Solution to EE106 Problem set 6

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P.1

(a) For this, we need to show ~~two~~ two things: first, that $N_0 e^{-kt}$ satisfies the given DE, and second, that its value at $t = 0$ is N_0. The second is easy:

$$N(0) = N_0 e^{-k \cdot 0} = N_0 e^0 = N_0$$

since $e^0 = 1$. To show it satisfies the DE, we use the chain rule with $f(t) = N_0 e^t$ and $g(t) = -kt$. Thus, $N(t) = f(g(t))$ gives

$$N'(t) = f'(g(t))\, g'(t) = N_0 e^{g(t)} (-k) = N_0 e^{-kt}(-k)$$

$$= -k(N_0 e^{-kt}) = -kN(t)$$

since $f'(t) = N_0 e^t$ and $g'(t) = -k$. Thus, $\boxed{N(t) = N_0 e^{-kt}}$ is the correct formula for the number of atoms at time t.

(b) If 33% of the atoms have decayed, this means that 66%, or two-thirds, of the initial sample remain. Thus, if this time is denoted by $t_{2/3}$, then

$$N(t_{2/3}) = (66\% \text{ of initial sample}) = \frac{2}{3} N_0 .$$

Since $N(t_{2/3}) = N_0 e^{-kt_{2/3}}$ comes from our function, we have

$$N_0 e^{-kt_{2/3}} = \frac{2}{3} N_0 \Rightarrow e^{-kt_{2/3}} = \frac{2}{3} \Rightarrow -kt_{2/3} = \ln(2/3)$$

$$\Rightarrow t_{2/3} = -\frac{\ln(2/3)}{k}.$$

We know that the half-life of this element is $T = 33150y$, and we also know that the relation between the decay constant and the half-life is $k = \frac{\ln(2)}{T}$. Thus,

$$t_{2/3} = \frac{-\ln(2/3)}{(\ln(2)/T)} = -\frac{\ln(2/3)}{\ln(2)} \cdot T$$

$$= -\frac{(-0.405)}{(0.693)}(33150y) = \boxed{19392y.}$$

(Obviously, it is less than T - it takes less time for one-third of the atoms to decay than it does for one-half to decay.)

P. 2

(a) To show this satisfies the DE, we need $f'(x)$ and $f''(x)$. To compute these, we show a general result: let a be any constant. Then

$$\frac{d}{dx}(e^{ax}) = ae^{ax}.$$

To see this, use the chain rule: $g(x) = e^x$ and $h(x) = ax$ give $g'(x) = e^x$ and $h'(x) = a$, so

$$\frac{d}{dx}(e^{ax}) = \frac{d}{dx}(g(h(x))) = g'(h(x))\,h'(x) = e^{h(x)}(a)$$
$$= ae^{ax}.$$

Thus, if $f(x) = Ae^{x/2} + Be^{-3x}$, then

$$f'(x) = \left(\tfrac{1}{2}\right)Ae^{x/2} + (-3)Be^{-3x} = \tfrac{1}{2}Ae^{x/2} - 3Be^{-3x}.$$

Another derivative gives

$$f''(x) = \frac{d}{dx}\left[\tfrac{1}{2}Ae^{x/2} - 3Be^{-3x}\right] = \tfrac{1}{2}A\left(\tfrac{1}{2}e^{x/2}\right) - 3B\left(-3e^{-3x}\right)$$
$$= \tfrac{1}{4}Ae^{x/2} + 9Be^{-3x}$$

So now we put these into the DE:

$$2f''(x) + 5f'(x) + 3f(x) = 2\left(\tfrac{1}{4}Ae^{x/2} + 9Be^{-3x}\right) + 5\left(\tfrac{1}{2}Ae^{x/2} - 3Be^{-3x}\right)$$
$$- 3\left(Ae^{x/2} + Be^{-3x}\right)$$
$$= \tfrac{1}{2}Ae^{x/2} + 18Be^{-3x} + \tfrac{5}{2}Ae^{x/2} - 15Be^{-3x} - 3Ae^{x/2} - 3Be^{-3x}$$
$$= \left(\tfrac{1}{2}A + \tfrac{5}{2}A - 3A\right)e^{x/2} + (18B - 15B - 3B)e^{-3x}$$
$$\boxed{= 0}$$

so $f(x)$ solves the DE no matter what A and B might be.

