CHAPTER II
Differential calculus

§ 1. Derivatives

HISTORICALLY derivatives grew out of a desire to have a general way to draw or construct a tangent to any curve. If the curve was a circle, an ellipse, or a parabola tangents could easily be constructed using geometrical rules but none of these rules gave a hint to a general method. We shall see now that the key technical step to drawing a general tangent is to think of it as a *limit*.

A tangent to a curve at a point p is a straight line which just touches the curve at p cf. fig 12.

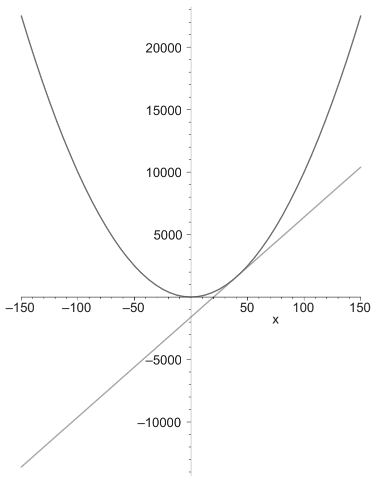


Fig. 12: The curve $f(x) = x^2 + 3$ and one of its tangents

Unfortunately we cannot *define* a tangent to a curve as a straight line which just touches it in one point. To see why this is so examine the two graphs shown in fig. 13. The first graph shows a straight line which is a tangent to a curve but the line touches the curve in two points only one of which—the rightmost one—is a tangential point. The second graph shows a parabola with two lines—one vertical and one slanted—passing through the same point on the parabola. However note that, though both these lines touch the parabola at just one point, only one of these lines—the slanted one—is a tangent.

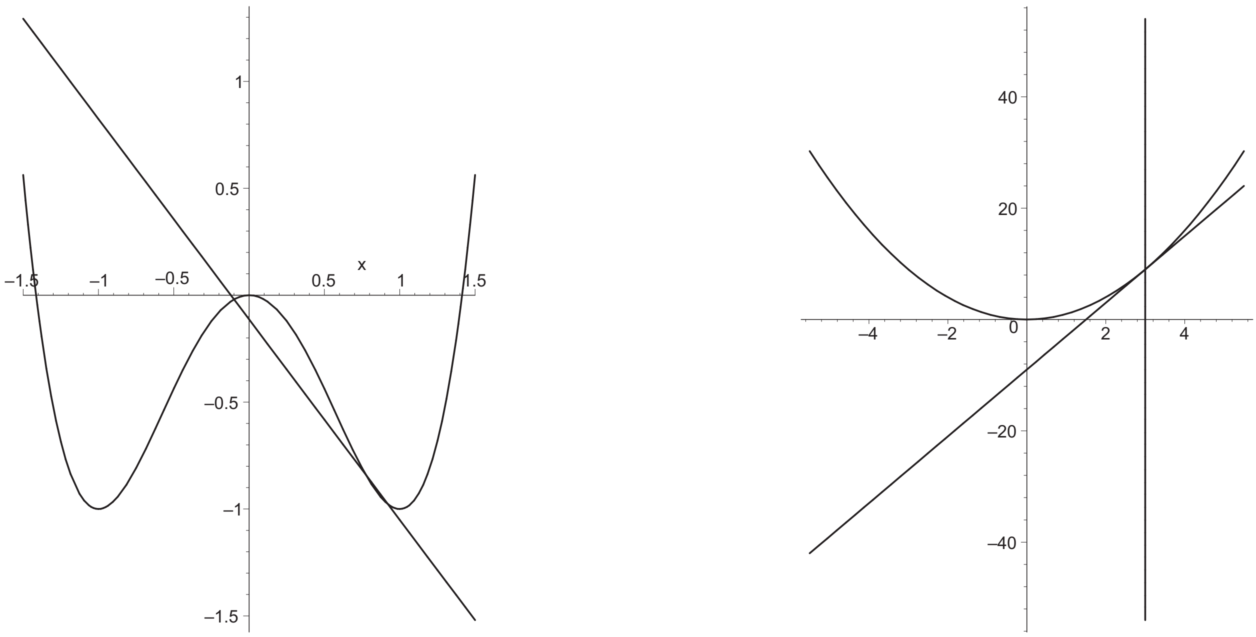


Fig. 13: The curves $f(x) = x^4 - 2x^2$ and $f(x) = x^2$ teaching a lesson about tangents

We now turn to the method of tangent construction that actually works; we shall see that this method will require us to use a limit.

Suppose then that we want to construct a tangent to the function $f(x)$ at the point x and then calculate its slope. We accomplish this task by doing just two things: First we draw a straight line through the two points with coordinates $(x, f(x))$ and $(x+h, f(x+h))$ as shown in fig. 14.

This line is not yet a tangent to the curve, but in any case, its slope is

$$\frac{f(x+h) - f(x)}{h} \tag{2.1}$$

Secondly we send $h \to 0$ which makes our line become the tangent to $f(x)$ at x; its slope is then given by the value of the expression 2.1 as $h \to 0$ i.e. by the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{2.2}$$

It is this quantity which is called the **derivative** of $f(x)$ at x—we record this symbolically by writing

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{2.3}$$

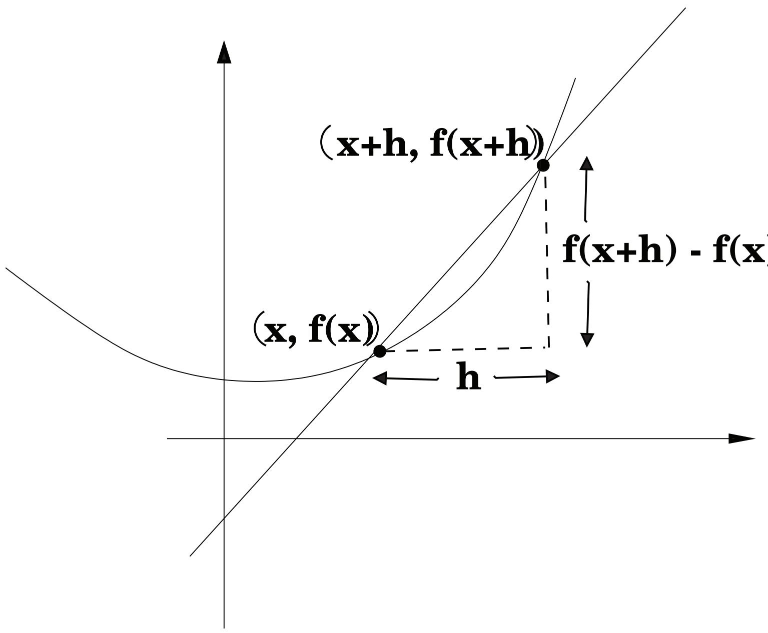


Fig. 14: The curve $(f(x)$ and the line through the points $(x, f(x))$ and $(x+h, f(x+h))$

The derivative is also often written as f' or $f'(x)$. To summarise the notation we have

$$f' = \frac{df}{dx} \tag{2.4}$$

We proceed at once to our first calculation of a derivative.

Example *The derivative of* x^n *for* $n \in \mathbf{N}$

So our task is to calculate[1]

$$\frac{df}{dx}, \quad \text{where} \quad f(x) = x^n,\ n \in \mathbf{N} \tag{2.5}$$

We have

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^n - x^n}{h} \\ &= \frac{\left(x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n-1}xh^{n-1} + h^n\right) - x^n}{h} \\ &= \binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + \binom{n}{n-1}xh^{n-2} + h^{n-1} \end{aligned} \tag{2.6}$$

[1] We shall need the binomial theorem for this calculation so we remind the forgetful that it says that

$$(x+y)^m = x^m + \binom{m}{1}x^{m-1}y + \binom{m}{2}x^{m-2}y^2 + \cdots + \binom{m}{m-1}xy^{m-1} + y^m$$

But now the limit is easy and we find that

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\left\{\binom{n}{1}x^{n-1}+\binom{n}{2}x^{n-2}h+\cdots+\binom{n}{n-1}xh^{n-2}+h^{n-1}\right\}\\ &= \binom{n}{1}x^{n-1}\\ &= nx^{n-1}, \quad \text{since } \binom{n}{1}=n\end{aligned} \tag{2.7}$$

Hence our result has the following simple form

$$\frac{dx^n}{dx} = nx^{n-1},\ n\in\mathbf{N} \tag{2.8}$$

We can even allow the n in x^n to have the value 0 because it is a simple matter to check that [2]

$$\frac{dx^0}{dx} \equiv \frac{d1}{dx} = 0 \tag{2.10}$$

Still more is true: the n in x^n can have *any* real value: positive, negative, rational or irrational [3]; so we have

$$\frac{dx^n}{dx} = nx^{n-1},\ n\in\mathbf{R} \tag{2.11}$$

It is easy to check that derivatives have the following useful properties

$$\begin{aligned}\frac{d}{dx}\left(f(x)+g(x)\right) &= \frac{df(x)}{dx}+\frac{dg(x)}{dx}, \quad \text{for any two functions } f(x) \text{ and } g(x)\\ \frac{d}{dx}\left(Cf(x)\right) &= C\frac{df(x)}{dx}, \quad \text{for any constant } C \text{ and function } f(x)\end{aligned} \tag{2.12}$$

These properties 2.12 make it instantly possible to compute many more derivatives. For instance using the first equality in 2.12 we can immediately conclude that

$$\begin{aligned}\frac{d}{dx}(x^3+x^7) &= \frac{dx^3}{dx}+\frac{dx^7}{dx}\\ &= 3x^2+7x^6\end{aligned} \tag{2.13}$$

[2] The argument goes like this

$$\begin{aligned}\frac{d1}{dx} &= \lim_{h\to 0}\frac{(1-1)}{h}\\ &= 0\end{aligned} \tag{2.9}$$

[3] The proof of this statement for irrational n is not quite as trivial as might be assumed; it requires a proof of the binomial theorem for irrational n and we do not give the details here.

while using 2.12 part two we find that

$$\frac{d}{dx}(100x^{10}) = 1000x^9 \tag{2.14}$$

Finally applying both properties together we have the useful result that

$$\frac{d}{dx}\left(100x^{10} + x^3 + x^7\right) = 1000x^9 + 3x^2 + 7x^6 \tag{2.15}$$

All this means that we can now differentiate any *polynomial* in x—i.e. any expression of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \tag{2.16}$$

where the quantities $a_0, a_1, \ldots a_n$ are constants. The derivative of this polynomial is given by

$$\frac{d}{dx}\left(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0\right) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2 x + a_1 \tag{2.17}$$

§§ 1.1 The derivatives of the expressions $1/f(x)$ and $f(x)g(x)$

We must forge ahead in learning more about how to differentiate. The first task we can polish off quickly is the derivative of

$$\frac{1}{f(x)} \tag{2.18}$$

This is how it goes

$$\begin{aligned}
\frac{d}{dx}\left(\frac{1}{f(x)}\right) &= \lim_{h\to 0}\left\{\frac{1}{f(x+h)} - \frac{1}{f(x)}\right\}\frac{1}{h} \\
&= \lim_{h\to 0}\left\{\frac{f(x) - f(x+h)}{f(x+h)f(x)}\right\}\frac{1}{h} \\
&= -\lim_{h\to 0}\left\{\frac{f(x+h) - f(x)}{h}\frac{1}{f(x+h)f(x)}\right\} \\
&= -\lim_{h\to 0}\left\{\frac{f(x+h) - f(x)}{h}\right\}\lim_{h\to 0}\left\{\frac{1}{f(x+h)f(x)}\right\} \\
&= -\left(\frac{1}{f(x)}\right)^2\frac{df}{dx}
\end{aligned} \tag{2.19}$$

So we have established that

$$\begin{aligned}
\frac{d}{dx}\left(\frac{1}{f(x)}\right) &= -\left(\frac{1}{f(x)}\right)^2\frac{df}{dx} \\
&= -\frac{f'}{f^2}
\end{aligned} \tag{2.20}$$

We can now display our latest new skill in the following small calculation

Example *The derivative of*

$$\frac{1}{(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)} \tag{2.21}$$

Using 2.20 above we easily calculate that

$$\frac{d}{dx}\left\{\frac{1}{(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)}\right\} = -\frac{(na_n x^{n-1} + \cdots + 2a_2 x + a_1)}{(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)^2} \tag{2.22}$$

Turning to our next job which concerns the product $f(x)g(x)$, we use the simple trick of adding and subtracting the quantity $f(x+h)g(x)$ and calculate that

$$\begin{aligned}
\frac{d}{dx}(f(x)g(x)) &= \lim_{h\to 0}\left\{\frac{f(x+h)g(x+h) - f(x)g(x)}{h}\right\} \\
&= \lim_{h\to 0}\left\{\frac{(f(x+h)g(x+h) - f(x+h)g(x)) + (f(x+h)g(x) - f(x)g(x))}{h}\right\} \\
&= \lim_{h\to 0} f(x+h)\left\{\frac{g(x+h) - g(x)}{h}\right\} + \lim_{h\to 0}\left\{\frac{f(x+h) - f(x)}{h}\right\} g(x) \\
&= \left\{\lim_{h\to 0} f(x+h)\right\} \lim_{h\to 0}\left\{\frac{g(x+h) - g(x)}{h}\right\} + \lim_{h\to 0}\left\{\frac{f(x+h) - f(x)}{h}\right\} g(x) \\
&= f(x)\frac{dg(x)}{dx} + \frac{df(x)}{dx}g(x)
\end{aligned} \tag{2.23}$$

This formula is sometimes called the *Leibnitz product formula* and, to summarise, we have just proved that

$$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{dg(x)}{dx} + \frac{df(x)}{dx}g(x) \tag{2.24}$$

The product formula can be put to use at once; let us have a look at one instance of it in action.

