CHAPTER I
Introductory analysis

§ 1. Notation

W E shall use $\mathbf{N}$ to denote the set of *natural numbers* $1, 2, \ldots$, in other words $\mathbf{N}$ is the set of *positive integers.* We shall use $\mathbf{Z}$ to denote *all* the integers $\ldots, -2, -1, 0, 1, 2, \ldots$ Lastly $\mathbf{R}$ and $\mathbf{C}$ will denote the real and complex numbers respectively. This same information is often, and more conveniently, displayed by writing

$$\begin{aligned}
\mathbf{N} &= \{1, 2, \ldots\} \\
\mathbf{Z} &= \{\ldots, -2, -1, 0, 1, 2, \ldots\} \\
\mathbf{R} &= \{x : x \text{ is a real number}\} \\
\mathbf{C} &= \{z : z \text{ is a complex number}\}
\end{aligned} \tag{1.1}$$

Let me just remind the reader that an equation like

$$\mathbf{R} = \{x : x \text{ is a real number}\} \tag{1.2}$$

when read aloud yields

$\mathbf{R}$ *is the set of x such that x is a real number*

and similarly for the other examples given above.

§ 2. Sequences

A collection of objects arranged in some particular order is called a *sequence.* Let us look at a few examples of sequences.

Example *The first ten positive integers (arranged in increasing order)*

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \tag{1.3}$$

Notice that these integers are arranged in *increasing order.* This order is a vital property of the sequence: if we *change the order* we create a *new sequence.* For instance we could reverse the order in our sequence $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ so as to give the new sequence

Example *The first ten positive integers (arranged in decreasing order)*

$$10, 9, 8, 7, 6, 5, 4, 3, 2, 1 \tag{1.4}$$

We see that we could make many more sequences from the first ten positive integers. In fact, we can now see, that there are as many such sequences as there are ways of *ordering* these integers.

If we recall that the number of ways of ordering n things is $n!$ (pronounced n factorial) and

$$n! = n(n-1)(n-2)\cdots 2 \cdot 1 \tag{1.5}$$

then we compute that the number of sequences that contain the ten integers $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ is

$$\begin{aligned} 10! &= 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\ &= 3628800 \end{aligned} \tag{1.6}$$

which is a pretty large number.

Sequences need not consist of mere numbers but can be made from any sort of object as long as one provides a rule for ordering them. For example one could have a sequence consisting of the top 100 golfers in the world ordered by their earnings in the previous financial year, or the top 10 football teams in a football league ordered by their point totals and so on.

Thus to discuss an arbitrary sequence it is useful to use an algebraic notation and write a sequence of, say, 5 objects or elements, as

$$s_1, s_2, s_3, s_4, s_5 \tag{1.7}$$

Suppose then that we have a sequence consisting of the first seven odd positive integers then we would write this as

$$1, 3, 5, 7, 9, 11, 13 \tag{1.8}$$

But we could also write

$$\begin{aligned} s_1, s_2, s_3, s_4, s_5, s_6, s_7 \\ \text{where } s_1 = 1, \ s_2 = 3, \ s_3 = 5, \ \text{etc.} \end{aligned} \tag{1.9}$$

More compactly still we could write

$$\begin{aligned} s_1, s_2, s_3, s_4, s_5, s_6, s_7 \\ \text{where } s_n = 2n - 1 \\ \text{and } n = 1, 2, 3, 4, 5, 6, 7 \end{aligned} \tag{1.10}$$

The point being that, as n runs through the 7 values $1, \ldots 7$, the expression $2n + 1$ runs through the 7 odd numbers $1, 3, \ldots 13$ comprising the sequence.

All the preceding sequences contained a *finite* number of elements; this need not be so: many important sequences contain an *infinite* number of elements. We have

Example *An infinite sequence*

$$2, 4, 6, 8, 10, 12, \ldots \tag{1.11}$$

The sequence 1.11 above consists of all the even positive integers and is clearly infinite.

When a sequence is finite it may be quite important just how many sequences with the same number of elements can be constructed.

Example *A National Lottery*

For example consider a national lottery where one chooses 6 numbers from the first 42 positive integers. [1]. We might then construct a sequence of all possible choices of 6 numbers from 42, in this case we are not interested in the order the choices are placed in just the number of elements in the sequence. This number—the number of ways of choosing 6 things from 42—is

$$\begin{aligned} \binom{42}{6} &= \frac{42!}{6!\, 36!} \\ &= 5245786 \end{aligned} \tag{1.12}$$

which we see is slightly under 5.25 million.

So the odds against winning the jackpot in this lottery are

$$1 \text{ in } 5245786 \tag{1.13}$$

Let us calculate how expensive it would be to bet on all the possible combinations. If we get 1 bet for € 1 euro then 5245786 bets would cost

$$€\, 5,245,786 \tag{1.14}$$

This tells us something quite interesting: namely if the jackpot exceeds € 5.24 million or so—which it occasionally does—then one can, in theory, emerge a net winner by betting on all the possible 5245786 combinations. There are of course some drawbacks to this: how does one place such a large number of bets in a reasonable time; also one might still lose if someone else also had the winning combination thus causing the jackpot to be split. These drawbacks would be somewhat offset by the large number of winning match 4 and match 5 combinations which would be achieved though.

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[1] Nowadays our lottery uses 45 rather than 42, the reader can easily adjust the analysis here to fit that case.

