x)\, dx$ and $\int_b^c f(x)\, dx$.

It is also useful to observe that if one *interchanges* a *and* b in the integral $\int_a^b f(x)\, dx$ then the integral *changes sign*. In other words we have

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx \tag{3.42}$$

This fact is evident already in the statement 3.5 of the fundamental theorem of calculus where we see at once that the RHS changes sign if a and b are interchanged.

§ 2. Difficult integrals...?

Unlike derivatives, integrals can often be impossible to compute in terms of well known functions[2] and then one has to resort to numerical methods to deal with them.

[2] This may seem a slightly vague statement since the phrase "well known functions" is certainly not precise. However mathematicians do have a more precise set of functions in mind. This sets consists of all functions that can be obtained by addition, multiplication, division and composition of: polynomials, trigonometric functions and their inverses, and the functions ln and exp. These functions are sometimes then referred to as the *elementary functions*.

For example, there is no elementary function F which satisfies

$$\frac{dF(x)}{dx} = e^{-x^2} \tag{3.43}$$

In everyday language we say *we can't do the integral*

$$\int e^{-x^2}\,dx \tag{3.44}$$

Another example of an integral *"we can't do"* is

$$\int \sqrt{x}\sin(x)\,dx \tag{3.45}$$

Despite this we *can do* the very similar integral

$$\int x\sin(x)\,dx \tag{3.46}$$

for differentiation readily verifies the correctness of the statement that

$$\int x\sin(x)\,dx = \sin(x) - x\cos(x) \tag{3.47}$$

Thus a small change in the function to be integrated can make a big difference.

§ 3. Some well known integrals

It is always useful to have a list of integrals for functions that one comes across often. For this reason we provide a short table of integrals on p. 86.

§ 4. Integration techniques

There are two main techniques for doing integrals and these are
(a) Integration by change of variables or substitution.
(b) Integration by parts. We shall study both of these methods beginning with (a).

§§ 4.1 Integration by change of variable or integration by substitution

Next we illustrate the technique of integration by *change of variable* (also called integration by *substitution*). Consider then the following example.

Example *The integral*

$$\int_0^{\pi/2} \sin^5(x)\cos(x)\,dx \tag{3.48}$$

$f(x)$	$\int f(x)\,dx$
$x^n,\ (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln(x)$
e^x	e^x
a^x	$\frac{a^x}{\ln(a)}$
$\ln(x)$	$x\ln(x) - x$
$\sin(x)$	$-\cos(x)$
$\cos(x)$	$\sin(x)$
$\tan(x)$	$-\ln(\cos(x))$
$\sec(x)$	$\ln(\sec(x) + \tan(x))$
$\mathrm{cosec}(x)$	$-\ln(\mathrm{cosec}(x) + \cot(x))$
$\cot(x)$	$\ln(\sin(x))$
$\sec^2(x)$	$\tan(x)$
$\sec(x)\tan(x)$	$\sec(x)$
$\sinh(x)$	$\cosh(x)$
$\cosh(x)$	$\sinh(x)$
$\tanh(x)$	$\ln(\cosh(x))$
$\frac{1}{1+x^2}$	$\arctan(x)$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x)$
$-\frac{1}{\sqrt{1-x^2}}$	$\arccos(x)$

A table of useful integrals

We change variable from x to u where

$$u = \sin(x) \tag{3.49}$$

So that we have

$$\begin{aligned} \frac{du}{dx} &= \cos(x) \\ \Rightarrow du &= \cos(x)dx \\ \Rightarrow dx &= \frac{du}{\cos(x)} \end{aligned} \tag{3.50}$$

So now, if we *temporarily* omit the limits of integration, we can substitute this information into the integral giving us the equation

$$\begin{aligned} \int \sin^5(x)\cos(x)\,dx &= \int u^5 \cos(x)\frac{du}{\cos(x)} \\ &= \int u^5 du \end{aligned} \tag{3.51}$$

Finally to restore the limit of integration we notice that, since $u = \sin(x)$, we have

$$\begin{aligned} x = 0 &\Rightarrow u = 0 \\ x = \frac{\pi}{2} &\Rightarrow u = \sin(\pi/2) = 1 \end{aligned} \tag{3.52}$$

Hence our limits in terms of u are 0 and 1 and, restoring the limits to both integrals, we have the equality

$$\begin{aligned} \int_0^{\pi/2} \sin^5(x)\cos(x)\,dx &= \int_0^1 u^5\,du \\ &= \left[\frac{u^6}{6}\right]_0^1 \\ &= \left[\frac{1}{6} - 0\right] \\ &= \frac{1}{6} \end{aligned} \tag{3.53}$$

and our integral has been completed. We move on.