Example *The derivative of*

$$(1 + 3x^2 + 6x^4 + 10x^8)(1 + x + x^2 + x^3) \tag{2.25}$$

Using 2.24 we obtain the result

$$\begin{aligned}
\frac{d}{dx}\big((1 + 3x^2 + 6x^4 + 10x^8)(1 + x + x^2 + x^3)\big) = (1 + 3x^2 + 6x^4 + 10x^8)(1 + 2x + 3x^2) + \\
(6x + 24x^3 + 80x^7)(1 + x + x^2 + x^3)
\end{aligned} \tag{2.26}$$

where we note the convenience of not having to multiply out all the brackets.

§§ **1.2 The derivative of** f/g

An immediate application of the two preceding results 2.20 and 2.24 is to the computation of the derivative of the *quotient* of two functions, that is the quantity

$$\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} \tag{2.27}$$

So, using 2.24 we obtain

$$\begin{aligned}\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} &= f(x)\frac{d}{dx}\left\{\frac{1}{g(x)}\right\} + \frac{df(x)}{dx}\frac{1}{g(x)} \\ &= -f(x)\frac{1}{g^2(x)}\frac{dg(x)}{dx} + \frac{df(x)}{dx}\frac{1}{g(x)}, \qquad \text{using 2.20} \\ &= \frac{g(x)\frac{df(x)}{dx} - f(x)\frac{dg(x)}{dx}}{g^2(x)} \\ &= \frac{gf' - fg'}{g^2}\end{aligned} \tag{2.28}$$

and this is sometimes called the *quotient rule.* As a token of what we are now able to differentiate we just quote some expressions—we shall leave the actual computation of their derivatives to the reader.

Example *Some expressions we can now differentiate*

$$\frac{(1-22x^2-77x^4+10x^8)}{(1-x+x^2-x^3+x^4)}, \frac{(x+3x^3-12x^4+x^6)(1+x^2+x^6+x^{24})}{(1-x+x^3-x^5-x^7)} \\ \frac{(1+x^2+x^4+x^6+x^8))(x^5-5x^7+x^{11})}{(1+x+3x^2-5x^7+6x^9)(x-x^2+x^3+x^6)} \tag{2.29}$$

It is now time for the derivative of just one more variety of function.

§§ **1.3 The chain rule: The derivative of** $f(g(x))$

The final formula we routinely need for working with derivatives is called the *chain rule.* This is the formula that tells us how to differentiate a function which is presented as *a function of a function* so that it looks like

$$f(g(x)) \tag{2.30}$$

For example if one had $f(x) = \sin(x)$ and $g(x) = x^4$ then we would have

$$f(g(x)) = \sin(x^4) \tag{2.31}$$

Steaming ahead we compute that

$$\begin{aligned}\frac{df(g(x))}{dx} &= \lim_{h\to 0}\left\{\frac{f(g(x+h))-f(g(x))}{h}\right\}\\ &= \lim_{h\to 0}\left\{\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\right\}\left\{\frac{g(x+h)-g(x)}{h}\right\}\\ &= \lim_{h\to 0}\left\{\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\right\}\lim_{h\to 0}\left\{\frac{g(x+h)-g(x)}{h}\right\}\\ &= \lim_{h\to 0}\left\{\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\right\}\frac{dg}{dx}\end{aligned} \tag{2.32}$$

Now if we just use the elementary fact that

$$\begin{aligned} g(x+h) &= g(x)+g(x+h)-g(x)\\ &= g(x)+k, \quad \text{where } k=g(x+h)-g(x)\\ \Rightarrow f(g(x+h)) &= f(g(x)+k)\end{aligned} \tag{2.33}$$

and substitute $f(g(x+h)) = f(g(x)+k)$ in the last line of 2.32 we find that

$$\begin{aligned}\frac{df(g(x))}{dx} &= \lim_{h\to 0}\left\{\frac{f(g(x)+k)-f(g(x))}{k}\right\}\frac{dg}{dx}\\ &= \frac{df(g)}{dg}\frac{dg}{dx}\end{aligned} \tag{2.34}$$

and our limits have been evaluated. Hence our result is that

$$\frac{df(g(x))}{dx} = \frac{df(g)}{dg}\frac{dg}{dx} \tag{2.35}$$

and this formula 2.35 is called the *chain rule* or sometimes the derivative of *a function of a function*. Bringing these four useful formulae 2.20, 2.24, 2.28 and 2.35 together, and using the f' style of notation instead of df/dx in the first three, we get the rather easier to remember compact forms

$$\begin{aligned}\left(\frac{1}{f}\right)' &= -\frac{f'}{f^2}\\ (fg)' &= fg'+f'g\\ \left(\frac{f}{g}\right)' &= \frac{gf'-fg'}{g^2}\\ \frac{df(g(x))}{dx} &= \frac{df(g)}{dg}\frac{dg}{dx}\end{aligned} \tag{2.36}$$

§ 2. Derivatives of trigonometric and exponential functions

The trigonometric functions—all of which we would like to be able to differentiate—are all built out of $\sin(x)$ and $\cos(x)$ so we shall now compute the derivatives of $\sin(x)$ and $\cos(x)$. Actually since

$$\cos(x) = \sqrt{1-\sin^2(x)} \tag{2.37}$$

we only have to compute the derivative of $\sin(x)$ and the derivatives of every other trigonometric function will then follow from this one computation.

§§ 2.1 The derivatives of $\sin(x)$ and $\cos(x)$

Proceeding forward, if we bear in mind the formula

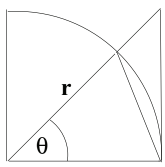
$$\sin(a) - \sin(b)) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right) \tag{2.38}$$

we have

$$\begin{aligned} \frac{d}{dx}(\sin(x)) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\ &= \lim_{h\to 0}\frac{2\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} \\ &= \cos(x)\lim_{h\to 0}\frac{\sin(\frac{h}{2})}{\frac{h}{2}} \\ &= \cos(x),\ \text{ since } \lim_{h\to 0}\frac{\sin(\theta)}{\theta} = 1 \end{aligned} \tag{2.39}$$

So we have found that [4]

[4] We just quoted the result $\lim_{h\to 0}\frac{\sin(\theta)}{\theta} = 1$. For the reader who is interested in its proof we sketch its main idea which is geometric: Consider the sector of the circle and the two triangles shown in the figure.



Since the large triangle contains the sector, which in turn contains the smaller triangle, then computation of their three areas gives the inequality

$$\begin{aligned} &\frac{1}{2}r^2\sin(\theta) < \frac{r^2\theta}{2} < \frac{1}{2}r^2\tan(\theta) \\ &\Rightarrow 1 < \frac{\theta}{\sin(\theta)} < \frac{1}{\cos(\theta)} \\ &\Rightarrow \cos(\theta) < \frac{\sin(\theta)}{\theta} < 1, \quad \text{if we invert everything} \end{aligned}$$

and the limit then easily follows on sending $\theta \to 0$.

$$\frac{d}{dx}(\sin(x)) = \cos(x), \qquad \text{or } \sin'(x) = \cos(x) \tag{2.40}$$

We can now move on to $\cos(x)$. We shall use the the chain rule as follows

$$\begin{aligned} \cos(x) &= \sqrt{1-\sin^2(x)} \\ \Rightarrow \frac{d}{dx}(\cos(x)) &= \left(\frac{1}{2}\right)\left(1-\sin^2(x)\right)^{-1/2}\frac{d}{dx}(1-\sin^2(x)) \\ &= \left(\frac{1}{2}\right)\frac{(-2\sin(x)\sin'(x))}{\sqrt{1-\sin^2(x)}} \\ &= -\frac{\sin(x)\cos(x)}{\cos(x)} \\ &= -\sin(x) \end{aligned} \tag{2.41}$$

so the result is

$$\frac{d}{dx}(\cos(x)) = -\sin(x), \qquad \text{or } \cos'(x) = -\sin(x) \tag{2.42}$$

§§ 2.2 The derivatives of the other trigonometric functions

It is now perfectly straightforward to calculate the derivatives of the remaining standard trigonometric functions. These functions, together with their definitions for the forgetful, are

$$\tan(x),\ \sec(x),\ \cot(x) \text{ and } \operatorname{cosec}(x) \tag{2.43}$$

and have the definitions

$$\tan(x) = \frac{\sin(x)}{\cos(x)},\ \sec(x) = \frac{1}{\cos(x)},\ \cot(x) = \frac{1}{\tan(x)} \text{ and } \operatorname{cosec}(x) = \frac{1}{\sin(x)} \tag{2.44}$$

We leave it to the reader to verify the formulae for their derivatives given below

$$\begin{array}{ll} \tan'(x) = \sec^2(x), & \sec'(x) = \sec(x)\tan(x) \\ \cot'(x) = -\operatorname{cosec}^2(x), & \operatorname{cosec}'(x) = -\operatorname{cosec}(x)\cot(x) \end{array} \tag{2.45}$$

§§ 2.3 The derivatives of exponential functions

An *exponential function* is a function $f(x)$ of the form [5]

$$f(x) = a^x, \qquad \text{where } a \text{ is a constant} \tag{2.46}$$

[5] Actually one allows such a function to be multiplied by a constant C so

$$f(x) = Ca^x$$

is also an exponential function

it acquires its name from the fact that the power x in a^x is also sometimes called an *exponent*. The most famous exponential function is the one obtained by setting

$$\begin{gathered} a = e, \\ \text{where } e = 2.71828\ldots \text{ is the base of natural logarithms} \end{gathered} \tag{2.47}$$

This function

$$e^x \tag{2.48}$$

is often called **the exponential function** and is often denoted by

$$\exp(x) \tag{2.49}$$

so remember that e^x and $\exp(x)$ stand for the *same* function, i.e.

$$e^x = \exp(x) \tag{2.50}$$

The exponential function e^x has the following infinite series expansion

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \end{aligned} \tag{2.51}$$

The derivative of e^x turns out to be equal to e^x—in other words e^x is its own derivative[6] This is easy to prove if we differentiate both sides of 2.51: Doing this gives

$$\begin{aligned} \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) &= \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) \\ &= \left(0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \cdots\right) \\ &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= e^x \end{aligned} \tag{2.52}$$

[6] In fact, instead of defining e^x by the infinite series 2.51, we could search for all functions $f(x)$ which are their own derivative—that is $f' = f$—and we would then find that f has to be given by the series

$$f(x) = C \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

So setting $C = 1$ gives $f = e^x$.

So

$$\frac{d}{dx}e^x = e^x \tag{2.53}$$

Before leaving e^x we shall also find the derivative of the logarithm function [7] $\ln(x)$ We remind the reader that $\ln(x)$ is the *inverse function* to e^x and vice-versa—i.e. we have

$$\ln(e^x) = x \quad \textbf{and} \quad e^{\ln(x)} = x \tag{2.54}$$

In any case we can find the derivative of $\ln(x)$ by differentiating the equation $e^{\ln(x)} = x$: if we do this—and remember to use the chain rule 2.35—we get

$$\begin{aligned} \frac{d}{dx}e^{\ln(x)} &= \frac{d}{dx}x \\ \Rightarrow e^{\ln(x)}\frac{d}{dx}\ln(x) &= 1 \\ \Rightarrow \frac{d}{dx}\ln(x) &= \frac{1}{e^{\ln(x)}} \\ \Rightarrow \frac{d}{dx}\ln(x) &= \frac{1}{x}, \quad \text{since } e^{\ln(x)} = x \end{aligned} \tag{2.55}$$

So we now know that

$$\frac{d}{dx}\ln(x) = \frac{1}{x} \tag{2.56}$$

The other exponential function we want to differentiate is simply $f = a^x$ for any constant a. If we use the log function $\ln(x)$ then we have

$$f = a^x = e^{x\ln(a)} \tag{2.57}$$

Hence we can do the differentiation immediately and the result is that

$$\begin{aligned} \frac{d}{dx}a^x &= \frac{d}{dx}e^{x\ln(a)} \\ &= e^{x\ln(a)}\frac{d}{dx}(x\ln(a)) \\ &= e^{x\ln(a)}\ln(a) \\ &= a^x\ln(a) \end{aligned} \tag{2.58}$$

Recapitulating, we have found that

$$\frac{d}{dx}e^x = e^x, \ \frac{d}{dx}a^x = a^x\ln(a), \ \frac{d}{dx}\ln(x) = \frac{1}{x} \tag{2.59}$$

[7] The notation $\ln(x)$ denotes the logarithm of x to the base e also called the *natural logarithm* of x

§§ 2.4 The hyperbolic functions $\sinh(x)$, $\cosh(x)$ and $\tanh(x)$

Closely related to the trigonometric functions $\sin(x)$, $\cos(x)$ and $\tan(x)$ are the functions $\sinh(x)$, $\cosh(x)$ and $\tanh(x)$ which are officially called *hyperbolic sine*, *hyperbolic cosine* and *hyperbolic tan* respectively. However sinh, cosh and tanh are usually pronounced as "sinsh", "cosh" and "tansh"respectively. These functions are made of various combinations of $e^{\mp x}$; their definitions are

$$\begin{aligned} \sinh(x) &= \frac{e^x - e^{-x}}{2} \\ \cosh(x) &= \frac{e^x + e^{-x}}{2} \\ \tanh(x) &= \frac{\sinh(x)}{\cosh(x)} \end{aligned} \tag{2.60}$$

They have quite a few analogous properties to their trigonometric cousins sin, cos and tan but, unlike sin, cos and tan, they are *not* periodic functions. Among their properties we quote the following, which the reader should find easy to verify from their definitions

(i) $\cosh^2(x) - \sinh^2(x) = 1$
(ii) $\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$
(iii) $\sinh'(x) = \cosh(x)$
(iv) $\cosh'(x) = \sinh(x)$
(v) $\tanh'(x) = \text{sech}^2(x)$, where $\text{sech}(x) = 1/\cosh(x)$.