§ 3. Series

If we have a sequence whose elements are real or complex numbers then we can *add together* all its terms obtaining what is called a *series*. So a *sequence* such as

$$1, 4, 7, 10, 13 \tag{1.15}$$

becomes the *series*

$$1 + 4 + 7 + 10 + 13 \tag{1.16}$$

If we denote the series by S then we can write

$$S = 1 + 4 + 7 + 10 + 13 \tag{1.17}$$

and since

$$1 + 4 + 7 + 10 + 13 = 35 \tag{1.18}$$

we describe this fact by saying that "S is a series consisting of 5 terms whose sum is 35". We also labour the obvious somewhat by noting that the symbol S is also equal to the sum of the series so that

$$S = 35 \tag{1.19}$$

§ 4. Arithmetic series

Notice that in the series 1.17 above consecutive terms all differ by the same amount namely 3. Any series which has this property of consecutive terms all differing by the same amount is referred to as an *arithmetic series*. Let us now examine this property in some more detail some algebraic notation will be helpful as we can then be more general.

We only need to specify two things in order to know an arithmetic series completely: these are the *first term* and the *difference between consecutive terms*. Let us denote the first term by

$$a \tag{1.20}$$

and the difference between consecutive terms by

$$d \tag{1.21}$$

Now suppose there are exactly n terms in the arithmetic series then we shall denote a general term in the series by a_i where where each a_i is given by the formula

$$a_i = a + id$$
$$i = 0, \ldots (n-1) \tag{1.22}$$

We can now summarise all this by writing a completely arbitrary *arithmetic series* as the expression

$$a_0 + a_1 + a_2 + \cdots + a_{n-1}$$
$$\text{where } a_i = a + id \tag{1.23}$$
$$i = 0, \ldots (n-1)$$

To see how this works in practice we return to the series 1.17 which is

$$S = 1 + 4 + 7 + 10 + 13 \tag{1.24}$$

and we see that the first term is 1, so that $a = 1$ and the difference between consecutive terms 3 so $d = 3$. Thus, using $a_i = a + id$ we find that

$$a_0 = a + 0 \cdot d \Rightarrow a_0 = 1$$
$$a_1 = a + d \Rightarrow a_1 = 4$$
$$a_2 = a + 2d \Rightarrow a_2 = 7$$
$$\vdots \tag{1.25}$$
$$a_4 = a + 3d \Rightarrow a_4 = 13$$

§§ 4.1 The sum of the terms in an arithmetic series

There is quite a simple formula for the sum of the terms in an arithmetic series. It is obtained by writing the series out twice: once in the normal order and the second time in reverse order. Let us see this in action: If we write out a general arithmetic series S with n terms out we get

$$S = a_0 + a_1 + a_2 + \cdots + a_{n-1} \tag{1.26}$$

where we note that the last term is a_{n-1} not a_n because otherwise we would have $n+1$ terms instead of n. Now writing S out in reverse order we get

$$S = a_{n-1} + a_{n-2} + \cdots a_1 + a_0 \tag{1.27}$$

The last step is place both expressions together, one on top of the other, and add them giving

$$S = a_0 + a_1 + a_2 + \cdots + a_{n-1}$$
$$S = a_{n-1} + a_{n-2} + \cdots a_1 + a_0 \tag{1.28}$$
$$\text{adding} \quad \Rightarrow 2S = (a_0 + a_{n-1}) + (a_1 + a_{n-2}) + \cdots (a_{n-1} + a_0)$$

where we draw attention to the fact that we have deliberately bracketed the terms together in pairs. The reason for this is that all the pairs are in fact *equal* to each other—indeed, if we use the property that

$$a_i = a + (i - 1)d \tag{1.29}$$

we observe that

$$(a_0 + a_{n-1}) = 2a + (n-1)d$$
$$(a_1 + a_{n-2}) = 2a + (n-1)d$$
$$\vdots$$
$$(a_{n-1} + a_0) = 2a + (n-1)d$$

$$(1.30)$$

Hence we conclude that

$$2S = \underbrace{(2a + (n-1)d) + (2a + (n-1)d) + \cdots + (2a + (n-1)d)}_{n \text{ terms}}$$

$$\Rightarrow 2S = n(2a + (n-1)d) \tag{1.31}$$

$$\Rightarrow S = \frac{n(2a + (n-1)d)}{2}$$

So the sum of the first n terms of an arithmetic series

$$a + (a+d) + (a+2d) + \cdots + (a + (n-1)d) \tag{1.32}$$

is S where

$$S = \frac{n(2a + (n-1)d)}{2} \tag{1.33}$$

Example *The sum of the arithmetic series 1.17*

Recall that 1.17 is the series

$$S = 1 + 4 + 7 + 10 + 13 \tag{1.34}$$

with $a = 1$ and $d = 3$ and application of our formula 1.33 gives

$$S = \frac{5(2 \cdot 1 + 4 \cdot 3)}{2} = 35 \tag{1.35}$$

just as it should.

Example *The first 10000 odd numbers*

For this example our series is

$$S = 1 + 3 + 5 + \cdots + 19999 \tag{1.36}$$

which can also be written as

$$S = a_0 + a_1 + \cdots + a_{n-1}$$
$$\text{with} \quad a_i = 2i + 1 \quad \text{and} \quad n = 10000$$

$$(1.37)$$

Evidently we have

$$a = 1, \quad d = 2 \tag{1.38}$$

so using 1.33 we compute that

$$S = \frac{n(2 + (n-1)2)}{2} \tag{1.39}$$
$$\Rightarrow S = n^2$$

which is a useful formulae in its own right. We are dealing with the case where $n = 10000$ though we see that have a completely general formula which gives the sum of the first n odd numbers; in any case for $n = 10000$ we find that

$$S = 10000^2 = 10^8 \tag{1.40}$$

§ 5. Geometric series

In this section we shall deal with what are called *geometric series*. Here is a example.

