Solution to EE106 Problem Set 7

P. 1

This is straightforward; it's just the integral of $1+2x^2-3x^4$ between -1 and 1, i.e. $\int_{-1}^{1}(1+2x^2-3x^4)dx$. The integrand is just a polynomial in x, meaning we use the formula

$$\int_a^b x^n dx = \frac{1}{n+1}x^{n+1}\Big|_a^b = \frac{1}{n+1}\left(b^{n+1}-a^{n+1}\right)$$

(for $n \neq -1$). Thus,

$$\int_{-1}^{1}(1+2x^2-3x^4)dx = \int_{-1}^{1}dx + 2\int_{-1}^{1}x^2dx - 3\int_{-1}^{1}x^4dx$$

$$= \left[x\Big|_{-1}^{1} + 2\left[\frac{1}{3}x^3\Big|_{-1}^{1} - 3\left[\frac{1}{5}x^5\Big|_{-1}^{1}\right.\right.\right.$$

$$= 1-(-1) + 2\left[\frac{1}{3} - \left(-\frac{1}{3}\right)\right] - 3\left[\frac{1}{5} - \left(-\frac{1}{5}\right)\right]$$

$$= 2 + 4/3 - 6/5 = \boxed{32/15}$$

P. 2

This is one way of finding the formula for the area of a right triangle; since the hypotenuse is described by $f(x) = h - \frac{hx}{b}$, and the base is between $x=0$ and $x=b$, then the area is just the integral of $f(x)$ from 0 to b, i.e.

$$\int_0^b f(x)dx = \int_0^b \left(h - \frac{hx}{b}\right)dx = h\int_a^b dx - \frac{h}{b}\int_0^b x\,dx$$

$$= h\left[x\Big|_0^b - \frac{h}{b}\left[\frac{1}{2}x^2\Big|_0^b = h(b-0) - \frac{h}{b}\left(\frac{1}{2}b^2 - 0\right)\right.\right.$$

$$= hb - \frac{1}{2}hb = \boxed{\frac{1}{2}hb}$$

as expected.

P. 3

This problem is one of differentiation, but it's necessary to prove what the integral $\int \csc(x)dx$ is.

$F(x) = -\ln(\csc(x) + \cot(x)) + C$. This is a function of a function, so if $g(x) = -\ln(x) + C$ & $h(x) = \csc(x) + \cot(x)$, then $F(x) = g(h(x))$, so $F'(x) = g'(h(x))\,h'(x)$ by the chain rule. Now, we know the derivative of $-\ln(x) + C$ is $-1/x$, and if we go to our table of common derivatives, we see that

$$\frac{d}{dx}\csc(x) = -\cot(x)\csc(x), \quad \frac{d}{dx}(\cot(x)) = -(\csc(x))^2$$

so

$$h'(x) = -\cot(x)\csc(x) - (\csc(x))^2 = -\csc(x)[\csc(x) + \cot(x)]$$

Thus,

$$F'(x) = -\frac{1}{h(x)}h'(x)$$

$$= -\frac{1}{\csc(x)+\cot(x)}\left(-\csc(x)[\csc(x)+\cot(x)]\right)$$

$$= \boxed{\csc(x)}$$

as desired. Thus, $\int \csc(x)dx$ is indeed $-\ln(\csc(x)+\cot(x)) + C$

P. 4

The key to this is realising that the Taylor series expansion of $\arctan(x)$ is a polynomial, and so it's easy to integrate, because if

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$$

Then

$$\int \frac{dx}{1+x^2} = \int dx - \int x^2 dx + \int x^4 dx - \int x^6 dx + \dots$$

$$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^6 + \dots + C.$$

But this is also $\arctan(x)$, so

$$\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^6 + \dots + C.$$

However, this isn't true for all constants C; to determine the value of C, put $x = 0$ into both sides. The right-hand side is obviously C, and since $\arctan(0) = 0$, we find $0 = C$. Thus, we conclude that

$$\boxed{\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots}$$

is the Taylor series expansion of $\arctan(x)$ around zero.