§§ 2.5 The inverse trigonometric functions $\arcsin(x)$, $\arccos(x)$ and $\arctan(x)$

The trigonometric functions all have inverses and we would like to be able to compute their derivatives as they occur in calculations from time to time. The inverse function to $\sin(x)$ is denoted by $\arcsin(x)$ and, because it is an *inverse function* it satisfies the pair of equations[8]

$$\arcsin(\sin(x)) = x, \ \sin(\arcsin(x)) = x \tag{2.61}$$

It is very useful to remember that $\arcsin(x)$ is simply *an angle*—hence if one writes

$$f = \arcsin(x) \tag{2.62}$$

it may help to read this aloud—or silently—as the phrase

"f is the angle whose sine is x"

The other inverse trigonometric functions are defined in the same way giving us

$$\begin{aligned} \arccos(\cos(x)) = x, \ \cos(\arccos(x)) = x \\ \arctan(\tan(x)) = x, \ \tan(\arctan(x)) = x \end{aligned} \tag{2.63}$$

[8] Sometimes, instead of $\arcsin(x)$, the notation $\sin^{-1}(x)$ is used but this is dangerous because it can be confused with $1/sin(x)$ to which it is definitely *not* equal.

Finally to compute their derivatives one uses the same method as was used to obtain the derivative of $\ln(x)$ in 2.55. We start, therefore, with

$$\begin{aligned} f &= \arcsin(x) \\ \Rightarrow \sin(f(x)) &= x \\ \Rightarrow \frac{d}{dx}\{\sin(f(x))\} &= 1 \\ \Rightarrow \frac{d\sin(f)}{df}\frac{df}{dx} &= 1 \\ \Rightarrow \cos(f)\frac{df}{dx} &= 1 \\ \Rightarrow \frac{df}{dx} &= \frac{1}{\cos(f)} \\ &= \frac{1}{\sqrt{1-\sin^2(f)}}, \text{ since } \cos(f) = \sqrt{1-\sin^2(f)} \\ &= \frac{1}{\sqrt{1-x^2}}, \text{ since } \sin(f) = x \end{aligned} \tag{2.64}$$

So we have shown that

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}} \tag{2.65}$$

The same technique can be used by the reader to calculate $\arccos'(x)$ and $\arctan'(x)$. These calculations will show that

$$\arccos'(x) = -\frac{1}{\sqrt{1-x^2}} \quad \text{and} \quad \arctan'(x) = \frac{1}{1+x^2} \tag{2.66}$$

§ 3. The significance of derivatives

Derivatives have many, many uses and we are now going to examine some of them.

We shall see that a relatively *small* knowledge of f' can reveal a *large* amount about a function f; this knowledge is usually acquired by studying the *maxima and minima* of f.

It is also important to realise that f', as well as being the slope of tangent, has an interpretation as a *rate of change.*

Thus, if f is a *distance* f' is a *velocity*, if f is a *velocity* f' is an *acceleration*, if f is the *amount of charge* in a conductor f' is the *current*, if f is momentum (in a given direction) f' is the force in that direction, if f is the energy of a *motor* f' is the *power*, if f is *data* inside a communications channel f' is the rate of data transmission in the channel, and so on.

§§ 3.1 Critical points, maxima and minima

We shall begin with a look at maxima and minima. The first fact that needs to be well noted is the significance of the *sign* of the derivative of any function $f(x)$. The *sign* of $f'(x)$—at a given point x—tells one whether $f(x)$ is *increasing* or *decreasing* at x. To see how this arises just remember that the $f'(x)$ is the tan of an angle and recall that $\tan(\theta)$ has the property that

$$\begin{aligned} \theta < \frac{\pi}{2} &\Rightarrow \tan(\theta) > 0 \\ \theta > \frac{\pi}{2} &\Rightarrow \tan(\theta) < 0 \end{aligned} \tag{2.67}$$

cf. fig. 15

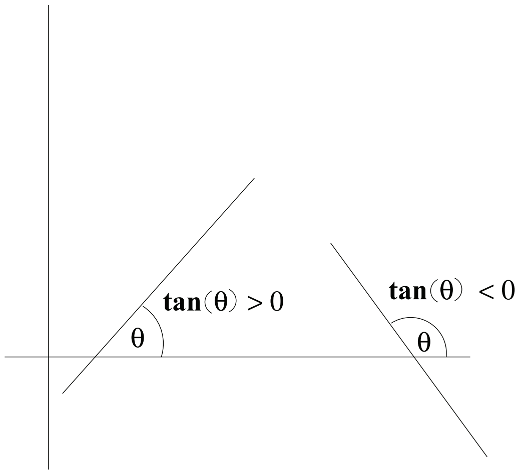


Fig. 15: The sign of $\tan(\theta)$

Now if we examine the graph of $f(x)$ in fig. 16 we see that at a point where $f(x)$ is *increasing* the slope of the tangent is that of an angle *less* than $\pi/2$; while, if $f(x)$ is *decreasing* the slope of the tangent is that of an angle *larger* than $\pi/2$.

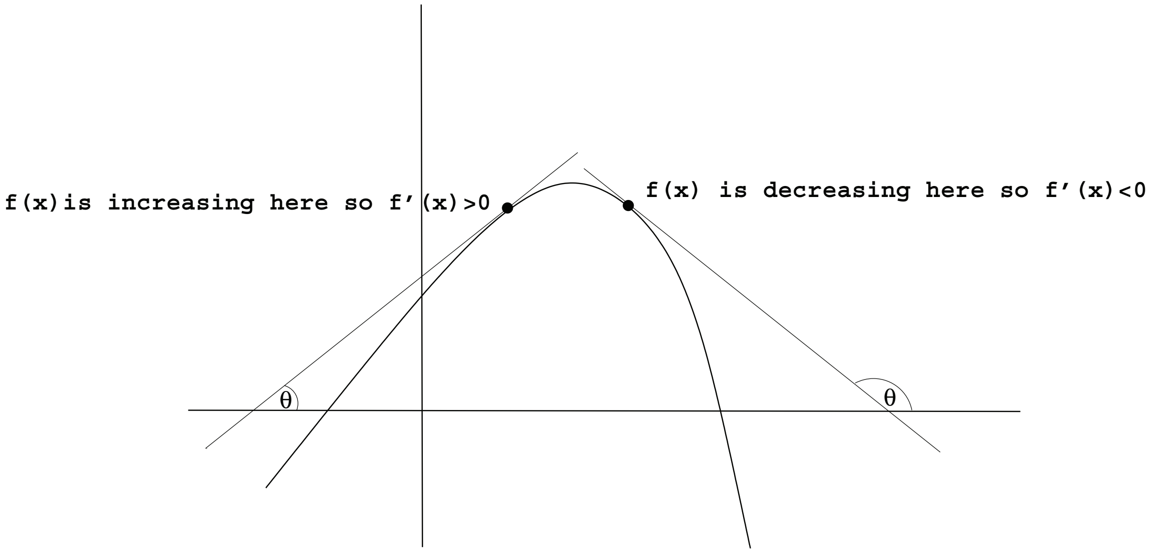


Fig. 16: The meaning of the sign of $f'(x)$

Hence we have the very useful statement that

$$\begin{aligned} f(x) \text{ increasing at the point } x &\Rightarrow f'(x) > 0 \\ f(x) \text{ decreasing at the point } x &\Rightarrow f'(x) < 0 \end{aligned} \tag{2.68}$$

However it is possible for an f, which is increasing and so has $f' > 0$, to *change over* and start decreasing rendering $f' < 0$. Thus the point a, say, at which that happens will be a *maximum* and so f' must pass through zero there; i.e. we must have

$$f'(a) = 0, \quad \text{when } a \text{ is a maximum} \tag{2.69}$$

In an exactly similar way a point a at which changes from being decreasing to increasing is a minimum and so f' must also pass through zero there; i.e. we must again have

$$f'(a) = 0, \quad \text{when } a \text{ is a minimum} \tag{2.70}$$

In general, then, when $f'(x)$ is *zero* at a we *usually*, but *not always*, find that a is a point where $f(x)$ has a *maximum* or a *minimum*. In any case a piece of terminology must now be defined: that of a *critical point* [9]

Definition (A critical point) *A critical point of a function f is a number x such that*

$$f'(x) = 0 \tag{2.71}$$

[9] Sometimes the term *stationary point* is used to mean a point where $f'(x) = 0$ but we shall use the more standard term *critical point*.

We move on at once to show how examples of maxima and minima and to give an example of a critical point which is neither a maximum nor a minimum.

Example *A critical point which is a minimum*

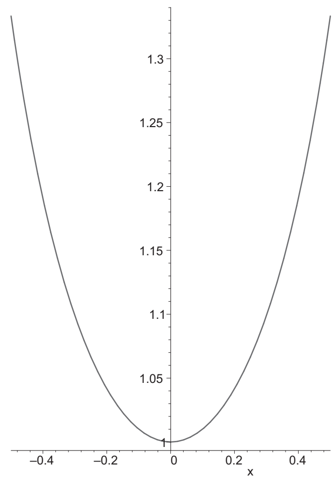


Fig. 17: The function $f = 1/(1-x^2)$. It has $f'(0) = 0$ and has a *minimum* at $x = 0$.

Example *A critical point which is a maximum*

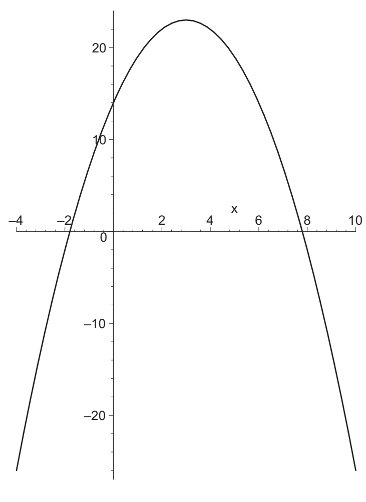


Fig. 18: The function $f = 14 - x^2 + 6x$. It has $f'(3) = 0$ and has a *maximum* at $x = 3$

Example *A critical point which is neither a maximum nor a minimum*

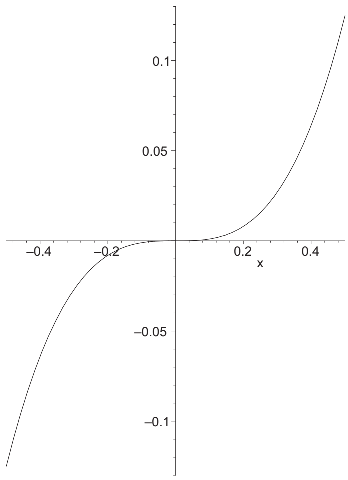


Fig. 19: The function $f = x^3$. It has $f'(0) = 0$ but has *neither maximum nor minimum* at $x = 0$.

We would like some systematic way of telling when critical points are maxima, minima or neither of the two. This is achieved by calculating the second derivative of f at the critical point. The result that we need is summarised in the following statement

$$\begin{aligned} f'(a) = 0 \text{ and } f''(a) < 0 \Rightarrow f \text{ has a } \textit{maximum} \text{ at } a \\ f'(a) = 0 \text{ and } f''(a) > 0 \Rightarrow f \text{ has a } \textit{minimum} \text{ at } a \end{aligned} \tag{2.72}$$

The reader should note carefully that 2.72 still fails to cover one case: this being the case where

$$f''(a) = 0 \tag{2.73}$$

Unfortunately when

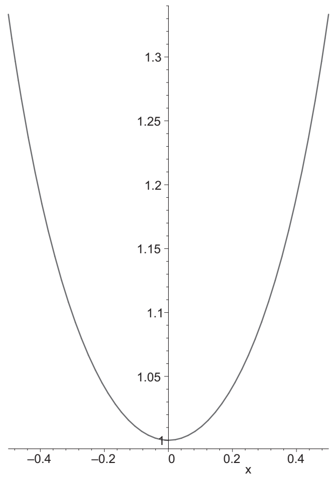
$$f'(a) = 0 \text{ and } f''(a) = 0 \tag{2.74}$$

then a may be a maximum, a minimum, or neither of the two.