Example *A geometric series*

$$S = 3 + 6 + 12 + 24 + 48 \tag{1.41}$$

In 1.41 above we see that the first term of the series is 3 and that the next term is got by multiplying 3 by 2 to get 6 and then 6 is multiplied by 2 to get its successor 12 and so on. We can rewrite this to make this pattern more obvious; doing this we obtain

$$S = 3 + 6 + 12 + 24 + 48$$
$$= 3 + 3 \cdot 2 + 3 \cdot 2^2 + 3 \cdot 2^3 + 3 \cdot 2^4 \tag{1.42}$$

Here is another geometric series

Example *A geometric series where the terms get smaller*

$$S = 5 + \frac{5}{3} + \frac{5}{3^2} + \frac{5}{3^3} + \frac{5}{3^4} + \frac{5}{3^5} \tag{1.43}$$

Now to display an arbitrary geometric series all we have to do is to replace the first term 3 by a and to replace the multiplier 2 by r. An arbitrary geometric series, with n terms, is now given by S where

$$S = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \tag{1.44}$$

again note that there are only n terms despite the fact that the last term is ar^{n-1}. A piece of *terminology* to be aware of is that the multiplier r is often referred to as the *ratio* of the series, this is because the ratio of successive terms is equal to r. In algebraic form we are simply observing that, if ar^i is a term in a geometric series then,

$$r = \frac{ar^i}{ar^{i-1}} \tag{1.45}$$

By returning to examples 1.41 and 1.43 we can easily verify that for 1.41 we have

$$a = 3,\ r = 2 \tag{1.46}$$

while for 1.43 we have

$$a = 5,\ r = \frac{1}{5} \tag{1.47}$$

Note that when $r > 1$ the terms in the series get successively *bigger* as the series progresses while for $r < 1$ the terms get *smaller*; it is clear, too, that were we to have $r = 1$ then the terms would stay the same and not change at all, thereby yielding a rather a boring series.

Notice that there is a certain similarity between arithmetic and geometric series namely: in arithmetic series the terms are constructed b repeated *adding* of the constant d, while in geometric series the terms are constructed by repeated *multiplication* by the constant r. Hence the main change in passing from arithmetic to geometric series is to trade addition for multiplication.

§§ 5.1 The sum of the terms in a geometric series

It is now time for us to derive a formula for the sum of the terms in a geometric series.

What we do is to write out S and then subtract from it the quantity rS. This gives us the expressions

$$\begin{aligned}
S &= a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \\
\Rightarrow rS &= ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n \\
\Rightarrow S - rS &= a - ar^n \\
\Rightarrow (1-r)S &= a - ar^n \\
\Rightarrow S &= \frac{(a - ar^n)}{(1-r)} \\
&= \frac{a(1-r^n)}{(1-r)}
\end{aligned} \tag{1.48}$$

So to summarise: the sum to n terms of a geometric series with first term a and ratio r is given by

$$S = \frac{a(1-r^n)}{(1-r)} \tag{1.49}$$

We can conclude this section by quickly computing the sums of the terms in our two examples 1.41 and 1.43.

For 1.41 where, $n = 5$, $a = 3$ and $r = 2$ we find that

$$S = \frac{3(1 - 2^5)}{(1 - 2)} = 3 \cdot 31 = 93 \tag{1.50}$$

a fact which can be verified by direct calculation.

Lastly for 1.43 we have $n = 6$, $a = 5$ and $r = 1/3$ giving

$$S = \frac{5\left(1 - \left(\frac{1}{3}\right)^6\right)}{\left(1 - \frac{1}{3}\right)} = \frac{5 \cdot 364}{3^5} = \frac{1820}{243} \tag{1.51}$$

a fact which is again verifiable by direct calculation.

§ 6. Limits and infinite series

A series can have an infinite number of terms: for example consider the series

Example *A nice infinite series*

$$\begin{aligned} S &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2} \end{aligned} \tag{1.52}$$

Such a series is called an *infinite series* and its sum S may or *may not* be finite. It turns out that this series 1.52 does have a finite value and in fact it is known that [2]

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{1.53}$$

Unfortunately S can also be infinite rendering the series not so useful as in the following.

Example *A not so nice infinite series*

$$\begin{aligned} S &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \end{aligned} \tag{1.54}$$

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[2] This is far from obvious and so we do not worry about how to prove this here; we merely want to use the result. It is easy to convince oneself experimentally of the result by using a calculator or a computer to sum up a finite number of terms.

For this series 1.54 one finds that[3] $S = \infty$. An infinite series can also have a sum S which refuses to settle down to any fixed value as one sums the terms but just *oscillates* as shown next.

Example *An oscillating series*

$$S = 2 - 2 + 2 - 2 + 2 - \cdots \tag{1.56}$$

It is clear that as we add up the terms of 1.56 the answer is 2 or 0 depending on whether we stop after an odd or an even numbers of terms. Thus S refuses to tend to any fixed value. Instead it *oscillates* between the values 0 and 2.

The terminology used to describe the behaviour in these three representative examples of infinite series is that a series such as 1.52 for which S has a *finite* value is called *convergent*; but a series such as 1.54 for which S tends to *infinity* is called *divergent*, the oscillatory series 1.56 is also called *divergent* even though no infinite value of S is encountered. Divergent and convergent series can be treated with more precision by introducing the idea of a limit to which we now turn.

A limit is an extremely widespread and fundamental concept in mathematics and it is time we met it. In the present context it is applied to a sequence but it is applied to functions as well and the same idea underlies both situations. We do not need the notion of the limit for *finite sequences* (we could use it but it we gain nothing) so we shall just explain what happens for *infinite sequences*.