Example *The integral*

$$\int \frac{dx}{x\ln(x)} \tag{3.54}$$

This time we set

$$\begin{aligned} u &= \ln(x) \\ \Rightarrow \frac{du}{dx} &= \frac{1}{x} \\ \Rightarrow dx &= x\,du \end{aligned} \tag{3.55}$$

Now we put this information into our integral and find that

$$\begin{aligned}\int \frac{dx}{x\ln(x)} &= \int \frac{x du}{xu} \\ &= \int \frac{du}{u} \\ &= \ln(u) \\ &= \ln(\ln(x))\end{aligned} \tag{3.56}$$

and so we have established that

$$\int \frac{dx}{x\ln(x)} = \ln(\ln(x)) \tag{3.57}$$

Time for our next integral which is

Example *The integral*

$$\int_0^a \sqrt{a^2 - x^2}\, dx \tag{3.58}$$

The trick here is to use a trigonometric substitution or change of variable; the one that works for this case [3] is

$$\begin{aligned} x &= a\sin(\theta) \\ \Rightarrow dx &= a\cos(\theta)\, d\theta \end{aligned} \tag{3.59}$$

We must also change the limits of integration to accommodate the new variable θ; to this end note that

$$\begin{aligned} x = 0 &\Rightarrow \theta = 0 \\ x = 1 &\Rightarrow \theta = \frac{\pi}{2} \end{aligned} \tag{3.60}$$

[3] For other closely related integrals such as

$$\int \left(a^2 \mp x^2\right)^{\mp 1/2} dx$$

one should also try a trigonometric substitution such as

$$x = a\sin(\theta),\ x = a\cos(\theta),\ x = a\tan(\theta)$$

$x = a\tan(\theta)$ being the one to use when $(a^2 + x^2)$, rather than $(a^2 - x^2)$, occurs in the integrand.

so the new limits are 0 and $\pi/2$ and we obtain

$$\begin{aligned}
\int_0^a \sqrt{a^2-x^2}\,dx &= \int_0^{\pi/2} \sqrt{a^2-a^2\sin(\theta)}\,a\cos(\theta)\,d\theta \\
&= \int_0^{\pi/2} \sqrt{a^2(1-\sin^2(\theta))}\,a\cos(\theta)\,d\theta \\
&= \int_0^{\pi/2} a\cos(\theta)a\cos(\theta)\,d\theta, \quad \text{using } (1-\sin^2(\theta)) = \cos^2(\theta) \\
&= a^2\int_0^{\pi/2} \cos^2(\theta)\,d\theta \\
&= a^2\int_0^{\pi/2} \frac{1+\cos(2\theta)}{2}\,d\theta, \quad \text{using } \cos^2(\theta) = \frac{1+\cos(2\theta)}{2} \\
&= a^2\left[\frac{\theta}{2}+\frac{\sin(2\theta)}{4}\right]_0^{\pi/2} = a^2\left[\frac{\pi}{4}+\frac{\sin(\pi)}{4}\right] = \frac{a^2\pi}{4}, \quad \text{since } \sin(\pi)=0
\end{aligned} \tag{3.61}$$

Now we consider

Example *The integral*

$$\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx \tag{3.62}$$

This time we shall try

$$\begin{aligned}
u &= \sqrt{e^x+1} \\
\Rightarrow du &= \frac{1}{2}\left(e^x+1\right)^{-1/2} e^x dx
\end{aligned} \tag{3.63}$$

Also

$$\begin{aligned}
u &= \sqrt{e^x+1} \\
\Rightarrow e^x &= u^2-1 \\
&\Rightarrow \begin{cases} e^x+1 = u^2 \\ e^{2x} = (u^2-1)^2 \\ dx = \dfrac{2u}{u^2-1}du \end{cases}
\end{aligned} \tag{3.64}$$

Our integral now displays the following transformation

$$\begin{aligned}
\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx &= \int \frac{(u^2-1)^2}{u}\frac{2u}{u^2-1}\,du \\
&= \int 2(u^2-1)\,du \\
&= \frac{2}{3}u^3-2u
\end{aligned} \tag{3.65}$$

But $u = \sqrt{e^x + 1}$ so we have deduced that

$$\int \frac{e^{2x}}{\sqrt{e^x + 1}}\, dx = \frac{2}{3}\left(e^x + 1\right)^{3/2} - 2\left(e^x + 1\right)^{1/2} \tag{3.66}$$

We are now ready for the other main integration technique which is integration by parts.