The way to understand all this is to look at some examples and in particular to look at the graphs of the functions involved. We begin by analysing the critical points in the graphs of figs. 17–19.

Fig. 17 shows the function $f(x) = 1/(1 - x^2)$ and so we calculate that

$$f(x) = \frac{1}{1-x^2}$$
$$f'(x) = \frac{2x}{(1-x^2)^2} \qquad (2.75)$$
$$f''(x) = \frac{6x^2+2}{(1-x^2)^3}$$



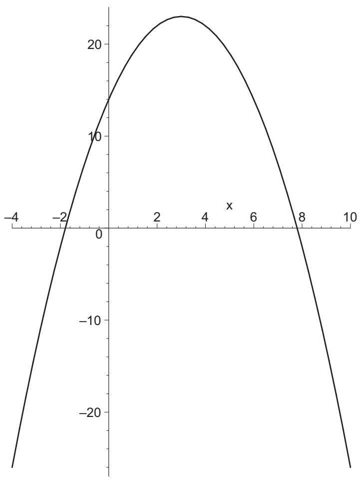
from which we find that

$$\left.\begin{array}{r} f'(0) = 0 \\ f''(0) = 2 \end{array}\right\} \Rightarrow x = 0 \text{ is a minimum} \qquad (2.76)$$

as we can see from the graph itself.

Fig. 18 is a graph of $f(x) = 14 - x^2 + 6x$ and we find that

$$f(x) = 14 - x^2 + 6x$$
$$f'(x) = -2x + 6 \qquad (2.77)$$
$$f''(x) = -2$$

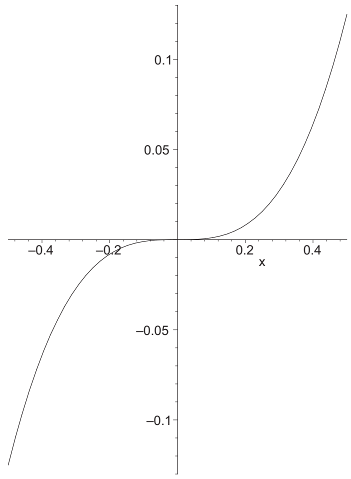


from which we find that

$$\left.\begin{array}{r} f'(3) = 0 \\ f''(3) = -2 \end{array}\right\} \Rightarrow x = 3 \text{ is a maximum} \qquad (2.78)$$

again as we knew already.

Now for fig. 19 for which we have



$$f(x) = x^3$$
$$f'(x) = 3x^2 \tag{2.79}$$
$$f''(x) = 6x$$

so that we obtain

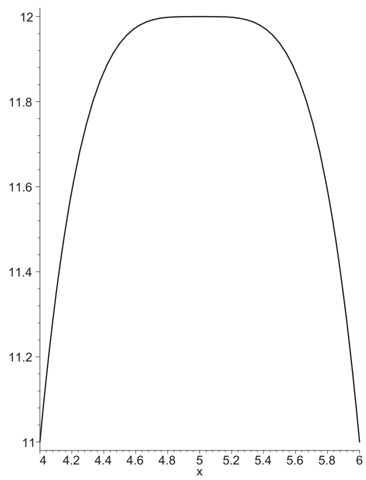
$$\left.\begin{array}{r} f'(0) = 0 \\ f''(0) = 0 \end{array}\right\} \textbf{But } x = 0 \text{ is neither maximum nor minimum} \tag{2.80}$$

which again is what we can see from the graph.

Now we examine two new functions, each with a critical point, the first of which is simply

$$f(x) = 12 - (x-5)^4 \tag{2.81}$$

Routine differentiation tells us that



$$f(x) = 12 - (x-5)^4$$
$$f'(x) = -4(x-5)^3 \tag{2.82}$$
$$f''(x) = -12(x-5)^2$$

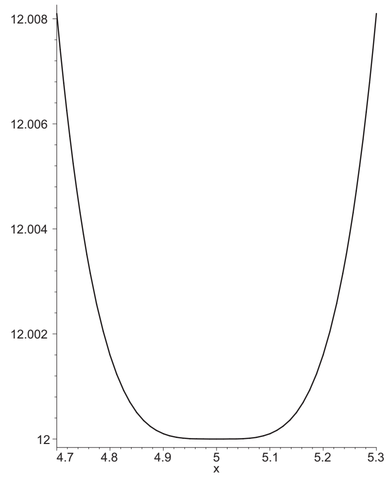
so that we obtain

$$\left.\begin{array}{r} f'(5) = 0 \\ f''(5) = 0 \end{array}\right\} \textbf{But } \ x = 5 \text{ is a } \textbf{maximum} \tag{2.83}$$

as we can read off from the graph.

Now if we just change the sign of the term $-(x-5)^4$ this maximum at $x = 5$ will turn into a minimum. Let's see how this works: we obtain

$$\begin{aligned} f(x) &= 12 + (x-5)^4 \\ f'(x) &= 4(x-5)^3 \\ f''(x) &= 12(x-5)^2 \end{aligned} \tag{2.84}$$



yielding

$$\left.\begin{aligned} f'(5) &= 0 \\ f''(5) &= 0 \end{aligned}\right\} \textbf{But} \quad x = 5 \text{ is a } \textbf{minimum} \tag{2.85}$$

as we can see for ourselves.

The point we are trying to emphasise in 2.79–2.84 is that when a critical point $x = a$ of a function f has $f''(a) = 0$—as well as the **obligatory** $f'(a) = 0$—then $x = a$ can be a maximum (as in 2.82), a minimum (as in 2.84) or neither of the two (as in 2.79).

§§ 3.2 Points of inflection

It turns out that the reason that the graph of 2.79 has a critical point which is neither maximum nor minimum can be traced to the fact that its second derivative $f''(x)$ *changes sign* as x passes through the critical point.

Let's just check this. The relevant data is

$$\begin{aligned} f(x) &= x^3 \\ f'(x) &= 3x^2 \\ f''(x) &= 6x \end{aligned}$$

so the critical point is $x = 0$ we must therefore see if the sign of $f''(x)$ changes as x passes through 0. But $f''(x)$ is simply $6x$ so we have

$$\begin{aligned} x < 0 &\Rightarrow 6x < 0 \\ x > 0 &\Rightarrow 6x > 0 \end{aligned} \tag{2.86}$$

So indeed $f''(x)$ did change sign as we passed through the critical point. Any point—not necessarily a critical point—with this property is called a *point of inflection*. Hence we have the following definition.

Definition (Point of inflection) *A point of inflection of a function f is a point a for which $f''(a) = 0$ and f'' changes sign as x passes through a.*

§§ 3.3 The shape of a graph: concavity and convexity

It turns out that the value of f'' is a very useful indicator of *the shape* of the graph of f. The way this works is quite simple and is summarised in fig. 20 below.

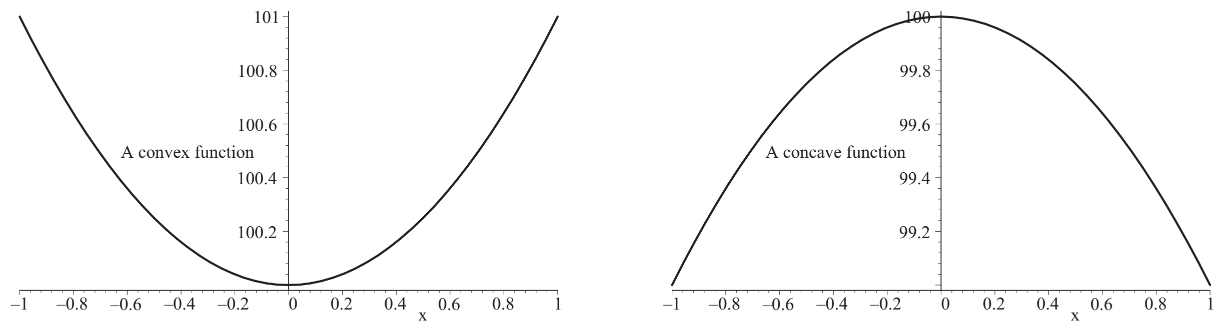


Fig. 20: The sign of f'' determines whether the graph of f is *concave* or *convex*: the graph on the left has $f'' > 0$ and the one on the right has $f'' < 0$.

In fig. 21 we see a graph which starts out concave and then becomes convex.

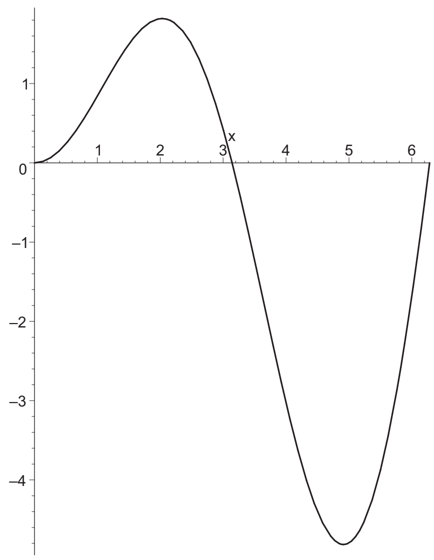


Fig. 21: A function with both concave and convex regions

This is all highly relevant for points of inflection. It is simple to see why this is so: At

a point of inflection $x = a$, say, f'' changes sign (by definition), hence there are just two possibilities
(i) f'' changes from positive to negative $\Rightarrow$ the shape of f changes from convex to concave
(ii) f'' changes from negative to positive $\Rightarrow$ the shape of f changes from concave to convex

Example *A function with two points of inflection*
The function

$$f(x) = \frac{1}{1+x^2} \tag{2.87}$$

has *two* points of inflection which means that it change its shape *twice.* It is plotted in fig. 22

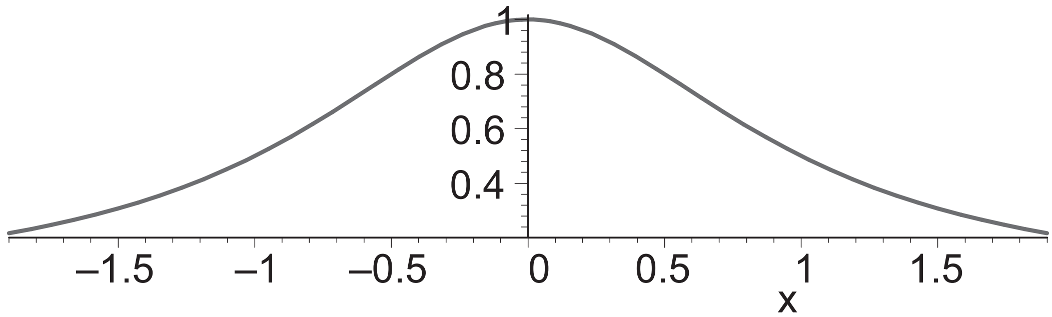


Fig. 22: This function has points of inflection at $x = \mp\frac{1}{\sqrt{3}}$

In fig. 22 $f = 1/(1+x^2)$ is convex until $x = -1/\sqrt{(3)} = -0.577$ then it becomes concave until $x = 1/\sqrt{(3)} = 0.577$ where it becomes convex again. We shall now prove this by doing a little calculation. What we need is f'' so, steaming ahead, we compute that

$$\begin{aligned} f &= \frac{1}{1+x^2} \\ \Rightarrow f' &= -\frac{2x}{(1+x^2)^2} \\ \Rightarrow f'' &= \frac{(1+x^2)^2(-2)-(-2x)(2(1+x^2)(2x)}{(1+x^2)^4}, \quad \text{(quotient rule)} \\ &= \frac{2(3x^2-1)}{(1+x^2)^3} \end{aligned} \tag{2.88}$$

The first pleasing thing to note about the formula 2.88 for f'' is that f'' *vanishes* when

$$3x^2 - 1 = 0 \Rightarrow x = \mp\frac{1}{\sqrt{3}} \tag{2.89}$$

Now, to make it really easy to see where f'' changes sign we shall simply plot it giving us fig. 23.

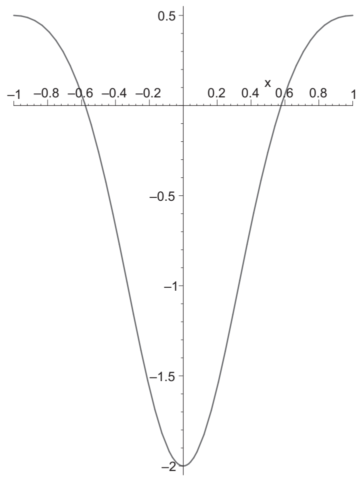


Fig. 23: The graph of f''

What we see in fig. 23 confirms our conclusions: f'' is at first positive then, at about $x = -0.57$ or so, it becomes negative and changes back to being positive at about $x = 0.57$. Thus we have confirmed all that we claimed about the shape of $f = 1/(1+x^2)$.