Our definition[4]

Definition (The limit of an infinite sequence) *An infinite sequence*

$$\{s_1, s_2, s_3, \ldots\} \tag{1.57}$$

approaches the limit s as n tends to infinity if we can make s_n as near as we wish to s by requiring that n be large enough. We then write this symbolically as

$$\lim_{n \to \infty} s_n = s \quad or \quad s_n \to s, \ as \ n \to \infty \tag{1.58}$$

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[3] For those interested in why this series diverges we digress briefly to give a little more information on the matter. If once sums up just N terms one finds that for N large one has

$$1 + \frac{1}{2} + \cdots \frac{1}{N} \to \ln N + \gamma \tag{1.55}$$

where γ is a constant. So as N increases S goes to infinity because $\ln N$ does.

[4] This definition is not as logically tight or rigorous as it would be in an advanced mathematics course but this is deliberate: it consumes more time to be rigorous, and would limit the amount of material we could cover. Rigour does have its place though in an appropriate mathematics course. The same remark will apply to subsequent definitions that we shall give in this course.

Return now to any infinite series

$$S = s_1 + s_2 + s_3 + \cdots$$
$$= \sum_{i=1}^{\infty} s_i \tag{1.59}$$

and first let us sum up only the first N terms giving what is called the *partial sum S_N* where

$$S_N = s_1 + s_2 + s_3 + \cdots + s_N$$
$$= \sum_{i=1}^{N} s_i \tag{1.60}$$

By doing this for successively larger and larger values of N we can construct an infinite *sequence* out of these partial sums, namely the sequence

$$\{S_1, S_2, S_3, \ldots\} \tag{1.61}$$

So this is the way we define convergence of the series S:

Definition (Convergence of an infinite series) *The infinite series*

$$s_1 + s_2 + s_3 + \cdots \tag{1.62}$$

converges to the value S if

$$\lim_{N \to \infty} S_N = S \quad \text{or equivalently} \quad S_N \to S, \text{ as } N \to \infty \tag{1.63}$$

In words one says that the series $s_1 + s_2 + s_3 + \cdots$ converges to S if its sequence of partial sums $\{S_1, S_2, S_3, \ldots\}$ has the limit S.

It is important to realise that when an infinite series does *not converge* this can be either because $S_N \to \infty$ as $N \to \infty$ as is the case in 1.54; or because S_N remains finite but *oscillates* as in 1.56. It is time to consider geometric series again.

Example *The convergence of the geometric series $a + ar + ar^2 + ar^3 + \cdots$ when $r < 1$*

If we take the infinite geometric series

$$a + ar + ar^2 + ar^3 + \cdots \tag{1.64}$$

then its partial sum S_N is simply what is stated in the formula of 1.49 so we have

$$S_N = \frac{a(1 - r^N)}{(1 - r)} \tag{1.65}$$

so

$$\lim_{N \to \infty} S_N = \lim_{N \to \infty} \frac{a(1 - r^N)}{(1 - r)} \tag{1.66}$$

But all we need to know on the RHS of 1.66 is $\lim_{N \to \infty} r^N$ about which we can immediately say that

$$\lim_{N \to \infty} r^N = \begin{cases} \infty, & \text{if } r > 1 \\ 1, & \text{if } r = 1 \\ 0, & \text{if } r < 1 \end{cases} \tag{1.67}$$

But we have [5] the condition $r < 1$ so we have $\lim_{N \to \infty} r^N = 0$ which on insertion into 1.66 gives

$$\lim_{N \to \infty} S_N = \frac{a}{(1 - r)} \tag{1.68}$$

So the sum S of the infinite series is given by

$$\begin{aligned} S &= a + ar + ar^2 + ar^3 + \cdots \\ &= \frac{a}{(1 - r)}, \text{ provided } r < 1 \end{aligned} \tag{1.69}$$

If $r \geq 1$ it is easy to verify that any geometric series diverges; it is also easy to check from the formula 1.33 that all arithmetic series diverge if we allow the number of terms to become infinite. We shall only consider convergent infinite series. We can do the computation of an infinite sum for a specific example.

Example *The infinite series*

$$S = 5 + \frac{5}{3} + \frac{5}{3^2} + \frac{5}{3^3} + \frac{5}{3^4} + \cdots \tag{1.70}$$

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Here we can observe straightaway that

$$a = 5, \quad \text{and} \quad r = \frac{1}{3} \tag{1.71}$$

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[5] Actually we also allow r to be negative as well as positive and to take account of that the condition $r < 1$ must be replaced by $-1 < r < 1$, i.e. $|r| < 1$. Still more: r could be complex but then the condition $|r| < 1$ is still sufficient though $|r|$ is now extended to complex numbers r cf. §6.

so 1.69 gives us at once the result that

$$S = \frac{5}{(1 - 1/3)} = \frac{15}{2} \tag{1.72}$$

Before leaving this subject we shall consider a slightly more exotic application of summing infinite geometric series.

Example *A mathematical snowflake*

First we must make the snowflake by using triangles as the building blocks. We start with the triangle
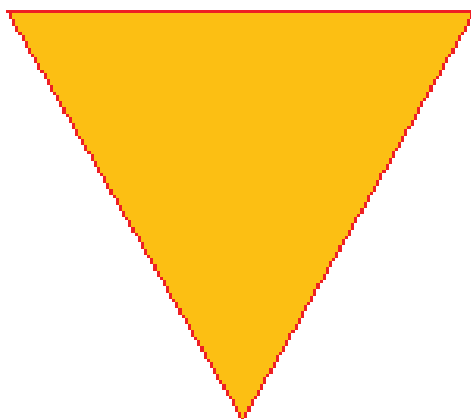

Fig. 1: The basic triangle for the snowflake.

Now we add 3 smaller triangles on to the sides of this basic triangle yielding what is shown in fig. 2

Fig. 2: The second figure for the snowflake.