(b) If we put in $x = 0$, we see that

$$f(0) = Ae^0 + Be^0 = A + B$$

$f'(x) = \tfrac{1}{2}Ae^{x/2} - 3Be^{-3x}$, so $f'(0) = \tfrac{1}{2}Ae^0 - 3Be^0 = \tfrac{1}{2}A - 3B$.

We are told $f(0) = 1$ and $f'(0) = -1$, so this means that

$A + B = 1$ and $\tfrac{1}{2}A - 3B = -1$. From the first, we see $B = 1 - A$, so if we substitute for B in the second, we see

$$\tfrac{1}{2}A - 3B = \tfrac{1}{2}A - 3(1 - A) = \tfrac{1}{2}A - 3 + 3A = \tfrac{7}{2}A - 3$$

This is -1, so $\tfrac{7}{2}A - 3 = -1 \Rightarrow \tfrac{7}{2}A = 2 \Rightarrow \boxed{A = 4/7}$.

And since $B = 1 - A$,

$$\boxed{B = 1 - 4/7 = 3/7.}$$

Thus,

$$f(x) = \tfrac{4}{7}e^{x/2} + \tfrac{3}{7}e^{-3x}$$

satisfies the DE and the conditions $f(0) = 1$ and $f'(0) = -1$.

P.3

The key is recognising that this DE has the form
(2nd derivative of function) + (positive number) (function) $= 0$.
This is the most common DE of all, and so it's worth remembering
that the two independent sol'ns are always
cos ((square root of positive number) . variable) and
sin ((square root of positive number) . variable). In this case, the
positive number is $196 = 14^2$ and the variable is t, so
the two sol'ns are

$$\boxed{\cos(14t) \text{ and } \sin(14t)}$$

P. 4

For convenience, let's define two constants A and B as

$$A = \frac{1}{a^2-2a+2}, \quad B = \frac{a-1}{a^2-2a+2}$$

so that our function is $y(x) = A\sin(x) + B\cos(x)$. This saves us the trouble of rewriting them all the time, and so we'll only use their full forms after we computed the left-hand side of the DE. Since $\frac{d}{dx}\sin(x) = \cos(x)$ and $\frac{d}{dx}\cos(x) = -\sin(x)$, we see

$$y'(x) = A\cos(x) - B\sin(x)$$

and another derivative gives

$$y''(x) = -A\sin(x) - B\cos(x).$$

Then the left-hand side of the DE is

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + ay = (-A\sin(x) - B\cos(x)) + (A\cos(x) - B\sin(x)) + a(A\sin(x) + B\cos(x))$$

$$= (-A - B + aA)\sin(x) + (-B + A + aB)\cos(x).$$

Now let's write A and B out in full, so that

$$-A - B + aA = -\frac{1}{a^2-2a+2} - \frac{a-1}{a^2-2a+2} + a\frac{1}{a^2-2a+2}$$

$$= \frac{-1-(a-1)+a}{a^2-2a+2} = \frac{-1-a+1+a}{a^2-2a+2} = 0$$

and

$$-B + A + aB = -\frac{a-1}{a^2-2a+2} + \frac{1}{a^2-2a+2} + a\frac{a-1}{a^2-2a+2}$$

$$= \frac{-(a-1)+1+a(a-1)}{a^2-2a+2} = \frac{-a+1+1+a^2-a}{a^2-2a+2}$$

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$$= \frac{a^2-2a+2}{a^2-2a+2} = 1.$$

Therefore,

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + ay = (0)\sin(x) + (1)\cos(x) = \boxed{\cos(x)}$$

and therefore this $y(x)$ is indeed a sol'n to the DE.