§§ 3.4 Limits revisited: L'Hôpital's rule

Calculus can sometimes be quite helpful in the evaluation of limits. Perhaps the most well known and useful application of calculus to limit evaluation is called *L'Hôpital's rule* and it helps when one has to calculate a limit of *a ratio* of two functions—that is something of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)} \tag{2.90}$$

Such a limit may be quite easy to calculate, it depends on the nature of f and g; L'Hôpital's rule only comes in if the numerator and denominator of f/g *both vanish* as $x \to a$. To see how this causes a problem choose the point a to be 0 and set

$$f(x) = \sin(x) \text{ and } g(x) = x \tag{2.91}$$

so that we have [10]

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{\sin(x)}{x} \tag{2.92}$$

Anyhow note that because

$$\sin(0) = 0 \text{ and } x \text{ vanishes if } x = 0 \tag{2.93}$$

we can say that

$$\lim_{x \to 0} \sin(x) = 0 \text{ and } \lim_{x \to 0} x = 0 \tag{2.94}$$

[10] We already saw in 2.39 that this particular limit has the value 1, we shall confirm that again here.

so we are in trouble if we say

$$\lim_{x\to 0}\frac{\sin(x)}{x} = \frac{\lim_{x\to 0}\sin(x)}{\lim_{x\to 0} x} = \frac{0}{0} \tag{2.95}$$

The trouble comes, clearly, because we do not know what to do with the quantity

$$\frac{0}{0} \tag{2.96}$$

Now we state (without proof) L'Hôpital's rule and show how our difficulty disappears.

Theorem (L'Hôpital's rule) *Suppose that we want to evaluate*

$$\lim_{x\to a}\frac{f(x)}{g(x)} \tag{2.97}$$

and we **also** *have*

$$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0$$

then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} \tag{2.98}$$

provided both limits exist.

In other words, if the conditions of the theorem apply, we can *replace* $f(x)$ and $g(x)$ by their derivatives. We go on immediately to show how L'Hôpital's rule removes our difficulty with $\sin(x)/x$.

Example *The evaluation of*

$$\lim_{x\to 0}\frac{\sin(x)}{x} \tag{2.99}$$

We have already noted above that

$$\lim_{x\to 0}\sin(x) = 0 \text{ and } \lim_{x\to 0} x = 0 \tag{2.100}$$

so L'Hôpital's rule gives us the result that

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(x)}{x} &= \lim_{x\to 0}\frac{\cos(x)}{1}, \quad \text{using } \begin{cases}\sin'(x) = \cos(x)\\ \dfrac{dx}{dx} = 1\end{cases}\\ &= \cos(0)\\ &= 1, \quad \text{since } \cos(0) = 1\end{aligned} \tag{2.101}$$

So we confirm the result first encountered in 2.39 that

$$\lim_{x\to 0}\frac{\sin(x)}{x}=1 \tag{2.102}$$

and we note that we no longer had to encounter the indeterminate quantity 0/0. We move on.

Example *The limit*

$$\lim_{x\to 1}\frac{x^3+x-2}{x^2-3x+2} \tag{2.103}$$

We see on examining numerator and denominator that they both vanish at the limiting point $x=1$; this means that this is a job for L'Hôpital's rule. Hence we can immediately say that

$$\begin{aligned}\lim_{x\to 1}\frac{x^3+x-2}{x^2-3x+2}&=\lim_{x\to 1}\frac{3x^2+1}{2x-3}\\&=\frac{4}{-1}\end{aligned} \tag{2.104}$$

In other words L'Hôpital's rule has saved the day and we have found that

$$\lim_{x\to 1}\frac{x^3+x-2}{x^2-3x+2}=-4 \tag{2.105}$$

We provide one more example.

Example *The limit*

$$\lim_{x\to 2}\frac{2-x}{4-x^2} \tag{2.106}$$

First we note that both numerator and denominator vanish at the limit point $x=2$ and so, computing their derivatives, we conclude that

$$\begin{aligned}\lim_{x\to 2}\frac{2-x}{4-x^2}&=\lim_{x\to 2}\frac{-1}{(-2x)}\\&=\frac{1}{4}\end{aligned} \tag{2.107}$$

and so we have our result, which is

$$\lim_{x\to 2}\frac{2-x}{4-x^2}=\frac{1}{4} \tag{2.108}$$

§ 4. Taylor series

We now come to a remarkable result which enables many functions to be expressed as convergent series in powers of x. These series are called *Taylor series*. Let us first quote the result and then give some examples.

If $f(x)$ is a function which satisfies certain appropriate conditions (to be elaborated on later) then $f(x)$ can be expanded in a so called *power series* given by

$$\begin{aligned} f(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \end{aligned} \tag{2.109}$$

We have already come across an instance of a Taylor series without any comment to that effect. Let us now give the details.

Example *The exponential series*

Recall the function e^x for which we know has the properties

$$\frac{de^x}{dx} = e^x \text{ and } e^0 = 1 \tag{2.110}$$

then we can say that

$$\frac{d^n e^x}{dx^n} = e^x, \quad \text{for } n = 1, 2, 3, \ldots \tag{2.111}$$

and, setting $x = 0$, we can further say that

$$\left.\frac{d^n e^x}{dx^n}\right|_{x=0} = 1, \quad \text{for } n = 1, 2, 3, \ldots \tag{2.112}$$

Hence if we set $f = e^x$ in 2.109 above, and use these results, we find that 2.109 simply becomes the statement that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{2.113}$$

which we recognise as being the same series as we had in 2.51. So the Taylor series for e^x has just reproduced the series for e^x that we knew already.

Example *The binomial series*

Another example of a series which the reader may not have realised is a Taylor series is that given by the binomial expansion of $(1 + x)^\alpha$ where α is *not necessarily an integer*. The binomial expansion says that [11]

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots \tag{2.114}$$

Now to use our formula 2.109 above we must compute

$$\frac{d^n(1+x)^\alpha}{dx^n}, \quad \text{for } n = 1, 2, 3, \ldots \tag{2.115}$$

This is easy and we find that

$$\begin{aligned} \frac{d(1+x)^\alpha}{dx} &= \alpha(1+x)^{\alpha-1} \\ \frac{d^2(1+x)^\alpha}{dx^2} &= \alpha(\alpha-1)(1+x)^{\alpha-2} \\ \vdots \quad & \quad \vdots \\ \frac{d^n(1+x)^\alpha}{dx^n} &= \alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n} \end{aligned} \tag{2.116}$$

Hence if we evaluate both sides at $x = 0$ we find that

$$\left.\frac{d^n(1+x)^\alpha}{dx^n}\right|_{x=0} = \alpha(\alpha-1)\cdots(\alpha-n+1) \tag{2.117}$$

Finally if we insert this information in the Taylor series formula 2.109 above we find that it gives us

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots \tag{2.118}$$

which is precisely the binomial expansion 2.114 above.

Next we need to be a bit more general because actually most suitable functions $f(x)$ can be expanded, not just in powers of x, but in powers of the quantity $(x - a)$ where a is any number. The result that describes the complete state of affairs is known as *Taylor's theorem* and we now quote it (again without proof).

[11] Note very carefully that if α **is** an integer, say $\alpha = n$, then this expansion will stop after $n+1$ terms; otherwise it does not terminate. The reader who for whom this fact is new should choose two values for α such as $\alpha = 3$ and $\alpha = 1/2$ and check that in the first case the series terminates after 4 terms but in the second case it goes on forever.

Theorem (Taylor's theorem) *Suppose that $f(x)$ is a function whose derivatives $f', f'', \ldots$ all exist then*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots \quad (2.119)$$

provided the series above converges and a certain remainder term $R_{n,a}(x) \to 0$ as $n \to \infty$.

The first thing the reader [12] should do is to notice that, if we set $a = 0$, then the formula 2.119 of Taylor's theorem reverts to the formula 2.109 that we had above.

A piece of *terminology* can be got out of the way here: this formula 2.109

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \quad (2.120)$$

is often called the *Maclaurin series* for f. In other words a Maclaurin series is the special case of a Taylor series that results if $a = 0$. We shall just use the term Taylor series for all such series as this is the more common practice.

Another linguistic point is that the Taylor expansion 2.119 is often referred to as "the Taylor expansion of f about $x = a$".

We need a few more examples to finish off with.

Example *The Taylor series for* $\sin(x)$

If we set $a = 0$ then Taylor's theorem tells us that

$$\sin(x) = \sin(0) + \sin'(0)x + \frac{\sin''(0)}{2!}x^2 + \cdots \quad (2.121)$$

Computing the derivatives that we need gives

$$\begin{aligned} \sin'(x) = \cos(x) &\Rightarrow \sin'(0) = \cos(0) = 1 \\ \Rightarrow \sin''(x) = -\sin(x) &\Rightarrow \sin''(0) = -\sin(0) = 0 \\ \Rightarrow \sin'''(x) = -\cos(x) &\Rightarrow \sin'''(0) = -\cos(0) = -1 \\ \Rightarrow \sin''''(x) = \sin(x) &\Rightarrow \sin''''(0) = \sin(0) = 0 \end{aligned} \quad (2.122)$$

[12] The reader should not worry about this remainder term or about the convergence of the series and the proof of the theorem. We shall always assume that the series converges and that the remainder term satisfies $R_{n,a}(x) \to 0$ as $n \to \infty$ and in our examples these properties will always hold. For those who may be interested the formula for $R_{n,a}(x)$ is $R_{n,a}(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\, dt$; we note that it involves an integral—an object we do not meet officially until the next chapter.

But this pattern of 4 derivatives now repeats itself ad infinitum so that we know that the next 4 are given by

$$\begin{aligned} \sin^{(5)}(0) &= 1 \\ \sin^{(6)}(0) &= 0 \\ \sin^{(7)}(0) &= -1 \\ \sin^{(8)}(0) &= 0 \end{aligned} \tag{2.123}$$

and so on. Hence we have established that

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \tag{2.124}$$

Example *The Taylor series for* $\cos(x)$

A similar but not quite identical calculation will give the following series for $\cos(x)$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \tag{2.125}$$

These series expansion for $\sin(x)$ and $\cos(x)$ can often prove extremely useful and are well worth memorising.

Example *Two expansions for* $(1+x)^{-1}$

We have already expanded $(1+x)^{\alpha}$. Now we set $\alpha = -1$ and consider the function $(1+x)^{-1}$. If we use 2.116 we obtain

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{1+x}\right) &= \frac{-1}{(1+x)^2} \\ \Rightarrow \frac{d^2}{dx^2}\left(\frac{1}{1+x}\right) &= \frac{(-1)^2\cdot 2}{(1+x)^3} \\ \Rightarrow \frac{d^3}{dx^3}\left(\frac{1}{1+x}\right) &= \frac{(-1)^3\cdot 2\cdot 3}{(1+x)^4} \\ &\vdots \\ \Rightarrow \frac{d^n}{dx^n}\left(\frac{1}{1+x}\right) &= \frac{(-1)^n n!}{(1+x)^{n+1}} \end{aligned} \tag{2.126}$$

Now setting $x = a$ gives

$$\left.\frac{d^n}{dx^n}\left(\frac{1}{1+x}\right)\right|_{x=a} = \frac{(-1)^n n!}{(1+a)^{n+1}} \tag{2.127}$$

and using this in 2.119 gives us the result that

$$\frac{1}{1+x} = \frac{1}{1+a} - \frac{(x-a)}{(1+a)^2} + \frac{(x-a)^2}{(1+a)^3} + \cdots + (-1)^n \frac{(x-a)^n}{(1+a)^{n+1}} + \cdots \tag{2.128}$$

Quite an interesting thing now happens and it is this: the ratio test will easily show that this series converges if

$$\begin{aligned} &\left|\frac{x-a}{1+a}\right| < 1 \\ \text{i.e.} \quad \text{if} \quad &|x-a| < |1+a| \end{aligned} \tag{2.129}$$

so by choosing *two different values of* a we can get *two different series* for $(1+x)^{-1}$ with *two different* ranges of x for which they converge.

To be specific let us choose the two values of a to be 0 and 1/2; we then get the two series

$$\begin{aligned} \frac{1}{1+x} &= 1 - x + x^2 - x^3 + \cdots \qquad \text{(convergent for } |x| < 1) \\ \frac{1}{1+x} &= \frac{1}{(3/2)} - \frac{(x-1/2)}{(3/2)^2} + \frac{(x-1/2)^2}{(3/2)^3} - \cdots \qquad \text{(convergent for } |x-1/2| < 3/2) \end{aligned} \tag{2.130}$$

So now notice that if

$$x = \frac{3}{2} \tag{2.131}$$

the first series *diverges* because $|x| > 1$ if $x = 3/2$—and so is useless—but the second series is still *convergent* because $|x - 1/2| < 3/2$ if $x = 3/2$.