We can do this yet again to get fig. 3

Fig. 3: The third figure for the snowflake.
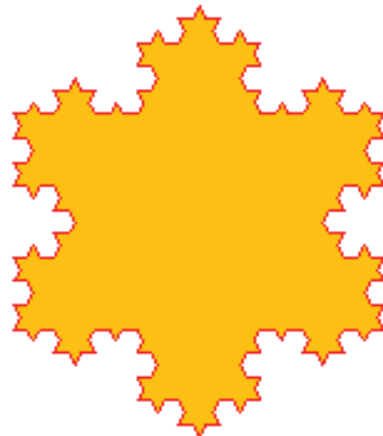
and doing it once more gives

Fig. 4: The fourth figure for the snowflake.

Now we have a figure which looks a bit like a snowflake and the idea is to repeat this process an infinite number of times. The figure that results from this process is then our snowflake [6]

Having made our snowflake the calculational task we set ourselves is to find the area contained inside it. This turns out to be given by an infinite geometric series which we now

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[6] This object is sometimes called the *Koch snowflake* after a mathematician called Koch. Its perimeter is actually infinitely long and this jagged perimeter is an example of what is called a *fractal*. Fractals turns out to have applications in many areas including telecommunications and image analysis and compression.

obtain and sum.

Example *The area of the snowflake*

Let the area of the large equilateral triangle shown in fig. 1 be

$$a \tag{1.73}$$

then the smaller triangles added on in fig. 2 are also equilateral and have a side which is $1/3$ of the length of that of the large triangle. Thus they have an area of $a/3^2$ or

$$\frac{a}{9} \tag{1.74}$$

Since three triangles were added on to get fig. 2 the area of the object depicted in fig. 2 is

$$a + \frac{3a}{9} \tag{1.75}$$

Next time, to get fig. 3, we add even smaller triangles whose area is $1/9$ that of the previous ones, that is their area is

$$\frac{a}{9^2} \tag{1.76}$$

and since a count shows that we add 2 of these for each vertex of fig. 2 then we add $6 \cdot 2 = 3 \cdot 4$ of these triangles giving the area of the "flake" of fig. 3 to be

$$a + \frac{3a}{9} + \frac{3 \cdot 4a}{9^2} \tag{1.77}$$

It is then easy to check, but we shall just accept, that the area, A say, after doing this infinitely many times is given by

$$\begin{aligned} A &= a + \frac{3a}{9} + \frac{3 \cdot 4a}{9^2} + \frac{3 \cdot 4^2 a}{9^3} + \cdots \\ &= a + \frac{3}{9}\left(a + \frac{4a}{9} + \frac{4^2 a}{9^2} + \cdots\right) \end{aligned} \tag{1.78}$$

But we can easily see that the expression

$$a + \frac{4a}{9} + \frac{4^2 a}{9^2} + \cdots \tag{1.79}$$

is an infinite geometric series. hence we know that

$$\begin{aligned} a + \frac{4a}{9} &+ \frac{4^2 a}{9^2} + \cdots \\ &= \frac{a}{(1 - 4/9)} = \frac{9a}{5} \end{aligned} \tag{1.80}$$

and if we substitute this result into 1.78 we find that

$$\begin{aligned} A &= a + \frac{3}{9}\frac{9a}{5} \\ &= \frac{8a}{5} \end{aligned} \tag{1.81}$$

So we have learned the interesting fact that the snowflake has an area given by

$$A = \frac{8a}{5} \tag{1.82}$$

where a is the area of the basic triangle of fig. 1.

§§ 6.1 Convergence of infinite series in general: the comparison and ratio tests

We have seen that an (infinite) geometric series converges when $r < 1$; for other infinite series there are several well known tests which are used to tell whether an infinite series converges. We shall just look at two of these: they are called *the comparison test* and *the ratio test*. Both are widely used.

§§ 6.2 The comparison test

Take two infinite series

$$\sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} b_n \tag{1.83}$$

and suppose that

$$0 \leq a_n \leq b_n \quad \text{for all } n \tag{1.84}$$

Then

$$\sum_{n=1}^{\infty} b_n \text{ convergent} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ convergent} \tag{1.85}$$

So the convergence of one series is decided by its comparison with another. Here is a sample of the the comparison test in action.

Example *Convergence of the two series*

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2} \tag{1.86}$$

For this example we set

$$a_n = \frac{1}{n^3} \text{ and } b_n = \frac{1}{n^2} \tag{1.87}$$

and we observe that

$$\frac{1}{n^3} \le \frac{1}{n^2}, \quad \text{for } n = 1, 2, 3, 4, \ldots \tag{1.88}$$

But we know already that $\sum_{n=1}^{\infty} (1/n^2)$ converges because we learn from consulting 1.53 that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{1.89}$$

Hence, applying the comparison test 1.85, we conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \quad \text{converges} \tag{1.90}$$

Incidentally the same argument would work with the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \quad \text{for } p > 2 \tag{1.91}$$

So the comparison test also gives the result that [7]

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{converges for } p > 2 \tag{1.92}$$

§§ **6.3 The ratio test**

Let

$$s_1 + s_2 + s_3 + \cdots$$
$$= \sum_{n=1}^{\infty} s_n \tag{1.93}$$

be an infinite series. Form the *ratio* s_{n+1}/s_n; now if we define r by writing

$$r = \lim_{n \to \infty} \left| \frac{s_{n+1}}{s_n} \right| \tag{1.94}$$

then the series $\sum_{n=1}^{\infty} s_n$ *converges* if

$$r < 1 \tag{1.95}$$

We can see how the ratio test at work in the following example.

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[7] Actually, just for information, we add that still more is known: this is that $\sum_{n=1}^{\infty} (1/n^p)$ converges precisely when $p > 1$.

