Solutions to EE106 Problem Set 8

(1)

P.1

The key to using trig substitution is that whenever a term like x^2+a^2, where a is a constant, appears in an integrand, we try the substitution $x = a\tan(\theta)$. (If a^2-x^2 appears, then use $x = a\sin(\theta)$). The reason this works is because

$$x^2+a^2 = (a\tan(\theta))^2+a^2 = a^2(\tan(\theta))^2+a^2$$
$$= a^2\left[(\tan(\theta))^2+1\right]$$
$$= a^2\left[\frac{(\sin(\theta))^2}{(\cos(\theta))^2}+1\right]$$
$$= a^2\left[\frac{(\sin(\theta))^2+(\cos(\theta))^2}{(\cos(\theta))^2}\right]$$
$$= a^2\left(\frac{1}{\cos(\theta)}\right)^2 = a^2(\sec(\theta))^2$$

and

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(a\tan(\theta)) = a(\sec(\theta))^2 \Rightarrow dx = a(\sec(\theta))^2\,d\theta.$$

Thus, for the integral we have,

$$\int \frac{dx}{x^2+a^2} = \int \frac{a(\sec(\theta))^2\,d\theta}{a^2(\sec(\theta))^2} = \frac{1}{a}\int d\theta = \frac{1}{a}\theta + C'$$

However, we want our answer in terms of x, not θ; however, we can do this:

$$x = a\tan(\theta) \Rightarrow \tan(\theta) = \frac{x}{a} \Rightarrow \theta = \arctan\left(\frac{x}{a}\right).$$

Therefore,

$$\boxed{\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C'.}$$

which you can confirm by computing $\frac{d}{dx}\left[\frac{1}{a}\arctan\left(\frac{x}{a}\right)+C'\right]$.

P.2

The integration by parts rule says that if you're trying to find the integral of a function of the form $f(x)g'(x)$, then it can sometimes help to rewrite it as

$$\int_a^b f(x)g'(x)\,dx = \left[f(x)g(x)\right]_a^b - \int_a^b f'(x)g(x)\,dx.$$

The key is to try to find $f(x)$ and $g'(x)$ such that $f'(x)$ and $g(x)$ are as simple as possible; here,

$f(x)g'(x) = x^2\sin(x)$, so let's pick $f(x) = x^2$, $g'(x) = \sin(x)$.
(This is by no means the only choice; for example, $f(x) = x$ and $g'(x) = x\sin(x)$ also give $f(x)g'(x) = x^2\sin(x)$. However, it's probably the best choice, ~~because we see~~

So with these choices, we see

$$f(x) = x^2 \Rightarrow f'(x) = 2x, \quad g'(x) = \sin(x) \Rightarrow g(x) = -\cos(x)$$

so the integration by parts rule says

$$\int_0^{\pi/2} x^2\sin(x)\,dx = \left[-x^2\cos(x)\right|_0^{\pi/2} - \int_0^{\pi/2} (2x)(-\cos(x))\,dx$$

$$= -\frac{\pi^2}{4}\cos\left(\frac{\pi}{2}\right) - \left(-0^2\cos(0)\right) + 2\int_0^{\pi/2} x\cos(x)\,dy$$

$$= 2\int_0^{\pi/2} x\cos(x)\,dx$$

since $\cos\left(\frac{\pi}{2}\right) = 0$. Now, we integrate by parts again, with the choice $f(x) = 2x$ and $g'(x) = \cos(x)$. This gives $f'(x) = 2$ and $g(x) = \sin(x)$, so

$$\int_0^{\pi/2} x^2\sin(x)\,dx = \left[2x\sin(x)\right|_0^{\pi/2} - \int_0^{\pi/2} 2\cdot\sin(x)\,dx$$

$$= \left(2\cdot\frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) - 0\right) - \left(-2\cos(x)\right|_0^{\pi/2}$$

$$= \pi\sin\left(\frac{\pi}{2}\right) - \left(-2\cos\left(\frac{\pi}{2}\right) + 2\cos(0)\right)$$

$$= \boxed{\pi - 2}$$

because $\sin\left(\frac{\pi}{2}\right) = \cos(0) = 1$.