Another interesting fact is that if we choose $x = 0.4$ then *both* series are convergent so does it matter which one we use. This is a practical matter which is of interest to an engineering readership; the answer is that it *does matter* if you want the series chosen to converge quickly so that you don't have to add up too many terms in order to get reasonable accuracy. The series chosen, if $x = 0.4$, should be the second one in 2.130 because it is an expansion in powers of the parameter [13]

$$x - \frac{1}{2} = 0.1, \quad \text{if } x = 0.4 \tag{2.132}$$

whereas the other series is an expansion in powers of the parameter

$$x = 0.4 \tag{2.133}$$

[13] Actually, if one looks more carefully, one sees that the expansion parameter is $(x - 1/2)/(3/2)$ which is even smaller than 0.1 when $x = 0.4$ since it has the value $0.0\dot{6}$; but this just makes things even better.

and a series in powers of a parameter will usually converge fastest when the parameter is smallest.

If the reader has access to computer mathematics packages such as *Maple*, *Mathematica* or *Matlab* all these divergence and convergence properties—as well as convergence rates—can be seen emerging numerically on the computer. Here are the results of doing such a calculation using the package *Maple*.

First of all note that if we work to 10 places of decimals then

$$\frac{1}{1+x} = 0.7142857143, \quad \text{when } x = 0.4 \tag{2.134}$$

We shall now sum the first 11 terms of each of the two series above and then set $x = 0.4$. We find that

$$\begin{aligned}
\sum_{n=0}^{10}(-x)^n &= 1 - x + x^2 - \cdots + x^{10} \\
&= 0.7143156736, \quad \text{when } x = 0.4 \\
\sum_{n=0}^{10}(-1)^n \frac{(x-1/2)^n}{(3/2)^{n+1}} &= \frac{1}{(3/2)} - \frac{(x-1/2)}{(3/2)^2} + \frac{(x-1/2)^2}{(3/2)^3} - \cdots + \frac{(x-1/2)^{10}}{(3/2)^{11}} \\
&= 0.7142857140, \quad \text{when } x = 0.4
\end{aligned} \tag{2.135}$$

Now we readily see the difference in accuracy of the two series: the first one differs from the actual value quoted in eq. 2.134 in the *fourth* decimal place; but the second one does not differ until the *tenth* decimal place—a considerable improvement.

Next we have two topics which, though not calculus applications, are of universal use in many mathematical calculations. These are *polar coordinates* and *complex numbers*.

§ 5. Plane polar coordinates

The usual Cartesian coordinates (x, y) of a point in the plane are not the only coordinates one can use. In calculations with some circular symmetry it is often convenient to introduce what are called *plane polar coordinates* or simply *polar coordinates*.

Polar coordinates consist of a *distance* r and an angle θ and are denoted by

$$(r, \theta) \tag{2.136}$$

Fig. 24 illustrates how the polar coordinates of a point are obtained.

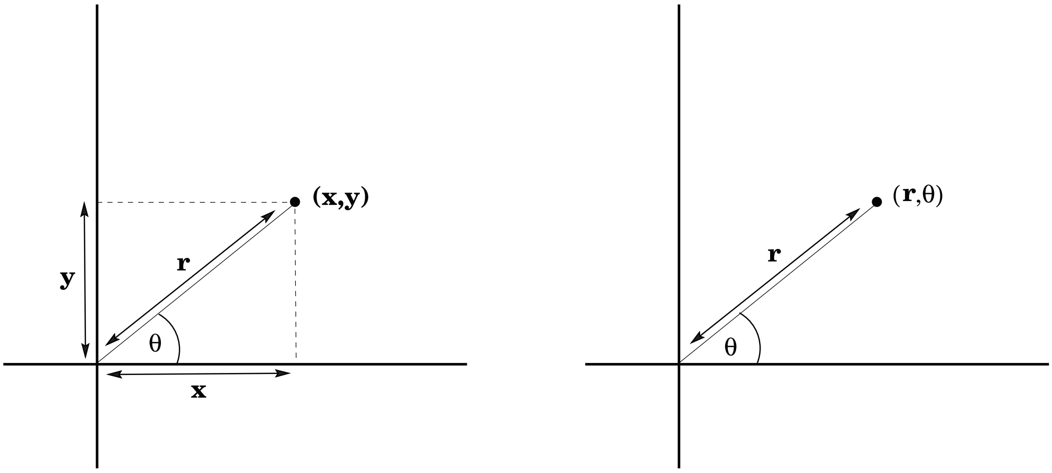


Fig. 24: The Cartesian and polar coordinates (r, θ) of a point

If we use fig. 24 we can see that the Cartesian coordinates x and y are related to r and θ by the pair of equations

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned} \tag{2.137}$$

We can also see that $x^2 + y^2 = r^2$ and $\tan(\theta) = y/x$; from which we deduce that

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \theta &= \arctan\left(\frac{y}{x}\right) \end{aligned} \tag{2.138}$$

With this last pair of equations we can start with Cartesian coordinates (x, y) and compute their corresponding polar coordinates (r, θ) or start with polar coordinates (r, θ) and convert them into Cartesian coordinates (x, y).

§§ 5.1 Some old and new equations expressed in polar coordinates

Example *A circle*

Let us consider the Cartesian equation of a circle of radius a which is

$$x^2 + y^2 = a^2 \tag{2.139}$$

But this is the same as

$$r^2 = a^2 \tag{2.140}$$

which we might as well simplify to

$$r = a \tag{2.141}$$

which is the polar form of the equation of a circle and we see that it is simpler than the Cartesian form. This will not always be the case however—the circle is a special case.

Example *A parabola*

Recall that a parabola has the equation

$$y = Ax^2, \qquad A \text{ a constant} \tag{2.142}$$

so this becomes

$$\begin{aligned} r\sin(\theta) &= Ar^2\cos^2(\theta) \\ \Rightarrow r &= \frac{\sin(\theta)}{A\cos^2(\theta)}, \quad (r \geq 0) \end{aligned} \tag{2.143}$$

Notice that this equation has become more complicated when expressed in polar coordinates.

Example *The four leaved rose*

Now purely for fun we show the plot of a *four leaved rose* which comes from plotting a very simple function when expressed in polar coordinates. One just plots

$$r = |\cos(2\theta)| \tag{2.144}$$

and the result is shown in fig. 25, note that we have modulus signs round $\cos(2\theta)$ to make sure that r cannot go negative [14]

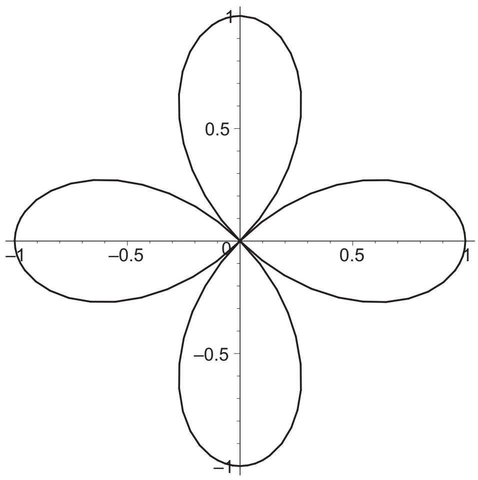


Fig. 25: A four leaved rose obtained by plotting $r = |\cos(2\theta)|$

[14] Negative r is sometimes allowed: the coordinates (r, θ) with $r < 0$ are then interpreted to mean a point with *positive* r and θ incremented by π (this incrementation reflects a point in the origin and can be derived by going back to Cartesian coordinates), i.e. to mean the point with coordinates $(-r, \theta + \pi)$ for $r < 0$. Please do not worry about this.

§ 6. Complex numbers

The reader is presumed to already have some familiarity with complex numbers so we shall only give a brief discussion of their origins and basic properties.

The first number to deal with is $\sqrt{-1}$ which is not a real number. The need to discuss $\sqrt{-1}$ arises when one tries to solve the very simple quadratic equation

$$x^2 + 1 = 0 \tag{2.145}$$

which seems to demand that we write its solution as

$$x = \mp\sqrt{-1} \tag{2.146}$$

One denotes $\sqrt{-1}$ by i so that [15]

$$\begin{aligned} i &= \sqrt{-1} \\ \Rightarrow i^2 &= -1 \end{aligned} \tag{2.147}$$

and if we include i in our calculations we can solve other quadratic equations such as, say,

$$x^2 + 9 = 0 \tag{2.148}$$

whose solutions are

$$x = \mp 3i \tag{2.149}$$

This means that we must allow $\mp 3i$ as possible numbers; more generally we allow now any number of the form

$$a + bi, \quad a, b \in \mathbf{R} \tag{2.150}$$

By the way one can write $a + bi$ or $a + ib$ or $ib + a$ etc. and they all denote the *same* complex number.

The set of all such numbers

$$a + bi, \quad a, b \in \mathbf{R} \tag{2.151}$$

is called the set of *complex numbers* which we denote by

$$\mathbf{C} \tag{2.152}$$

to distinguish it from the set $\mathbf{R}$ of real numbers.

[15] The reader who is an electrical engineer will find that, in the engineering literature, $\sqrt{-1}$ is frequently denoted by

$$j$$

instead of i. This is because i is already in such widespread use for electric current and it is felt that too much confusion would result if i was also used for $\sqrt{-1}$.

It turns out that *all* polynomial equations such as

$$x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots a_1 x + a_0 = 0 \tag{2.153}$$

where the a_i are real or complex constants, can always be solved by numbers x in $\mathbf{C}$. Hence no more new numbers need to be adjoined to $\mathbf{R}$ once we have enlarged the real numbers $\mathbf{R}$ to the complex numbers $\mathbf{C}$.

Complex numbers are often denoted by z so we may write

$$z = x + iy, \quad (x, y \in \mathbf{R}) \tag{2.154}$$

When we do this x is called the real part of z—and denoted by $Re\, z$— and y is called the imaginary part of z—and denoted by $Im\, z$—summarising the notation is

$$z = x + iy, \quad x = Re\, z,\, y = Im\, z \tag{2.155}$$

The *complex conjugate* of a complex number z is denoted by $\bar{z}$ and is defined by

$$\bar{z} = x - iy \tag{2.156}$$

The *absolute value* or *modulus* of z is denoted by $|z|$ and is defined by

$$|z| = \sqrt{x^2 + y^2} \tag{2.157}$$

It can be quite useful to notice that

$$z\bar{z} = |z|^2 \tag{2.158}$$

§§ 6.1 Complex numbers and polar coordinates

Let θ be the angle of polar coordinates and consider next the function

$$e^{i\theta} \tag{2.159}$$

Now if we use the usual series expansion 2.51 for e^x with $x = i\theta$ we find that

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{i\theta^5}{5!} - \cdots\right) \end{aligned} \tag{2.160}$$

But recall that when doing Taylor's theorem we found that (cf. 2.124 and 2.125)

$$\begin{aligned} \cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots \\ \sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{i\theta^5}{5!} - \cdots \end{aligned} \tag{2.161}$$

Hence we have just shown the very important and useful result that

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{2.162}$$

If we replace θ by $-\theta$ and remember that $\sin(-\theta) = -\sin(\theta)$ and that $\cos(-\theta) = \cos(\theta)$ we obtain

$$e^{-i\theta} = \cos\theta - i\sin\theta \tag{2.163}$$

Now if we successively add and subtract these two formulae for $e^{i\theta}$ and $e^{-i\theta}$ we get two marvelous and very important formulae: one for $\cos\theta$ and one for $\sin\theta$. These are

$$\begin{aligned} \cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\ \sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i} \end{aligned} \tag{2.164}$$

We can now apply some of this information to obtain a very nice formula for complex numbers in polar coordinates. We start with

$$z = x + iy \tag{2.165}$$

and then use the polar coordinate information that $x = r\cos\theta$ and $y = r\sin\theta$ giving

$$\begin{aligned} z &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta}, \quad \text{using 2.162} \end{aligned} \tag{2.166}$$

Hence the form of a complex number $z = x + iy$ in polar coordinates is

$$z = re^{i\theta} \tag{2.167}$$

a result of considerable use and importance. Note that the angle θ is referred to as the *argument* of z and is denoted by $\arg(z)$, i.e.

$$\text{when } z = re^{i\theta} \quad \text{then } \arg(z) = \theta \text{ and } |z| = r \tag{2.168}$$

Two more useful properties of complex numbers are

$$\begin{aligned} |zw| &= |z||w| \\ \arg(zw) &= \arg(z) + \arg(w) \end{aligned} \tag{2.169}$$

The reader will find these easy to verify if polar form is used for z and w—i.e. write $z = r\exp[i\theta]$, $w = \rho\exp[i\theta]$ and substitute into 2.169 above.

§ 7. Some common differential equations

A differential equation is any equation for an unknown function f which involves at least one derivative of f for example

$$\frac{df}{dx} - kf = 0, \quad k \text{ a constant} \tag{2.170}$$

is a differential equation for f.

If a particle undergoes *simple harmonic motion* or SHM its displacement $x(t)$ at time t obeys the differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{2.171}$$

where ω is the frequency of the oscillations.

Actually equation 2.170 occurs in the study or radioactive decay and also in population growth. It is worth having a look at this.