Example *Convergence of the series*

$$\sum_{n=1}^{\infty} \frac{1}{n!} \tag{1.96}$$

Forming the desired ratio we see that we have to evaluate the limit

$$
\begin{aligned}
r &= \lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \right| \\
&= \lim_{n\to\infty} \frac{1}{(n+1)} \\
&= 0
\end{aligned} \tag{1.97}
$$

So we have found that $r < 1$ and so

$$\sum_{n=1}^{\infty} \frac{1}{n!} \tag{1.98}$$

converges. In fact most of you probably know already that if $e = 2.7182818284\ldots$ is the base of natural logarithms then

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} \tag{1.99}$$

and

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{1.100}$$

so for e to be a finite number at all the series 1.98 has to converge.

§ 7. Limits, functions and continuity

So far we have considered limits that arose in the study of sequences and series; however limits also arise all the time when working with functions. Still more important, as we shall see later, is the fact that the whole of differential and integral calculus is based on appropriate limits of functions.

We begin with a definition similar to that used for sequences in 1.57 above.

Definition (The limit of a function) *A function $f(x)$ approaches the limit L near $x = a$ if we can make $f(x)$ as close as we like to L by requiring x to be close enough to a. Symbolically we then write*

$$\lim_{x\to a} f(x) = L \quad \text{or} \quad f(x) \to L, \text{ as } x \to a \tag{1.101}$$

The first point to notice is that if function $f(x)$ has a graph which passes through a point a and has a value L there then its limit at a is indeed L. For example consider the perfectly

ordinary graph shown in fig. 5 below. This is the graph of the function $f(x) = x^2 - 10x + 2$ and when $x = 12$ this function $f(x)$ has the value 26.
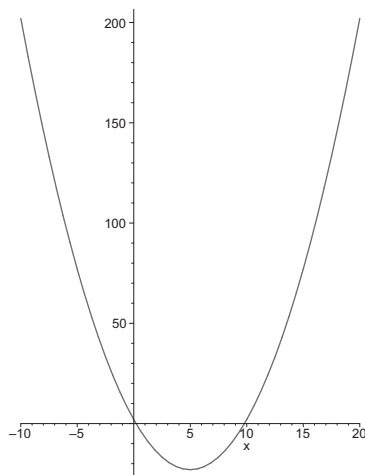

Fig. 5: The function $x^2 - 10x + 2$.

So, for this function $x^2 - 10x + 2$, we can say

$$\lim_{x \to 12} x^2 - 10x + 2 = 26 \tag{1.102}$$

The second point to notice about limits of functions (and this one is not so trivial as the first one) is that a function may have a limit L but my never attain the value L but just get arbitrarily close to it. This phenomenon is shown in fig. 6.
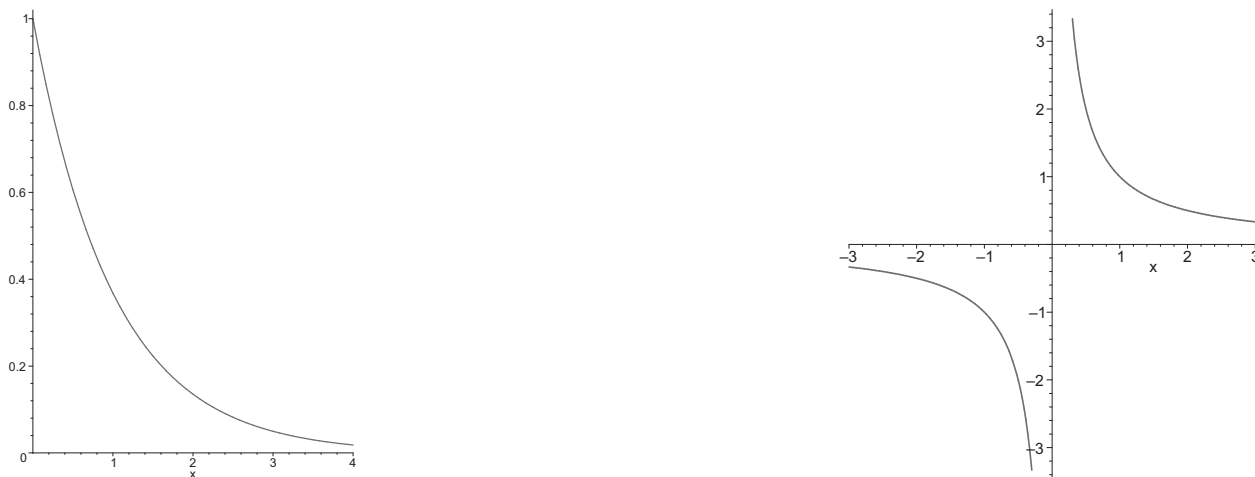

Fig. 6: The functions e^{-x} and $1/x$.

Figure 6 shows the graph of e^{-x} on the left and the graph of $1/x$ on the right.

For the case of e^{-x} we see that, as $x \to \infty$, then $e^{-x} \to 0$; but e^{-x} never actually *equals* zero (unless $x = \infty$ which is not allowed) it just gets closer and closer to zero. Hence we can

say that

$$\lim_{x \to \infty} e^{-x} = 0 \tag{1.103}$$

While for the graph of $1/x$ we observe that, if we set x equal to a *positive value*, say $x = 1$, and then let $x \to 0$, then $1/x$ tends to $+\infty$ and at $x = 0$ itself $1/x$ is infinite and so undefined.