P.3

The partial fractions method is handy when we have an integrand that's the ratio of two polynomials, which is what we have here. Recall that the trick is to factorise the denominator and express the ratio as inverses of these factors plus a constant; in this case, $x^2 - 100 = (x+10)(x-10)$ is the denominator, so we look for constants A, B and C such that

$$\frac{x^2}{x^2-100} = A + \frac{B}{x+10} + \frac{C}{x-10}.$$

To find these constants, multiply through by $x^2 - 100$ and equate powers of x; this gives

$$x^2 = A(x^2-100) + \frac{B(x^2-100)}{x+10} + \frac{C(x^2-100)}{x-10}$$

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$$= Ax^2 - 100A + B(x-10) + C(x+10)$$
$$= Ax^2 + (B+C)x + (-100A - 10B + 10C)$$

If this is equal to x^2, then

$$A = 1,\ B+C = 0,\ -100A - 10B + 10C = 0.$$

The second gives $B = -C$, so putting this and $A = 1$ into the third gives

$$-100 - 10(-C) + 10C = 0 \Rightarrow 20C = 100 \Rightarrow C = 5.$$

Thus $A = 1$, $B = -5$ and $C = 5$, or

$$\frac{x^2}{x^2-100} = 1 - \frac{5}{x+10} + \frac{5}{x-10},$$

so

$$\int \frac{x^2}{x^2-100}\,dx = \int dx - 5\int \frac{dx}{x+10} + 5\int \frac{dx}{x-10}.$$

The first integral is easy - it's just x - and the second and third are easy as well, once we prove a general fact: let a be any constant; what's $\int \frac{dx}{x+a}$?

If we define a new variable $u = x+a$, then $\frac{du}{dx} = 1$, so $dx = du$. Thus,

$$\int \frac{dx}{x+a} = \int \frac{du}{u} = \ln(u) + C = \ln(x+a) + C.$$

The second integral has $a = 10$, the third has $a = -10$, so we put everything together to get

$$\int \frac{x^2}{x^2-100}\,dx = x - 5\ln(x+10) + 5\ln(x-10) + C.$$

P.4

This perhaps looks a bit ugly, but we can do it. Recall the general bit of advice: when confronted with an integral involving "ugly" functions, try to substitute a new variable in for the "ugliest" part. Now, $\ln(x)$ is actually not that "ugly" a function, but it's not as nice as, say, x. So let's try a substitution of the form $u = \ln(x)$. This gives $x = e^u$, so $\frac{dx}{du} = e^u \Rightarrow dx = e^u\,du$. Substituting u in for x thus gives

$$\int \frac{dx}{x(\ln(x))^3} = \int \frac{(e^u du)}{(e^u)(u)^3} = \int \frac{du}{u^3} = \int u^{-3}\,du$$

Since $\int u^n\,du = \frac{1}{n+1}u^{n+1} + C$ for all n except -1, this gives

$$\int \frac{dx}{x(\ln(x))^3} = \frac{1}{-3+1}u^{-3+1} + C = -\frac{1}{2}u^{-2} + C = -\frac{1}{2u^2} + C$$

$$= \boxed{-\frac{1}{2(\ln(x))^2} + C.}$$

To show you that there's generally more than one way to do an integral, let's do it a different (and more complicated) way using integration by parts. We want to find $f(x)$ and $g(x)$ such that
$f(x)g'(x) = \frac{1}{x(\ln(x))^3}$.
We can certainly get this if we choose $f(x) = -\frac{1}{\ln(x)}$ and $g'(x) = -\frac{1}{x(\ln(x))^2}$.
We need $f'(x)$ and $g(x)$ from these: the reciprocal rule tells us that

$$f'(x) = -\left(-\frac{\frac{d}{dx}(\ln(x))}{(\ln(x))^2}\right) = -\left(-\frac{1/x}{(\ln(x))^2}\right) = \frac{1}{x(\ln(x))^2}.$$

But we note this is just $-g'(x)$, which means $g(x) = \frac{1}{\ln(x)}$. Thus, integration by parts tells us that

$$\int \frac{dx}{x(\ln(x))^3} = \int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

$$= \left(-\frac{1}{\ln(x)}\right)\left(\frac{1}{\ln(x)}\right) - \int \left(\frac{1}{x(\ln(x))^2}\right)\left(\frac{1}{\ln(x)}\right)dx$$

$$= -\frac{1}{(\ln(x))^2} - \int \frac{dx}{x(\ln(x))^3} + C'$$

The second integral is just the same as the left-hand side, except with a minus sign. Thus, adding it to both sides gives

$$2\int \frac{dx}{x(\ln(x))^3} = -\frac{1}{(\ln(x))^2} + C$$

so

$$\boxed{\int \frac{dx}{x(\ln(x))^3} = -\frac{1}{2(\ln(x))^2} + C/2}$$

which is exactly what we had using the substitution method. There are other ways to get the same answer as well; I leave it to you to show this to yourself.