Solutions to EE106 Problem Set 9

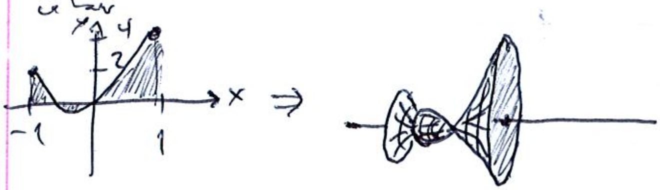
1

P.1

Recall that if we have a curve defined by a (positive) function $f(x)$, then the volume of the solid of revolution when the region under the curve between a and b is rotated around the x-axis is

$$V = \int_a^b \pi [f(x)]^2 dx .$$

For this problem, $f(x) = x + 3x^2$ and $a = -1, b = 1$. If we sketch this, we have the curve to the left; The solid looks roughly like the sketch to the right of that.



curve solid

The volume of this solid is thus

$$V = \int_{-1}^1 \pi (x+3x^2)^2 dx = \int_{-1}^1 \pi (x^2 + 6x^3 + 9x^4) dx$$

$$= \pi \int_{-1}^1 x^2 dx + 6\pi \int_{-1}^1 x^3 dx + 9\pi \int_{-1}^1 x^4 dx$$

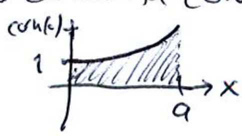
$$= \pi \left[\tfrac{1}{3} x^3\right|_{-1}^1 + 6\pi \left[\tfrac{1}{4} x^4\right|_{-1}^1 + 9\pi \left[\tfrac{1}{5} x^5\right|_{-1}^1$$

$$= \pi \left[\tfrac{1}{3} 1^3 - \tfrac{1}{3}(-1)^3\right] + 6\pi \left[\tfrac{1}{4} 1^4 - \tfrac{1}{4}(-1)^4\right] + 9\pi \left[\tfrac{1}{5} 1^5 - \tfrac{1}{5}(-1)^5\right]$$

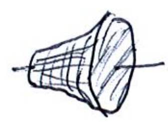
$$= \pi \cdot \tfrac{2}{3} + 6\pi \cdot 0 + 9\pi \cdot \tfrac{2}{5} = \boxed{\tfrac{64\pi}{15}}$$

P.2

The next three problems all deal with the curve $\cosh(x)$ between 0 and a; it looks like



First we get a solid of revolution, one that looks like



Its volume is

$$V = \int_a^b \pi (f(x))^2 dx = \int_0^a \pi (\cosh(x))^2 dx .$$

Using the identity given, $(\cosh(x))^2 = \frac{1}{2} + \frac{1}{2}\cosh(2x)$, so

$$V = \pi \int_0^a \left(\tfrac{1}{2} + \tfrac{1}{2}\cosh(2x)\right) dx = \frac{\pi}{2} \int_0^a dx + \frac{\pi}{2} \int_0^a \cosh(2x)\, dx$$

The first integral is just $\int_0^a dx = [x]_0^a = a - 0 = a$, while the second is

$$\int_0^a \cosh(2x)\,dx = \int_0^{2a} \cosh(u)\cdot \frac{1}{2}du = \left[\frac{1}{2}\sinh(u)\right]_0^{2a} = \frac{1}{2}\sinh(2a) - \frac{1}{2}\sinh(0)$$
$$= \frac{1}{2}\sinh(2a)$$

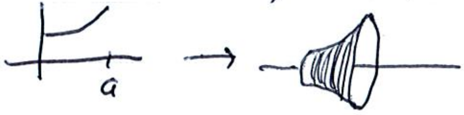
where we used the substitution $u = 2x$ in the second integral.

Thus, the volume of the solid is

$$V = \frac{\pi}{2}\cdot a + \frac{\pi}{2}\cdot\frac{1}{2}\sinh(2a) = \boxed{\frac{\pi}{4}\left[2a + \sinh(2a)\right]}.$$

P.3

Now we rotate only the curve, and not the region under it, and thus obtain a "hollow" surface of revolution.



The formula for the surface area of such an object is

$$S = \int_a 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx.$$

Here, $f(x) = \cosh(x)$, so let's compute the square root first:

$$\sqrt{1+(f'(x))^2} = \sqrt{1+(\sinh(x))^2} = \cosh(x)$$

since $\frac{d}{dx}\cosh(x) = \sinh(x)$ and $(\cosh(x))^2 - (\sinh(x))^2 = 1$. Therefore, the surface area is

$$S = \int_0^a 2\pi\cdot\cosh(x)\cdot\cosh(x)\,dx = 2\left(\int_0^a \pi(\cosh(x))^2dx\right)$$

But this integral is exactly the one we had in P.2 with an extra factor of 2 in front! Thus, we don't have to redo all the integrals, and can just say

$$S = 2\left(\frac{\pi}{4}\left[2a+\sinh(2a)\right]\right) = \boxed{\frac{\pi}{2}\left[2a+\sinh(2a)\right]}.$$

P.4

We've already done most of the work for this one: the length of a curve $f(x)$ between a and b is

$$L = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

For $f(x) = \cosh(x)$, we showed above that $\sqrt{1+(f'(x))^2} = \cosh(x)$, so

$$L = \int_0^a \cosh(x)\,dx = [\cosh(x)]_0^a = \cosh(a) - \cosh(0)$$

$$= \boxed{\cosh(a) - 1}.$$