Alternatively we also can see that if we set x equal to a *negative value*, say $x = -1$ then $1/x$ tends to $-\infty$ and is still undefined at $x = 0$. Hence $1/x$ is trying to have *two* illegal values at 0: The first is seen when we approach 0 through positive values from the right we write this one as

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \tag{1.104}$$

where the notation $x \to 0^+$ means that 0 is approached from the *right* by going through *positive* values. The second is seen by approaching 0 through negative values and is written symbolically as

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty \tag{1.105}$$

where the notation $x \to 0^-$ this time means that 0 is approached from the *left* through *negative* values.

We summarise matters by noting what was in common about the limits of the two functions e^{-x} and $1/x$ just discussed: this was that for each function the limit evaluated was never properly attained by the function.

In the first case this was because x would have to have an illegal value (namely ∞) to allow the function to reach its limiting value; while in the second case x takes on a perfectly legal limiting value but the function would have to take on illegal values (namely $\mp\infty$) for the limit to be attained. The upshot is the same in both cases and can be summarised by writing [8]

$$\lim_{x \to a} f(x) = L \quad \textbf{BUT} \quad \begin{cases} f(a) \neq L \\ \text{because either } a \text{ or } L \text{ is an illegal value} \end{cases} \tag{1.106}$$

This matter of whether $f(a) = L$ or not, when $\lim_{x \to a} f(x) = L$, is really what underlies the notion of *continuity*—a notion we are now ready to examine.

§§ 7.1 Continuity

Informally we can describe a continuous function as being one whose graph, once begun, can be completed without lifting the writing implement from the paper. With this description we

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[8] For the reader who needs reminding and doesn't want to turn the pages back we gives the values of a and L for both examples. For $f(x) = e^{-x}$ we had $a = \infty$ and $L = 0$ and for $f(x) = 1/x$ we had $x = 0$ and $L = \mp\infty$.

can immediately display an example of a continuous and a discontinuous function cf. fig. 7 below.

Fig. 7: The functions $x^4 - 2x^2$ and $f(x) = \begin{cases} x, & \text{if } x > 1 \\ x - 1, & \text{if } x \leq 1 \end{cases}$

In fig. 7 the function on the *left* is continuous while the function $f(x)$ on the *right* is discontinuous since it 'jumps' at the value $x = 1$. The value $x = 1$ is then called *a point of discontinuity* of $f(x)$. We can easily use limits to get to the bottom of what happens at the point of discontinuity of f.

We simply investigate the value $x = 1$ by evaluating the pair of limits

$$\lim_{x \to 1^-} f(x) \quad \text{and} \quad \lim_{x \to 1^+} f(x) \tag{1.107}$$

What we find from fig. 17 is that the limit from the *left* $\lim_{x \to 1^-} f(x)$ is given by

$$\lim_{x \to 1^-} f(x) = 0 \tag{1.108}$$

while the limit from the *right* $\lim_{x \to 1^+} f(x)$ is given by

$$\lim_{x \to 1^+} f(x) = 1 \tag{1.109}$$

and we note that these disagree. This then the is 'signature' of the discontinuity namely the fact that

$$\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x) \tag{1.110}$$

The more precise definition of continuity can now be unveiled.

Definition (Continuity) *A function $f(x)$ is continuous at $x = a$ if* [9]

$$\lim_{x \to a} f(x) = f(a) \tag{1.111}$$

It is clear too for $\lim_{x \to a} f(x) = f(a)$ then both the left and right limits must both *exist and agree* and they cannot do this when $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$.

For convenience of examination we display together our two examples of discontinuous functions in fig. 8

Fig. 8: The function $1/x$ and the function $f(x) = \begin{cases} x, & \text{if } x > 1 \\ x - 1, & \text{if } x \leq 1 \end{cases}$

$$\frac{1}{x} \text{ has } \begin{cases} \lim\limits_{x \to 0^-} \dfrac{1}{x} = -\infty \\ \lim\limits_{x \to 0^+} \dfrac{1}{x} = \infty \end{cases} \quad \text{and} \quad f(x) \text{ has } \begin{cases} \lim\limits_{x \to 1^-} f(x) = 0 \\ \lim\limits_{x \to 1^+} f(x) = 1 \end{cases} \tag{1.112}$$

It is now easy to see take in the visual signs of the discontinuities from the graphs and also the more precise information given in the caption 1.112.

Discontinuous functions are sometimes thought to be abnormal or in some way undesirable. This is a mistake. In fact discontinuous functions arise quite naturally in switching electrical systems on and off. Also the current practice of *digitising* many forms of transmitted data such as telephone conversations, modem data, and radio and television signals shows that functions which take *integer* values as opposed to any *real* value are very common.

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[9] It is important to realise that the point $x = a$ at which a function may or may not be continuous is of necessity *finite*. one does not evaluate functions at infinite points. This means that the example of 1.103 where we pointed out that $\lim_{x \to \infty} e^{-x} = 0$ is not an example of a discontinuity, it is simply an example of a function which does not attain one of its limiting values. In fact the function e^{-x} is continuous *everywhere*—that is it is continuous for all finite x.