§§ 4.2 Integration by parts

All integrations by parts rest on a clever use of the same formula. This formula is very simply obtained: one just integrates the formula for the derivative of a product. More precisely we begin with the formula

$$\frac{d}{dx}\left(f(x)g(x)\right) = f(x)\frac{dg(x)}{dx} + \frac{df(x)}{dx}g(x) \tag{3.67}$$

which we rewrite as

$$f(x)\frac{dg(x)}{dx} = \frac{d}{dx}\left(f(x)g(x)\right) - \frac{df(x)}{dx}g(x) \tag{3.68}$$

and then we integrate both sides from a to b yielding

$$\int_a^b f(x)\frac{dg(x)}{dx}\, dx = \int_a^b \frac{d}{dx}\left(f(x)g(x)\right)\, dx - \int_a^b \frac{df(x)}{dx}g(x)\, dx$$

$$\Rightarrow \int_a^b f(x)\frac{dg(x)}{dx}\, dx = \left[f(x)g(x)\right]_a^b - \int_a^b \frac{df(x)}{dx}g(x)\, dx$$

Or more compactly

$$\int_a^b fg'\, dx = [fg]_a^b - \int_a^b f'g\, dx \tag{3.69}$$

This last formula is the one we want and we emphasise its importance by quoting it as a theorem.

Theorem (Integration by parts) *If f and g are two differentiable functions then*

$$\int_a^b fg'\, dx = [fg]_a^b - \int_a^b f'g\, dx$$

or

$$\int fg'\, dx = fg - \int f'g\, dx, \qquad \textit{without limits} \tag{3.70}$$

Now we need some examples of integration by parts in action. We begin with something simple.

Example *The integral*

$$\int xe^x \, dx \tag{3.71}$$

The method of integration by parts then consists of equating the integrand to fg' and then using 3.70. So we write

$$xe^x = fg' \tag{3.72}$$

and immediately we are faced with the task of deciding which part of xe^x should we equate to f and which to g'. The answer to this problem is that one proceeds partially by trial and error and partially by previous experience. This time we choose to set

$$f = x \tag{3.73}$$

which forces

$$g' = e^x \tag{3.74}$$

With this choice for f and g we use 3.70 giving us

$$\begin{aligned} \int xe^x \, dx &= xe^x - \int 1 \cdot e^x \, dx \\ &= xe^x - e^x \end{aligned} \tag{3.75}$$

and so we are finished.

A favourite trick in integration by parts is to set g' equal to 1 and then use the formula 3.70. This can be seen at work in our next calculation.

Example *The integral*

$$\int \ln(x) \, dx \tag{3.76}$$

We set

$$g' = 1 \tag{3.77}$$

We must therefore set $f = \ln(x)$ and 3.70 gives

$$\begin{aligned} \int \ln(x) \cdot 1 \, dx &= \ln(x) \cdot x - \int \frac{1}{x} \cdot x \, dx \\ &= x \ln(x) - x \end{aligned} \tag{3.78}$$

and so we have our integral and the result agrees, as it must, with that quoted in our integral table on p. 86.

A second popular trick is to try to express $\int f\,dx$ in terms of itself and solve the resulting formula. We illustrate this next.

Example *The integral*

$$\int \frac{\ln(x)}{x}\,dx \tag{3.79}$$

Setting $f = \ln(x)$, and thus $g' = 1/x$, we obtain the equation

$$\begin{aligned} \int \frac{\ln(x)}{x}\,dx &= \ln(x)\ln(x) - \int \frac{1}{x}\ln(x)\,dx \\ \Rightarrow 2\int \frac{\ln(x)}{x}\,dx &= \ln(x)\ln(x) \\ \text{or} \qquad \int \frac{\ln(x)}{x}\,dx &= \frac{\ln^2(x)}{2} \end{aligned} \tag{3.80}$$

and we note the appearance of $\int \ln(x)/x\,dx$ on both sides of the equation in the first line of 3.80. Another integral of this type is