Example *Radioactive decay and population growth*

Consider a sample of radioactive material with $N(t)$ atoms at time t. Radioactive decay causes $N(t)$ to *decrease* as time goes on, hence

$$\frac{dN(t)}{dt} < 0 \tag{2.172}$$

Now *experimentally* it is found that the number of atoms decaying per unit time is doubled if the size of the sample—i.e .the number of atoms in it—is doubled. In other words the *decay rate*—which is just dN/dt—is proportional to the *number of atoms* N. This is simply the statement that

$$\frac{dN(t)}{dt} = -kN(t), \quad k \text{ a positive constant} \tag{2.173}$$

so the proportionality constant is $-k$ and we see that the minus sign is there to keep dN/dt negative. This constant k is called the *decay constant* of the radioactive element.

Now we must *solve* this differential equation. To do this we proceed informally as follows: we simply observe that if a function $f(t)$ is given by

$$f(t) = Ce^{-kt}, \qquad \text{where } C \text{ is any constant} \tag{2.174}$$

then, if we differentiate $f(t)$, we find that

$$\frac{df(t)}{dt} = -kf(t) \tag{2.175}$$

i.e. $f(t$ is a solution [16] to our differential equation and we now set

$$\begin{aligned} & N(t) = f(t) \\ \text{i. e.} \qquad & N(t) = Ce^{-kt} \end{aligned} \tag{2.177}$$

So now we know what $N(t)$ looks like; notice that if $t = 0$ then $N(t) = C$ so the constant C is the number of atoms present at the beginning $t = 0$, let us therefore *rename* C by writing

$$C = N_0$$

so that we have

$$N(t) = N_0 e^{-kt} \tag{2.178}$$

and N_0 is the number of atoms present at $t = 0$.

The well known number called the *half life* of the radioactive substance can now be calculated. The half life is the time taken for exactly half the atoms to decay. If this time is denoted by T then we have

$$N(T) = \frac{N_0}{2} \tag{2.179}$$

[16] Actually it is easy to *derive* this solution if we use just a little integration. We shall meet integration in the next chapter, but for those of you who already know some integration here is the proof:

$$\begin{aligned} \frac{dN(t)}{dt} &= -kN(t) \\ \Rightarrow \frac{dN}{N} &= -k \\ \Rightarrow \int \frac{dN}{N} &= -k \int dt \\ \Rightarrow \ln(N) &= -kt + c, \quad c \text{ a constant of integration} \\ \Rightarrow N &= e^{-kt+c} \\ \Rightarrow N &= e^{-kt}e^{c} \\ \Rightarrow N(t) &= N_0 e^{-kt}, \quad \text{where } N_0 = e^{c} \end{aligned} \tag{2.176}$$

But using our formula $N(t) = N_0 e^{-kt}$ this means that

$$\begin{aligned} N_0 e^{-kT} &= \frac{N_0}{2} \\ \Rightarrow e^{-kT} &= \frac{1}{2} \\ \Rightarrow \ln(e^{-kT}) &= \ln(1/2) = -\ln(2) \\ \Rightarrow -kT &= -\ln(2) \\ \Rightarrow T &= \frac{\ln(2)}{k} \end{aligned} \tag{2.180}$$

and we have our formula for the half life T in terms of the decay constant k.

Now for Uranium 235, Carbon 14 and Iodine 120 the decay constants k have the values

$$0.9845840634 \times 10^{-9}\, y^{-1}, \quad 1.212 \times 10^{-4}\, y^{-1}, \quad 0.513\, h^{-1} \tag{2.181}$$

respectively where y stands for years and h for hours. This means that their respective half lives are

$$7.04 \times 10^8,\ \text{years} \quad 5715,\ \text{years}, \quad 1.35\ \text{hours} \tag{2.182}$$

We can also use our formula for $N(t)$ to work out how long it takes for a certain amount of the material to decay: suppose there is 45% of the radioactive material left then if the age of the sample is T', say we have

$$\begin{aligned} N(T') &= 0.45 N_0 \\ \Rightarrow N_0 e^{-kT'} &= 0.45 N_0 \\ \Rightarrow T' &= -\frac{\ln(0.45)}{k} \end{aligned} \tag{2.183}$$

Hence if we were dealing with Carbon 14 we would find that the age of the sample was

$$-\frac{\ln(0.45)}{1.212 \times 10^{-4}} = 6583.69\,\text{years} \tag{2.184}$$

We can also study population growth with this our equation

$$\frac{dN(t)}{dt} = -kN(t), \quad k \text{ a positive constant} \tag{2.185}$$

we just change the sign in front of the constant k and write

$$\frac{dN(t)}{dt} = kN(t), \quad k \text{ a positive constant} \tag{2.186}$$

So, now,

$$\frac{dN(t)}{dt} > 0 \tag{2.187}$$

and $N(t)$ *increases* with time. Then, depending on the size of k, we have a certain *doubling time* for the population instead of a *half life*. Note that this population may be one of people or bacteria or anything else which obeys our differential equation.

Finally we consider differential equations which describe the behaviour of some simple electrical circuits.

Example *Two electrical circuits and their differential equations: The discharging and charging of a capacitor*

Examine the two circuits shown in fig. 26.

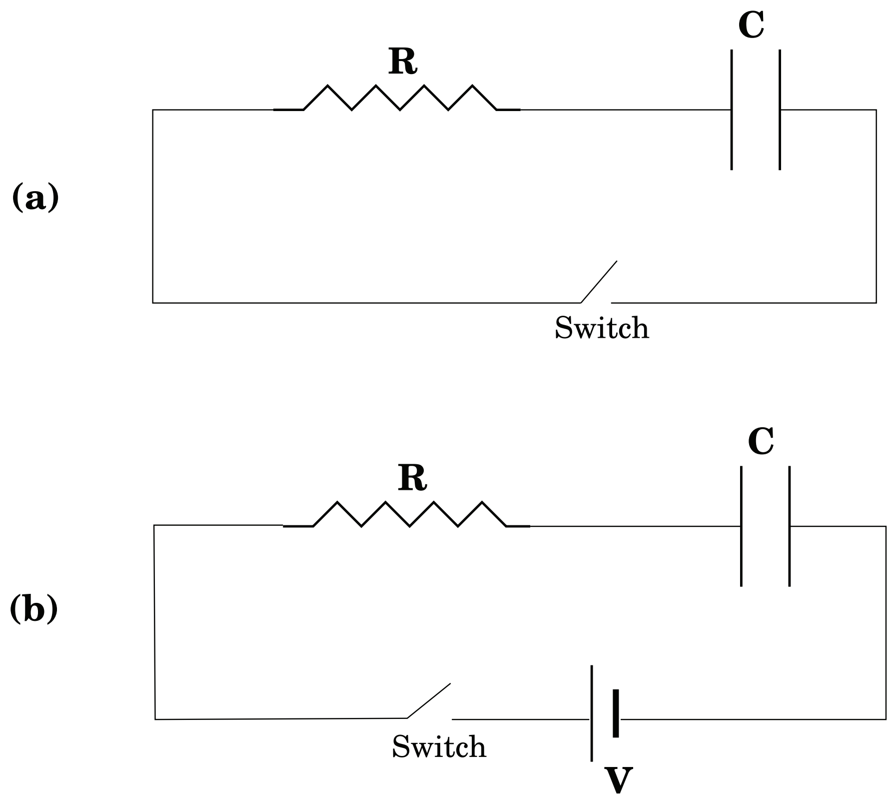


Fig. 26: The discharging and charging of a capacitor

Figure 26 (a) shows the circuit for a *discharging* capacitor while figure 26 (b) shows the circuit for a *charging* capacitor. We now examine each of these circuits in turn.

A discharging capacitor

Consider figure 26 (a) it shows a capacitor of capacitance C and a resistance R and an open switch. The resistance R represents the *necessarily non-zero* internal resistance of the capacitor, the switch and the wiring of the circuit.

We start with the *switch open* and the capacitor possessing a quantity of charge Q. Now we close the switch, thus making a closed circuit, and the capacitor begins to discharge: after t seconds the charge on the capacitor has decreased to some value $Q(t)$ and it is known that $Q(t)$ obeys the differential equation

$$R\frac{dQ(t)}{dt} + \frac{Q(t)}{C} = 0 \qquad (2.188)$$

But we can rewrite this equation 2.188 as

$$\frac{dQ(t)}{dt} = -kQ(t), \qquad \text{where } k = \frac{1}{RC} \qquad (2.189)$$

But we see that this is an instance of equation 2.175 above and so we immediately know that the solution to 2.189 is of the form

$$Q(t) = Ce^{-kt}, \qquad \text{where } k = \frac{1}{RC} \qquad (2.190)$$

If we set $t = 0$ it easy to see that the constant C is equal to the charge on the capacitor at $t = 0$—i.e. the quantity Q—and so we have finally

$$Q(t) = Qe^{-\frac{t}{RC}} \qquad (2.191)$$

but using the notation

$$\exp[x] = e^x \qquad (2.192)$$

we display the solution more clearly as

$$Q(t) = Q\exp\left[-\frac{t}{RC}\right] \qquad (2.193)$$

We end with figure 27 which shows the typical exponential decay shape of the graph of a discharging capacitor; the values of Q, R and C are 5, 2 and 3 respectively.

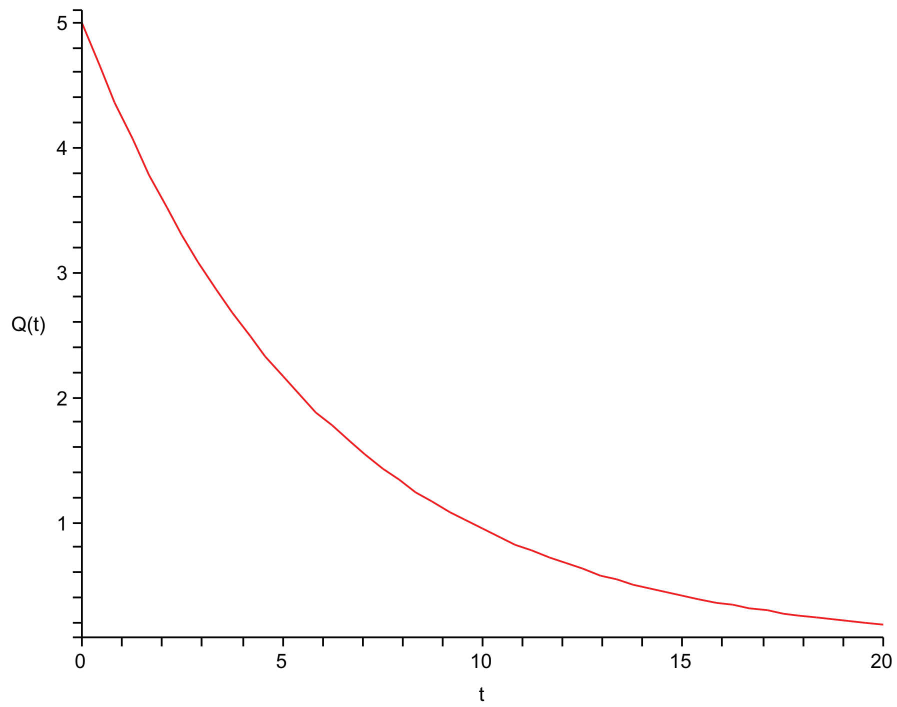


Fig. 27: A discharging capacitor

Now it is time to move on to the charging case.

A charging capacitor

Figure (b) shows the circuit relevant for a charging capacitor: one sees a capacitor of capacitance C, a battery supplying a *constant* voltage V and the usual resistance R representing the combined internal resistance (*necessarily non-zero*) of the battery, the capacitor and the wiring.

We suppose that we begin with a completely *discharged* capacitor: i.e. before the the switch in the circuit is closed the charge Q in the capacitor is *zero*. However, after the switch has been closed for t seconds, the charge $Q(t)$ in the capacitor grows to some non-zero value and obeys the differential equation

$$R\frac{dQ(t)}{dt} + \frac{Q(t)}{C} = V \tag{2.194}$$

To solve this differential equation 2.194 requires only a small tweaking of what we have done above, here is how it goes. First we note that

$$\begin{aligned}
R\frac{dQ(t)}{dt} + \frac{Q(t)}{C} &= V \\
\Rightarrow \frac{dQ(t)}{dt} &= -\frac{1}{RC}Q(t) + \frac{V}{R} \\
\Rightarrow \frac{dQ(t)}{dt} &= -\frac{1}{RC}\left(Q(t) - CV\right)
\end{aligned} \tag{2.195}$$

Now introduce the function $F(t)$ defined by

$$F(t) = Q(t) - CV \tag{2.196}$$

and notice that

$$\begin{aligned}
\frac{dF(t)}{dt} &= \frac{d}{dt}\left(Q(t) - CV\right) \\
&= \frac{dQ(t)}{dt}, \qquad \text{since } CV \text{ is a constant}
\end{aligned} \tag{2.197}$$

Next substitute 2.196 and 2.197 into the last line of 2.195 thereby obtaining

$$\frac{dF(t)}{dt} = -\frac{1}{RC}F(t) \tag{2.198}$$

which we recognise as 2.175 yet again and so the solution is

$$F(t) = F\exp\left[-\frac{t}{RC}\right] \tag{2.199}$$

where F is the value of $F(t)$ at $t = 0$; and one can check by substituting $t = 0$ into 2.196 above that this means that

$$F = -CV \tag{2.200}$$

But since

$$F(t) = Q(t) - CV \tag{2.201}$$

the final solution is given by

$$Q(t) - CV = F\exp\left[-\frac{t}{RC}\right], \quad F = -CV \tag{2.202}$$

which, after a little tidying and rearranging, can be written as

$$Q(t) = CV\left(1 - \exp\left[-\frac{t}{RC}\right]\right) \tag{2.203}$$

The reader should use the formula 2.203 above to check that the *initial* charge on the capacitor is indeed 0: i.e. check that at $t = 0$ we do have $Q(t) = 0$; it is also of interest to note that

$$\lim_{t\to\infty} Q(t) = CV \tag{2.204}$$

In other words the *final* charge on the capacitor is CV. The graph of a charging capacitor is one we have already displayed in figure 28 but here is is again (it has been obtained from formula 2.203 by using the values $C = 2$, $V = 1/2$ and $R = 0.111$).