Example *A simple switch*

If we take just a electrical switch which is simply turned on and off repeatedly generating a constant voltage V when on and 0 when off the we get the simple square wave graph below which is manifestly discontinuous.
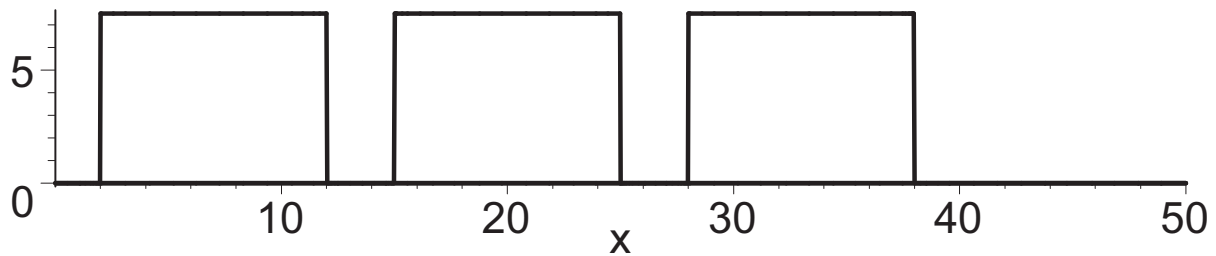

Fig. 9: A square wave

Example *A digital thermometer*

Another discontinuous function is obtained if we take the output in ^0C, say, for a series of measurements made with a thermometer placed inside a car engine; the output of this device being digital. One can imagine that this output being fed into the energy management computer—a feature of all present day cars—so that it may make adjustments to other engine parameters such as fuel supply rate, air mixture rate, spark plug and valve timings and so on. In any case the graph is just a somewhat periodic looking collection of dots and is depicted in fig. 10.
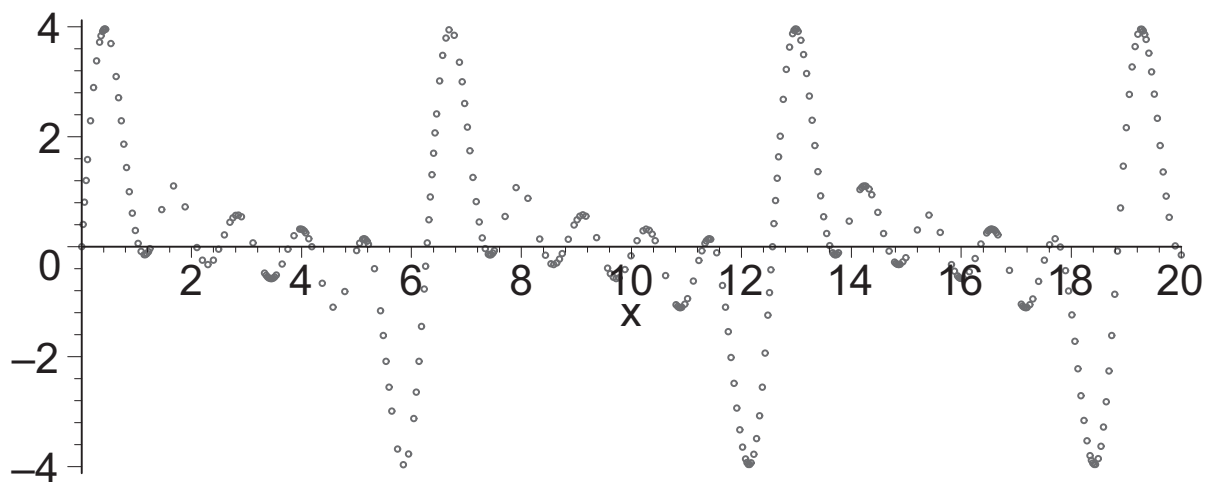

Fig. 10: A digital thermometer

The preceding two examples illustrate that functions with discontinuities are can be perfectly practical and real functions not just esoteric pathological mathematical examples. Note too that a graph such as that of the digital thermometer, which just consists of dots, is discontinuous *everywhere* not just at one point as in the second graph of fig. 8, or at a series

of isolated points as in fig. 9.

Example *A capacitor being charged*

  As our final example we take a continuous function. The graph of 28 shows the function $Q(t$ as a function of time t where $Q(t)$ is the charge on a capacitor as it is being charged up. Incidentally the function $Q(t)$ being plotted is given by

$$Q(t) = 1 - \exp(-4.5t) \tag{1.113}$$

so that the capacitor will have a charge $Q = 1$ only if t becomes infinite. [10] Hence in theory, the capacitor takes an infinite time to charge fully; of course in practice it reaches 99% of its final charge in a finite time.
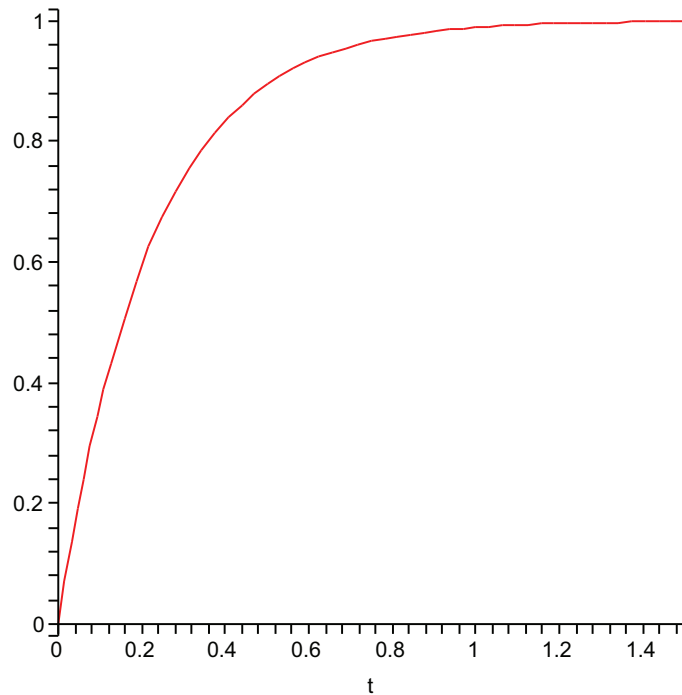

Fig. 11: A charging capacitor

  Having been introduced to the notions of limits and continuous functions we are ready to begin our work on calculus; this is the topic of the next chapter.

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[10] In the language of limits this is just the statement that $\lim_{t \to \infty} Q(t) = 1$.