Example *The integral*

$$\int \sin^2(x)\,dx \tag{3.81}$$

Choosing $f = \sin(x)$, and therefore $g' = \sin(x)$ also, gives

$$\begin{aligned} \int \sin^2(x)\,dx &= \sin(x)(-\cos(x)) - \int \cos(x)(-\cos(x))\,dx \\ &= -\sin(x)\cos(x) + \int \cos^2(x)\,dx \\ &= -\sin(x)\cos(x) + \int (1-\sin^2(x))\,dx, \quad \text{using } \cos^2(x) = 1-\sin^2(x) \\ \Rightarrow 2\int \sin^2(x)\,dx &= -\sin(x)\cos(x) + \int dx \\ \Rightarrow \int \sin^2(x)\,dx &= \frac{x - \sin(x)\cos(x)}{2} \end{aligned} \tag{3.82}$$

§ 5. A little more integration

We shall finish this chapter with a bit more integration practice.

Example *The integral*

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx \tag{3.83}$$

We use the substitution $u = \sqrt{x}$ and so find that

$$\begin{aligned} \frac{du}{dx} &= \frac{1}{2}x^{-1/2} \\ \Rightarrow dx &= 2\,x^{1/2}du \end{aligned} \tag{3.84}$$

Hence we obtain

$$\begin{aligned} \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx &= \int e^u \frac{2\,x^{1/2}du}{\sqrt{x}} \\ &= 2\int e^u\, du \\ &= 2e^u = 2e^{\sqrt{x}} \end{aligned} \tag{3.85}$$

and so we have shown that

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx = 2e^{\sqrt{x}} \tag{3.86}$$

Example *The integral*

$$\int x^2 e^x\, dx \tag{3.87}$$

We can do this by parts but we will apply the method twice to get to the end. Setting

$$g' = e^x \tag{3.88}$$

we get

$$\int x^2 e^x\, dx = x^2 e^x - \int 2xe^x\, dx \tag{3.89}$$

This leaves us with the integral $\int xe^x\, dx$ still to do; so for this integral we again set

$$g' = e^x \tag{3.90}$$

and obtain

$$\begin{aligned} \int xe^x\, dx &= xe^x - \int 1 \cdot e^x\, dx \\ &= xe^x - e^x \end{aligned} \tag{3.91}$$

Substituting 3.91 into 3.89 we find the result that

$$\int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x \tag{3.92}$$

Example *The integral*

$$\int \frac{e^x}{e^{2x} + 2e^x + 1} \, dx \tag{3.93}$$

For this integral note that the denominator of the integrand is a perfect square—that is

$$e^{2x} + 2e^x + 1 = (e^x + 1)^2 \tag{3.94}$$

This means that we can write

$$\int \frac{e^x}{e^{2x} + 2e^x + 1} \, dx = \int \frac{e^x}{(e^x + 1)^2} \, dx \tag{3.95}$$

and now if we set

$$u = e^x + 1 \tag{3.96}$$

we find that

$$\begin{aligned} \int \frac{e^x}{(e^x + 1)^2} \, dx &= \int \frac{e^x}{u^2} \frac{du}{e^x} \\ &= \int \frac{du}{u^2} \\ &= -\frac{1}{u} = -\frac{1}{e^x + 1} \end{aligned} \tag{3.97}$$

So we have deduced that

$$\int \frac{e^x}{e^{2x} + 2e^x + 1} \, dx = -\frac{1}{e^x + 1} \tag{3.98}$$

Example *The integral*

$$\int \sqrt{x} \ln(x) \, dx \tag{3.99}$$

For this example we use integration by parts and set

$$g' = \sqrt{x} \tag{3.100}$$

This yields the equation

$$\begin{aligned}\int \sqrt{x}\ln(x)\,dx &= \frac{2}{3}x^{3/2}\ln(x) - \int \frac{2}{3}x^{3/2}\frac{1}{x}\,dx \\ &= \frac{2}{3}x^{3/2}\ln(x) - \frac{2}{3}\int x^{1/2}\,dx \\ &= \frac{2}{3}x^{3/2}\ln(x) - \left(\frac{2}{3}\right)^2 x^{3/2}\end{aligned} \tag{3.101}$$

So we have found that

$$\int \sqrt{x}\ln(x)\,dx = \frac{2}{3}x^{3/2}\left(\ln(x) - \frac{2}{3}\right) \tag{3.102}$$