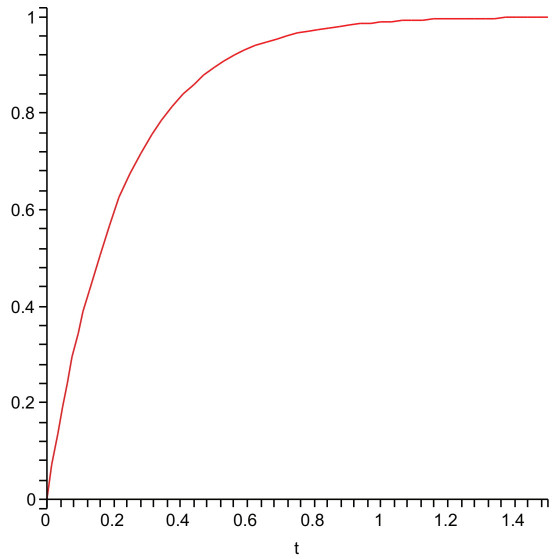


Fig. 28: A charging capacitor

Example *A second order differential equation*

Consider the differential equation

$$a\frac{d^2 f(x)}{dx^2} + b\frac{df(x)}{dx} + cf(x) = 0, \qquad a,\ b,\ c \text{ all constants} \tag{2.205}$$

This is called a *second order* differential equation because of the presence of the term $d^2 f/dx^2$—in general the order of any differential equation is called n when $d^n f/dx^n$ is the highest derivative appearing in it. We shall now learn how to solve this equation.

All one has to do is to substitute

$$f(x) = \exp[rx], \qquad r \text{ a constant} \tag{2.206}$$

into the differential equation 2.205. Doing this and demanding that the RHS be zero yields

$$\begin{aligned} ar^2 \exp[rx] + br \exp[rx] + c \exp[rx] &= 0 \\ \Rightarrow (ar^2 + br + c) \exp[rx] &= 0 \\ \Rightarrow (ar^2 + br + c) &= 0 \end{aligned} \tag{2.207}$$

Thus we have a solution of the form

$$f(x) = \exp[rx] \tag{2.208}$$

if r satisfies the quadratic equation

$$ar^2 + br + c = 0 \tag{2.209}$$

We immediately know that there are two solutions for r given by the standard formula

$$r = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a} \tag{2.210}$$

For convenience let us denote these two solutions by r_+ and r_- where

$$r_+ = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad r_- = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{2.211}$$

Hence we have two solutions to our differential equation and these are

$$f)x) = \exp[r_+ x] \quad \text{and} \quad f(x) = \exp[r_- x] \tag{2.212}$$

In fact the *general solution* to our differential equation is got by talking a linear combination of these two solutions by which we mean that *all solutions* to

$$a\frac{d^2 f(x)}{dx^2} + b\frac{df(x)}{dx} + cf(x) = 0, \qquad a,\ b,\ c \text{ all constants}$$

are given by

$$f(x) = A \exp[r_+ x] + B \exp[r_- x], \qquad \begin{cases} A \text{ and } B \text{ constants} \\ r_\mp = \dfrac{-b \mp \sqrt{b^2 - 4ac}}{2a} \end{cases} \tag{2.213}$$

All that one has to do to find the solution in a particular case is to give the constants A and B the right values.

Here is a concrete worked example

Example *The solutions to the differential equation*

$$\frac{d^2f(x)}{dx^2} - 7\frac{df(x)}{dx} + 12f(x) = 0 \tag{2.214}$$

We see that

$$a = 1, \quad b = -7, \quad c = 12 \tag{2.215}$$

and so 2.213 tells us that

$$\begin{aligned} r_{\mp} &== \frac{7 \mp \sqrt{7^2 - 4 \cdot 12}}{2} \\ \Rightarrow r_+ &= 4, \quad r_- = 3 \end{aligned} \tag{2.216}$$

and so we have the two independent solutions

$$f(x) = \exp[4x] \qquad \text{and } f(x) = \exp[3x] \tag{2.217}$$

and the general solution

$$f(x) = A\exp[4x] + B\exp[3x], \qquad A \text{ and } B \text{ any two constants} \tag{2.218}$$

Our next concrete example is the differential equation for simple harmonic motion or SHM for short.

Example *Simple harmonic motion or SHM*

The differential equation for a quantity y undergoing simple harmonic motion is

$$\frac{d^2y(t)}{dt^2} + \omega^2 y = 0 \tag{2.219}$$

Indeed we have already quoted this equation in 2.171 above. Note that the notation for unknown function of the differential equation has changed from $f(x)$ to $y(t)$, this is *only a notational change* and the reader must get used to such changes.

In any case, using our solution formula 2.213 above, we see that

$$a = 1, \quad b = 0, \quad c = \omega^2 \tag{2.220}$$

which gives the result that

$$\begin{aligned} r_{\mp} &= \mp\sqrt{-\omega^2} \\ \Rightarrow r_{\mp} &= \mp i\omega \\ \Rightarrow y(t) &= A\exp[i\omega t] + B\exp[-i\omega t] \end{aligned} \tag{2.221}$$

But now we utilise eq. 2.162 which says that

$$\exp[i\theta] = \cos(\theta) + i\sin(\theta) \tag{2.222}$$

and, on using this fact in 2.221 we find that

$$\begin{aligned} y(t) &= A\left(\cos(\omega t) + i\sin(\omega t)\right) + B\left(\cos(\omega t) - i\sin(\omega t)\right) \\ &= (A+B)\cos(\omega t) + i(A-B)\sin(\omega t) \\ \Rightarrow y(t) &= C\cos(\omega t) + D\sin(\omega t), \qquad \text{where } C = A+B, \quad D = i(A-B) \end{aligned} \tag{2.223}$$

Hence we see that the two independent solutions to the SHM differential equation are

$$\cos(\omega t) \qquad \text{and} \qquad \sin(\omega t) \tag{2.224}$$

and these functions have the well known oscillatory behaviour illustrated in fig. 29 below.

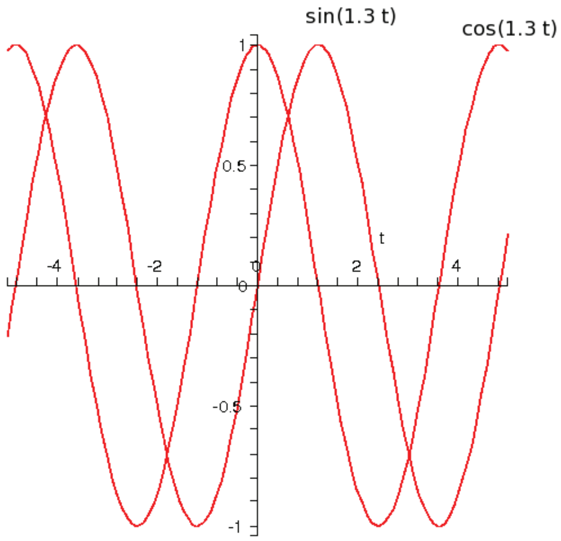


Fig. 29: The functions $\cos(\omega t)$ and $\sin(\omega t)$ $(\omega = 1.3)$

We end our section on differential equations with the differential equation for an electric circuit which possesses the fundamental trio of electrical properties which are resistance, capacitance and induction.

Example *Another electrical circuit and its differential equation: An LRC circuit*

The charge $Q(t)$ at time t on the capacitor for the circuit shown in fig. 30 obeys the differential equation

$$L\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{Q(t)}{C} = V(t) \tag{2.225}$$

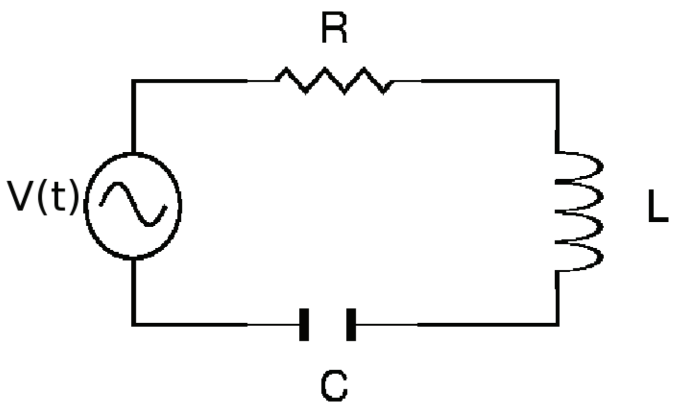


Fig. 30: An LRC circuit

$$L\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{Q(t)}{C} = V(t) \tag{2.226}$$

where L, R and C denote inductance, resistance and capacitance respectively.

The term $V(t)$ is called a *forcing term* and represents an applied voltage so let us take $V(t)$ to be an oscillatory, or AC, voltage given by

$$V(t) = V_0 \cos(\omega t) \tag{2.227}$$

First of all if

$$V_0 = 0 \tag{2.228}$$

so that there is *no applied voltage* then the nature of the solution $Q(t)$ is controlled by the sign of the parameter [17]

$$R^2 - \frac{4L}{C} \tag{2.229}$$

If we use the data in 2.213 we can verify that the possible behaviours are shown in fig. 31.

[17] The interested student may like to check that since $a = L$, $b = R$ and $c = 1/C$ then

$$r_{\mp} = \frac{-R \mp \sqrt{R^2 - \frac{4L}{C}}}{2L} = -\frac{R}{2L} \mp \frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}}$$

and it is just when $R^2 - \frac{4L}{C}$ becomes *negative* that the term $\sqrt{R^2 - \frac{4L}{C}}$ becomes pure imaginary: this is what gives an oscillatory part to the solution via the cosines and sines in the formula $\exp[i\theta] = \cos(\theta) + i\sin(\theta)$ above—indeed this happened in the SHM equation which corresponds here to the special case where $R = 0$ and $\omega^2 = 1/(LC)$.

Fig. 31: $Q(t)$ for $V_0 = 0$ with $R^2 - \frac{4L}{C} < 0$ on the left and $R^2 - \frac{4L}{C} > 0$ on the right

We see that for $R^2 - \frac{4L}{C} < 0$ we have a *damped oscillation* while for $R^2 - \frac{4L}{C} > 0$ we just have a decaying solution; if $R^2 - \frac{4L}{C} = 0$ the solution may grow a little at first but then also decays as time progresses.

Now we require $V_0 \neq 0$ so that the applied voltage *is present.*[18] In this case a principle property of the solution is that it can exhibit what is called *resonance*. This means an enhancement of the size of the solution when a certain parameter has a specified value cf. fig. 32

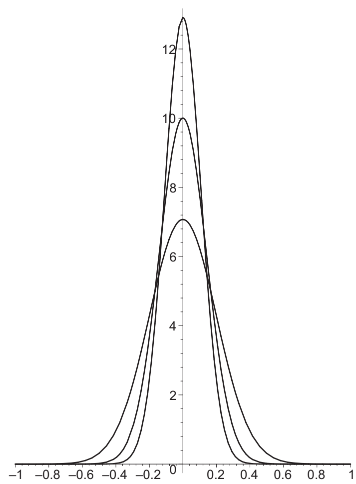


Fig. 32: Resonance: $Q(t)$ is more and more peaked as ω approaches $1/\sqrt{LC}$

Fig. 32 shows several graphs of $Q(t)$ each with a peak and a different value of ω: the more pronounced the peak the closer the driving frequency ω is to the value $1/\sqrt{LC}$. This value

[18] We have not shown how to derive a solution for such an equation with a *varying non zero* RHS so the reader must just accept what follows below without proof.

$1/\sqrt{LC}$ is called the *natural frequency* [19] of the circuit and the phenomenon of enhancement as $\omega \to 1/\sqrt{LC}$ is called *resonance.* The circuit designer usually chooses to adjust the value of the natural frequency so as to deliberately enhance or reject the applied voltage depending on his or her needs.

[19] This is the frequency that the circuit would oscillate with if there was no applied voltage and R were zero—the differential equation would then be that of simple harmonic motion.